A Convolution of the Normalized Differences of the Riemann Zeta Function Zeros with the Möbius Function

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Abstract

The convolution of the normalized differences of non-trivial Riemann zeta function zeros with the Möbius function appears to result in an ensemble of normal distributions. This hypothesis is tested using histograms, quantile-quantile plots, the Kolmogorov-Smirnov test, the Shapiro-Wilk test, and the Lilliefors test. The slopes and y-intercepts of the quantile-quantile plots are sufficiently close to the line $y = x$ that linear regression can also be done. As in testing of the Montgomery-Odlyzko Law, statistics of next-nearest neighbors are also investigated.

1 Introduction

Odlyzko's [1] tables of the zeros of the Riemann zeta function will be used. All known non-trivial zeros of the zeta function are of the form $\rho_n = \frac{1}{2} + i\gamma_n$, $\gamma_n \in R$. The normalized differences as given by Odlyzko [2] are $\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\log(\gamma_n/(2\pi))}{2\pi}$. δ_n have mean value 1 in the sense that for any positive integers N and M, $\sum_{n=N+1}^{N+M} \delta_n = M + O(\log NM)$. Let z_n denote $\sum_{i|n} \delta_i \mu(i)$ where μ is the Möbius function. The z_n values are grouped according to the prime factorizations of n. Let p, q, r, s, t, and u denote distinct primes (this number of distinct primes is sufficient for $n \leq 2,000,000$). The simplest grouping of z_n values is at n values of the form pq. (The grouping of z_n values at prime n values is not normally distributed.) The cumulative distribution function is denoted by $\Phi(p)$ where p are probabilities between 0 and 1. The probit function is the inverse cumulative distribution function for the standard normal distribution. Wichura's [3] algorithm (accurate to 16 decimal places) is used to compute the probit function. Wichura's criteria for a satisfactory implementation are met, that is, $Z_{.25} = -0.6744897501960817$, $Z_{.001} = -3.090232306167814$, and $Z_{10^{-20}} = -9.262340089798408$. The more general function $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ where μ and σ are the mean and standard deviation of the z_n distribution will be used to generate quantile-quantile plots. The Matlab command “normrnd” is used to generate random variates for the normal probability distribution.

2 The Ensemble of Normal Probability Distributions

For $n \leq 1,000,000$, there are 55 groups in the ensemble with a significantly large sample size. For a normal probability fit of the 209,867 z_n values for $n \leq 1,000,000$ and at n values of the form pq, the mean is -0.0480 with a 95% confidence interval of (-0.0507, -0.0452) and the standard deviation is 0.6372 with a 95% confidence interval of (0.6352, 0.6391). See Figure (1) for a plot of these sorted z_n values. The corresponding F^{-1} values are also shown in the plot. (The F^{-1} values are overlain with the sorted z_x values.) See Figure (2) for a quantile-quantile plot of the sorted z_n values and the F^{-1} values. For a linear least-squares fit of the curve, the slope is 1 with a 95% confidence interval of (0.9999, 1), the y-intercept is $-4.923 \cdot 10^{-5}$ with a 95% confidence interval of ($-7.316 \cdot 10^{-5}$, $-2.529 \cdot 10^{-5}$), SSE (sum of squared error) equals 6.532, R-squared (goodness-of-fit measure for linear regression) equals 0.9999, and RMSE (root mean square error) equals 0.005579. There are significant deviations from the reference line (the linear least-squares fit) in the tails of the quantile-quantile plot. See Figure (3) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 10.0 and offset by 50 bins to the right.) The histograms are almost the same. When the parameters are estimated from a sample, the Lilliefors test should be used. The null hypothesis is that the function is normal with unspecified mean and variance. At a significance level of 0.05, the null hypothesis is rejected by the Lilliefors test. The P-value (calculated probability of finding the observed results when the null hypothesis is true) is NaN (not-a-number), the test statistic is 0.0026 and the critical value is 0.0019. The test returns the approximate P-value via interpolation into the Lilliefors simulation table. A NaN is returned when P is not found within the interval $0.01 \leq p \leq .20$. These missing observations in the sample are ignored. A small P-value (typically less than or equal to 0.05) indicates strong evidence against the null hypothesis. A large P-value (greater than 0.05) indicates weak evidence against the null hypothesis. Sample values are standardized for the Kolmogorov-Smirnov test by subtracting the mean and then dividing by the standard deviation. Standardizing the samples is equivalent to setting the mean and variance of the reference distribution equal to that of the sample estimate. This changes the null distribution of the test statistic. Empirical evidence collected for the distributions considered here indicates that the test statistic of the Lilliefors test is approximately equal to that of the Kolmogorov-Smirnov test and that the critical value of the Kolmogorov-Smirnov test is about 1.52 times as large as that of the Lilliefors test. At a significance level of 0.05, the null hypothesis that the data comes from a normal distribution is rejected by the Kolmogorov-Smirnov test. The test statistic is 0.0026 and the critical value is 0.0020 (0.0030/1.52). The Kolmogorov-Smirnov statistic requires a relatively large number of data points to reject the null hypothesis. If the sample size is sufficiently large, the test may detect even "trivial" departures from the null hypothesis. See Figure (4) for a plot of the empirical CDF (cumulative distribution function) and the standard normal CDF. There is no discernible difference.

For a normal probability fit of the 168 z_n values for $n \leq 1,000,000$ and at n values of the form p^2, the mean is -0.0816 with a 95% confidence interval of (-0.1388, -0.0244) and the standard deviation is 0.3755 with a 95% confidence interval of (0.3392, 0.4206). See Figure (5) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9603 with a 95% confidence interval of (0.9329, 0.9876) and the y-intercept is -0.003212 with a 95% confidence interval of (-0.01369, -0.007267), SSE=0.75, R-squared=0.9666, and RMSE=0.06721. At a significance level of 0.05, the null hypothesis is rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0935, and the critical value is 0.0684. At a significance level of 0.05, the null hypothesis is rejected by the Kolmogorov-Smirnoff test. The test statistic is 0.0935, and the adjusted critical value (0.1037/1.52) is 0.0682. See Figure (6) for a plot of the empirical CDF and the standard normal CDF. Even for a sample this small, the CDF's match fairly well.

For a normal probability fit of the 43,864 z_n values for $n \leq 1,000,000$ and at n values of the form p^2q, the mean is -0.0239 with a 95% confidence interval of (-0.0296, -0.0181) and the standard deviation is 0.6103 with a 95% confidence interval of (0.6063, 0.6143). See Figure (7) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9998 with a 95% confidence interval of (0.9997, 0.9999) and the y-intercept is $-4.727 \cdot 10^{-5}$ with a 95% confidence interval of ($-9.973 \cdot 10^{-5}$, $-5.187 \cdot 10^{-6}$), SSE=1.376, R-squared=0.9999, and RMSE=0.005601. See Figure (8) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 8.0 and offset by 50 bins to the right.) The histograms are almost the same (other than the peak of the z_n histogram being slightly shorter). At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is 0.0780, the test statistic is 0.0040, and the critical value is 0.0042.

For a normal probability fit of the 17,459 z_n values for $n \leq 1,000,000$ and at n values of the form p^3q, the mean is 0.0145 with a 95% confidence interval of (0.0056, 0.0234) and the standard deviation is 0.6002 with a 95% confidence interval of (0.5940, 0.6066). See Figure (9) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9994 with a 95% confidence interval of (0.9992, 0.9996), the y-intercept is $-8.338 \cdot 10^{-6}$ with a 95% confidence interval of $(-0.000138, 0.0001214)$, SSE=1.334, R-squared=0.9998, and RMSE=0.00874. There are significant deviations from the reference line in the tails of the quantile-quantile plot. See Figure (10) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 5.0 and offset by 50 bins to the right.) The histograms are almost the same. At a significance level of 0.05, the null hypothesis is rejected (just barely) by the Lilliefors test. The P-value is 0.0378, the test statistic is 0.0070, and the critical value is 0.0067.

For a normal probability fit of the 206,964 z_n values for $n \leq 1,000,000$ and at n values of the form pqr, the mean is 0.0158 with a 95% confidence interval of (0.0117, 0.0199) and the standard deviation is 0.9521 with a 95% confidence interval of (0.9492, 0.9550). See Figure (11) for a quantile-quantile plot of these sorted z_n values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9999 with a 95% confidence interval of (0.9999, 0.9999), the y-intercept is $1.321 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.224 \cdot 10^{-5}, 3.886 \cdot 10^{-5})$, SSE=0.0460, R-squared=1, and RMSE=0.005906. See Figure (12) for a histogram (with 100 bins) of these values overlain with a histogram of randomly generated values for a normal probability distribution with the same mean and standard deviation. (The values have been scaled by a factor of 9.0 and offset by 50 bins to the right.) The histograms are almost the same. At a significance level of 0.05, the null hypothesis is rejected (just barely) by the Lilliefors test. The P-value is 0.0460, the test statistic is 0.0020, and the critical value is 0.0019.

Similar results have been obtained for z_n values at n values of the forms p^4q, p^2q^2, p^5q, p^3q^2, p^2qr, p^6q, p^4q^2, p^3qr, $pqrs$, p^7q, p^8q, p^2q^2r, p^4qr, p^9q, p^3q^2r, p^2qrs, p^5qr, $p^{10}q$, $p^{11}q$, p^7qr, p^6qr, $p^2q^2r^2$, p^4q^2r, p^3q^3r, p^3qrs, $pqrst$, p^4qrs, p^2q^2rs, p^5q^2r, $p^3q^2r^2$, p^8qr, p^9qr, p^3q^2rs, p^2qrst, p^4q^3r, p^6q^2r, p^5qrs, p^5q^3r, p^7q^2r, p^4q^2rs, p^3qrst, p^6q^3r, p^6qrs, $pqrstu$, $p^2q^2r^2s$, p^3q^3rs, p^7qrs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective test statistics and critical values (at a significance level of 0.05) are (0.0074, 0.0098), (0.0305, 0.0522), (0.0117, 0.0140), (0.0399, 0.0631), (0.0024, 0.0030), (0.0155, 0.0195), (0.0614, 0.0853), (0.0044, 0.0049), (0.0051, 0.0029), (0.0242, 0.0270), (0.0405, 0.0370), (0.0076, 0.0095), (0.0071, 0.0073), (0.0556, 0.0503), (0.0138, 0.0113), (0.0064, 0.0037), (0.0087, 0.0107), (0.0728, 0.0686), (0.0968, 0.0914), (0.0134, 0.0220), (0.0103, 0.0154), (0.0668, 0.0763), (0.0195, 0.0169), (0.0256, 0.0276), (0.0056, 0.0063), (0.0073, 0.0065), (0.0072, 0.0099), (0.0106, 0.0087), (0.0180, 0.0244), (0.0952, 0.0719), (0.0164, 0.0314), (0.0189, 0.0447), (0.0117, 0.0109), (0.0100, 0.0078), (0.0345, 0.0301), (0.0287, 0.0345), (0.0085, 0.0153), (0.0325, 0.0443), (0.0334, 0.0481), (0.0184, 0.0174), (0.0102, 0.0144), (0.0559, 0.0636), (0.0112, 0.0231), (0.0196, 0.0252), (0.0273, 0.0339), (0.0141, 0.0286), (0.0290, 0.0352), (0.0314, 0.0269), (0.0385, 0.0356), and (0.0204, 0.0167). The respective slopes of the linear least-squares fits (of the quantile-quantile plots) are 0.9989, 0.9836, 0.9981, 0.9757, 0.9998, 0.9964, 0.9646, 0.9997, 0.9998, 0.9942, 0.9895, 0.9988, 0.9993, 0.9804, 0.9984, 0.9996, 0.9987, 0.9686, 0.948, 0.9996, 0.9978, 0.9623, 0.9971, 0.9935, 0.9993, 0.9992, 0.9988, 0.999, 0.9951, 0.9683, 0.9929, 0.9874, 0.9982, 0.9992, 0.9914, 0.9911, 0.9975, 0.9874, 0.9852, 0.9964, 0.9998, 0.9767, 0.9955, 0.9946, 0.9899, 0.9935, 0.9898, 0.9927, 0.9892, and 0.9971. The slopes are usually relatively small (less than 0.99) for small sample sizes. The respective y-intercepts of the linear least-square fits are $3.287 \cdot 10^{-5}$, -0.000383, 0.0001179, 0.0004057, $-2.202 \cdot 10^{-5}$, 0.0004119, 0.002177, $2.8 \cdot 10^{-5}$, $3.156 \cdot 10^{-5}$, 0.0006472, 0.001361, $-2.704 \cdot 10^{-5}$, $6.111 \cdot 10^{-5}$, 0.002654, $1.086 \cdot 10^{-5}$, $3.495 \cdot 10^{-5}$, 0.000229, 0.005501, 0.009329, 0.0006415, 0.0003692, 0.002545, $6.304 \cdot 10^{-5}$, -0.0003208, 0.0001188, 0.0001706, 0.0002333, $6.16 \cdot 10^{-5}$, 0.0001465, 0.004169, 0.001378, 0.001982, 0.0001907, $9.476 \cdot 10^{-5}$, -0.0004179, 0.0004685, 0.0005859, -0.0002908, 0.001071, 0.0005006, 0.0003821, 0.0002159, 0.0009297, -0.0003097, 0.0006202, 0.0007539, 0.002325, 0.001273, 0.00082, and 0.0001185. The absolute values of the y-intercepts are usually relatively large for small sample sizes. The respective R-squared values are 0.9997, 0.9974, 0.9994, 0.9921, 1, 0.9988, 0.994, 0.9999, 0.9997, 0.9985, 0.9963, 0.9994, 0.9997, 0.9893, 0.9992, 0.9996, 0.9995, 0.9837, 0.9678, 0.9994, 0.9996, 0.9791, 0.9989, 0.9977, 0.9995, 0.9993, 0.9996, 0.9994, 0.9989, 0.9864, 0.999, 0.9984, 0.9986, 0.9996, 0.9952, 0.9977, 0.999, 0.9981, 0.9968, 0.9978, 0.9995, 0.9949, 0.999, 0.9985, 0.9948, 0.9984, 0.9957, 0.9957, 0.9948, and 0.9987. The respective sample sizes are 8,115, 288, 4,017, 197, 87,338, 2,058, 108, 33,201, 92,966, 1,077, 574, 8,629, 14,676, 310, 6,185, 55,930, 6,883, 167, 94, 1,619, 3,312, 135, 2,735, 1,028, 19,538, 18,387, 7,930, 10,282, 1,318, 152, 797, 393, 6,648, 12,753, 867, 661, 3,367, 400, 340, 2,585, 3,807, 194, 1,465, 1,235, 683, 963, 633, 1,087, 618, and 2,825. This accounts for about 91.0% of the z_n values for n less than or equal to 1,000,000. Presumably, similar results will be obtained (when enough data is available) for all other curves except z_n values at n values of the form p.

The groups where the null hypothesis is rejected are at n values of the forms pq, p^2, p^3q, pqr, $pqrs$, p^8q, p^9q, p^3q^2r, p^2qrs, $p^{10}q$, $p^{11}q$, p^4q^2r, $pqrst$, p^2q^2rs, $p^3q^2r^2$, p^3q^2rs, p^2qrst, p^4q^3r, p^4q^2rs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective sample sizes are 209,867, 168, 17,459, 206,964, 92,966, 574, 310, 6,185, 55,930, 167, 94, 2,735, 18,387, 10,282, 152, 6,648, 12,753, 867, 2,585, 1,087, 618, and 2,825. This accounts for about 65.0% of the z_n values less than or equal to 1,000,000. (The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.496 (649,623/260,269)). Excluding the groups that almost passed (at n values of the forms p^3q and pqr) reduces this to about 42.5%. For the groups with a sample size less than 5,000, the null hypothesis is also rejected by the Shapiro-Wilk normality test. In addition, the Shapiro-Wilk test rejects the null hypothesis for the group at n values of the form $p^2q^2r^2s$ (consisting of 683 elements).

The $\delta_n + \delta_{n+1}$ values approach their asymptotic behavior more slowly than the δ_n values. In the following, adjacent z_n values are added after they are grouped.

Groups that pass (but previously failed) the Lilliefors test are at n values of the forms p^3q, pqr, p^8q, p^9q, p^2qrs, $p^{10}q$, $p^{11}q$, p^4q^2r, $pqrst$, p^2q^2rs, $p^3q^2r^2$, p^2qrst, p^4q^3r, p^4q^2rs, p^5q^2rs, $p^3q^2r^2s$, and p^2q^2rst. The respective sample sizes are 17,458, 206,963, 573, 309, 55,929, 167, 93, 2,734, 18,386, 10,281, 151, 12,752, 866, 2,584, 1,086, 617, and 2,824. This accounts for about 33.4% of the z_n values less than or equal to 1,000,000. Groups that fail (but previously passed) the Lilliefors test are at n values of the forms p^3qrs, p^6qrs, and p^7qrs. The respective sample sizes are 19,537, 1,464, and 632. This accounts for about 2.2% of the z_n values less than or equal to 1,000,000.

For a normal probability fit of the 8,114 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^4q, the mean is 0.1025 with a 95% confidence interval of (0.0846, 0.1204) and the standard deviation is 0.8221 with a 95% confidence interval of (0.8096, 0.8349). See Figure (13) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.999 with a 95% confidence interval of (0.9986, 0.9993), the y-intercept is 0.0001363 with a 95% confidence interval of (-0.0001379, 0.00041061), SSE=1.269, R-squared=0.9998, and RMSE=0.01251. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0076, and the critical value is 0.0098.

For a normal probability fit of the 287 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^2q^2, the mean is -0.0478 with a 95% confidence interval of (-0.1357, 0.0401) and the standard deviation is 0.7565 with a 95% confidence interval of (0.6992, 0.8240). See Figure (14) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9819 with a 95% confidence interval of (0.973, 0.9908), the y-intercept is -0.0008932 with a 95% confidence interval of (-0.007617, 0.005831), SSE=0.9507, R-squared=0.994, and RMSE=0.05776. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0285, and the critical value is 0.0523.

For a normal probability fit of the 4,016 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^5q, the mean is 0.1644 with a 95% confidence interval of (0.1392, 0.1896) and the standard deviation is 0.8146 with a 95% confidence interval of (0.7972, 0.8328). See Figure (15) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9982 with a 95% confidence interval of (0.9977, 0.9987), the y-intercept is 0.0002639 with a 95% confidence interval of (-0.0001542, 0.0006821), SSE=0.7046, R-squared=0.9997, and RMSE=0.01325. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0075, and the critical value is 0.0140.

For a normal probability fit of the 196 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^3q^2, the mean is 0.0417 with a 95% confidence interval of (−0.0661, 0.1495) and the standard deviation is 0.7650 with a 95% confidence interval of (0.6960, 0.8493). See Figure (16) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9759 with a 95% confidence interval of (0.9642, 0.9876), the y-intercept is 0.0009936 with a 95% confidence interval of (−0.007951, 0.009938), SSE=0.7798, R-squared=0.9929, and RMSE=0.0634. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0511, and the critical value is 0.0633.

For a normal probability fit of the 87,337 $z_n + z_{n+1}$ values for $n \leq 1,000,000$ and at n values of the form p^2qr, the mean is 0.0886 with a 95% confidence interval of (0.0801, 0.0972) and the standard deviation is 1.2953 with a 95% confidence interval of (1.2893, 1.3014). See Figure (17) for a quantile-quantile plot of these sorted $z_n + z_{n+1}$ values and the corresponding F^{-1} values. For a linear least-squares fit of the curve, the slope is 0.9998 with a 95% confidence interval of (0.9998, 0.9999), the y-intercept is $-3.539 \cdot 10^{-5}$ with a 95% confidence interval of $(-8.266 \cdot 10^{-5}, 1.188 \cdot 10^{-5})$, SSE=4.416, R-squared=1, and RMSE=0.007111. At a significance level of 0.05, the null hypothesis is not rejected by the Lilliefors test. The P-value is NaN, the test statistic is 0.0022, and the critical value is 0.0030.

3 The Ensemble of Normal Probability Distributions for Zeros Higher on the Line $x = 1/2$

A difficulty with computing z_n values for zeta function zeros higher on the line $x = 1/2$ is that all the zeros up to that height must be available. For $1,000,001 \leq n \leq 2,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, $pqrs$, p^2q^2r, p^2qrs, p^3qrs, $pqrst$, p^5q^3r, p^3qrst, and $p^3q^2r^2s$. The respective test statistics and critical values are (0.0034, 0.0020), (0.0032, 0.0027), (0.0104, 0.0104), (0.0049, 0.0036), (0.0073, 0.0060), (0.0063, 0.0057), (0.0505, 0.0487), (0.0137, 0.0121), and (0.0429, 0.0364). See Figures (18) through (26) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9998, 0.9998, 0.9997, 0.9995, 0.9995, 0.9822, 0.9983, and 0.9879. The respective y-intercepts are $-3.33 \cdot 10^{-5}$, $8.459 \cdot 10^{-6}$, $5.28 \cdot 10^{-5}$, $3.217 \cdot 10^{-6}$, $3.84 \cdot 10^{-5}$, 0.0001036, −0.0006766, 0.0002712, and 0.0002291. The respective R-squared values are 0.9999, 0.9998, 0.9996, 0.9997, 0.9997, 0.9997, 0.9915, 0.9991, and 0.9927. The respective sample sizes are 197,194, 104,322, 7,322, 60,656, 21,543, 24,568, 331, 5,398, and 593. This accounts for about 42.2% of the z_n values between 1,000,001 and 2,000,000. Excluding the group that almost passed (at n values of the form p^2q^2r) reduces this to about 41.5%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.8756 (422,065/482,018). This is a substantial improvement over the 65.0% that were rejected for z_n values less than or equal to 1,000,000. For $1,000,001 \leq n \leq 2,000,000$, about 90.4% of the z_n values are in the ensemble. This is slightly less than the percentage for $n \leq 1,000,000$, so a better measure of the improvement is the fail/pass ratio (2.496 for $n \leq 1,000,000$).

For $2,000,001 \leq n \leq 3,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, p^2q, $pqrs$, p^2qrs, p^3qrs, p^2q^2rs, p^2qrst, and p^7q^2r. The respective test statistics and critical values are (0.0034, 0.0020), (0.0050, 0.0046), (0.0048, 0.0027), (0.0048, 0.0036), (0.0069, 0.0060), (0.0092, 0.0087), (0.0068, 0.0065), and (0.0560, 0.0549). See Figures (27) through (34) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 0.9997, 0.9998, 0.9997, 0.9994, 0.999, 0.9993, and 0.9748. The respective y-intercepts are $-5.486 \cdot 10^{-5}$, $2.171 \cdot 10^{-5}$, $4.304 \cdot 10^{-5}$, $6.58 \cdot 10^{-5}$, 0.0001029, 0.0001181, $9.935 \cdot 10^{-5}$, and 0.0004454. The respective R-squared values are 0.9999, 0.9999, 0.9998, 0.9997, 0.9997, 0.9995, 0.9995, and 0.9825. The respective sample sizes are 193,116, 36,632, 108,253, 61,819, 22,100, 10,485, 18,761, and 260. This accounts for about 45.1% of the z_n values between 2,000,001 and 3,000,000. Excluding the group that almost passed (at n values of the form p^2qrst) reduces this to about 43.3%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.976 (451,426/462,489). For $2,000,001 \leq n \leq 3,000,000$, about 91.4% of the z_n values are in the ensemble.

For $3,000,001 \leq n \leq 4,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, p^2, p^3q, pqr, p^4q, p^4q^2, $pqrs$, p^7q, p^2q^2r, p^3q^2r, p^2qrs, $pqrst$, and p^3q^2rs. The respective test statistics and critical values are (0.0044, 0.0020), (0.2009, 0.1519), (0.0075, 0.0075), (0.0022, 0.0019), (0.0111, 0.0110), (0.1938, 0.1832), (0.0032, 0.0027), (0.0315, 0.0309), (0.0116, 0.0108), (0.0144, 0.0129), (0.0046, 0.0035), (0.0060, 0.0053), and (0.0128, 0.0107). See Figures (35) through (47) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 0.8576, 0.9992, 1, 0.9985, 0.8731, 0.9999, 0.9914, 0.9987, 0.9981, 0.9997, 0.9996, and 0.9985. The respective y-intercepts are $-1.64 \cdot 10^{-5}$, -0.01312, $3.608 \cdot 10^{-5}$, $-3.154 \cdot 10^{-5}$, $4.868 \cdot 10^{-5}$, -0.01215, $-1.583 \cdot 10^{-5}$, 0.001011, $5.006 \cdot 10^{-5}$, $6.088 \cdot 10^{-5}$, $4.512 \cdot 10^{-5}$, $3.828 \cdot 10^{-5}$, and 0.0001901. The respective R-squared values are 0.9998, 0.8613, 0.9997, 1, 0.9992, 0.9454, 0.9999, 0.9956, 0.9995, 0.9992, 0.9997, 0.9997, and 0.9992. The respective sample sizes are 190,506, 34, 14,110, 209,072, 6,499, 22, 110,409, 824, 6,695, 4,753, 62,558, 28,451, and 6,913. This accounts for about 64.1% of the z_n values between 3,000,001 and 4,000,000. Excluding the groups that almost passed (at n values of the forms p^3q, pqr, and p^4q) reduces this to about 41.1%. There is little deviation from the reference lines for the groups at n values of the forms pqr and $pqrs$. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about

2.345 (640,846/273,283). For $3,000,001 \leq n \leq 4,000,000$, about 91.4% of the z_n values are in the ensemble.

For $4,000,001 \leq n \leq 5,000,000$, the null hypothesis is rejected by the Lilliefors test for groups at n values of the forms pq, $pqrs$, p^4qr, p^2qrs, p^3qrs, $pqrst$, and p^2q^2rs. The respective test statistics and critical values are (0.0039, 0.0020), (0.0041, 0.0026), (0.0090, 0.0076), (0.0046, 0.0035), (0.0061, 0.0059), (0.0092, 0.0051), and (0.0089, 0.0087). See Figures (48) through (54) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9985, 0.9999, 0.9994, 0.9997, 0.9995, 0.9995, and 0.999. The respective y-intercepts are $-6.096 \cdot 10^{-5}$, $3.534 \cdot 10^{-5}$, 0.0001049, $-2.306 \cdot 10^{-5}$, $4.244 \cdot 10^{-5}$, 0.0001044, and 0.0001469. The respective R-squared values are 0.9999, 0.9999, 0.9998, 0.9998, 0.9997, 0.9996, and 0.9994. The respective sample sizes are 188,260, 111,897, 13,546, 63,148, 22,640, 29,655, and 10,342. This accounts for about 43.9% of the z_n values between 4,000,001 and 5,000,000. Excluding the groups that almost passed (at n values of the forms p^3qrs and p^2q^2rs) reduces this to about 40.6%. There is little deviation from the reference line for the group at n values of the form p^2qrs. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.9260 (439,448/474,583). For $4,000,001 \leq n \leq 5,000,000$, about 91.4% of the z_n values are in the ensemble.

For $5,000,001 \leq n \leq 6,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, pqr, $pqrs$, p^7q, p^2qrs, p^2q^2rs, and p^2qrst. The respective test statistics and critical values are (0.0034, 0.0020), (0.0024, 0.0019), (0.0036, 0.0026), (0.0315, 0.0314), (0.0046, 0.0035), (0.0089, 0.0087), and (0.0065, 0.0061). The null hypothesis is not rejected by the Shapiro-Wilk test for the group at n values of the form p^7q. See Figures (55) through (61) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9998, 1, 0.9999, 0.9927, 0.9998, 0.9991, and 0.9995. The respective y-intercepts of the reference lines are $-4.225 \cdot 10^{-5}$, $-1.291 \cdot 10^{-5}$, $4.015 \cdot 10^{-5}$, 0.0005051, $6.973 \cdot 10^{-5}$, $9.926 \cdot 10^{-5}$, and 0.0001058. The respective R-squared values are 0.9999, 1, 0.999, 0.9988, 0.9999, 0.9997, and 0.9998. The respective sample sizes are 187,276, 208,604, 113,555, 795, 63,377, 10,400, and 21,108. This accounts for about 60.5% of the z_n values between 5,000,001 and 6,000,000. Excluding the groups that almost passed (at n values of the forms pqr, p^7q, p^2q^2rs, and p^2qrst) reduces this to about 36.4%. There is little deviation from the reference lines for the groups at n values of the forms pq and pqr. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 1.9990 (605,115/302,710). For $5,000,001 \leq n \leq 6,000,000$, about 90.8% of the z_n values are in the ensemble.

For $6,000,001 \leq n \leq 7,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, p^5q, $pqrs$, p^2qrs, $p^{11}q$, p^3q^3r, $pqrst$, and $p^3q^2r^2s$. The respective test statistics and critical values are (0.0033, 0.0021), (0.0187, 0.0162), (0.0028, 0.0026), (0.0051, 0.0035), (0.1306, 0.1163), (0.0449, 0.0329), (0.0066, 0.0050), and (0.0403, 0.0394). See Figures (62) through (69) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9975, 0.9999, 0.9997, 0.9393, 0.9909, 0.9995, and 0.9886. The respective y-intercepts of the reference lines are $1.869 \cdot 10^{-5}$, 0.0001429, $2.921 \cdot 10^{-6}$, $5.509 \cdot 10^{-5}$, 0.0004251, -0.0002774, $3.099 \cdot 10^{-5}$, and 0.0007343. The respective R-squared values are 0.9999, 0.9992, 0.9999, 0.9998, 0.9814, 0.9961, 0.9996, and 0.9964. The respective sample sizes are 185,638, 2,996, 114,326, 63,670, 58, 725, 31,253, and 506. This accounts for about 39.9% of the z_n values between 6,000,001 and 7,000,000. Excluding the group that almost passed (at n values of the form $pqrs$) reduces this to about 28.5%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 0.7752 (399,172/514,949). For $6,000,001 \leq n \leq 7,000,000$, about 91.4% of the z_n values are in the ensemble.

For $7,000,001 \leq n \leq 8,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms pq, pqr, $pqrs$, p^2qrs, p^4q^2r, $pqrst$, p^2q^2rs, and p^3q^3rs. The respective test statistics and critical values are (0.0036, 0.0021), (0.0022, 0.0019), (0.0038, 0.0026), (0.0045, 0.0035), (0.0223, 0.0204), (0.0056, 0.0050), (0.114, 0.0088), and (0.0307, 0.0276). See Figures (70) through (77) for quantile-quantile plots of these groups. The respective slopes of the reference lines are 0.9999, 0.9999, 0.9998, 0.9998, 0.9961, 0.9996, 0.9988, and 0.9921. The respective y-intercepts of the reference lines are $2.729 \cdot 10^{-5}$, $-2.123 \cdot 10^{-5}$, $-1.139 \cdot 10^{-5}$, $2.311 \cdot 10^{-5}$, $4.667 \cdot 10^{-5}$, $3.697 \cdot 10^{-5}$, 0.000131, and 0.0006379. The respective R-squared values are 0.9998, 1, 0.9999, 0.9998, 0.9986, 0.9997, 0.999, and 0.9948. The respective sample sizes are 184,760, 208,784, 115,239, 64,000, 1,893, 31,885, 10,232, and 1,034. This accounts for about 61.8% of the z_n values between 7,000,001 and 8,000,000. Excluding the group that almost passed (at n values of the form pqr) reduces this to about 40.9%. The ratio of the number of z_n values that failed the Lilliefors test to those that passed is about 2.0844 (617,827/296,399). For $7,000,001 \leq n \leq 8,000,000$, about 91.4% of the z_n values are in the ensemble.

Hervé and Molin [4] give the formula $\frac{.895}{f_N}$, $f_N = \frac{.83+N}{\sqrt{N}} - .01$, $N > 50$, where N is the sample size for the critical value of the Lilliefors test at a confidence level of 0.05. For large sample sizes, f_N is approximately equal to $\sqrt{N}$. For $1 \leq n \leq 1,000,000$, $1,000,001 \leq n \leq 2,000,000$, $2,000,001 \leq n \leq 3,000,000$, $3,000,001 \leq n \leq 4,000,000$, $4,000,001 \leq n \leq 5,000,000$, $5,000,001 \leq n \leq 6,000,000$, $6,000,001 \leq n \leq 7,000,000$, and $7,000,001 \leq n \leq 8,000,000$ the respective percentages of the z_n values that failed the Lilliefors test (excluding groups that almost passed) are 42.5%, 41.5%, 43.3%, 41.1%, 40.6%, 36.4%, 28.5%, and 40.9%. The rejection rate appears to be decreasing, but the evidence is not that conclusive. The respective percentages of the z_n values that are in the ensemble are 91.0%, 90.4%, 91.4%, 91.4%, 91.4%, 90.8%, 91.4%, and 91.4%. Although this percentage is almost constant, the relative number of z_n values in the group at n values of the form pq (and other groups where n has a simple factorization) is decreasing. See Figure (78) for a plot of the number of z_n values in the groups at n values of the forms pq, p^2q, p^3q, pqr, p^2qr, and p^3qr. Except for the group at n values of the form pqr, the number of z_n values in the groups is decreasing. See Figure (79) for a plot of the number of z_n values in the groups at n values of the forms $pqrs$, p^2qrs, p^3qrs, $pqrst$, p^2q^2rs, and p^2qrst. Except for the group at n values of the form p^2q^2rs, the number of z_n values in the groups is increasing.

4 Another Ensemble of Normal Probability Distributions

The ensemble consists of 60 groups of z_n values at n values of the forms p^3, p^3q^3, p^5q^2, p^6q^2, p^5q^3, p^7q^2, $p^{12}q$, p^6q^3, p^4q^4, $p^{13}q$, $p^{14}q$, p^9q^2, p^8q^2, p^5q^4, p^7q^2, $p^{10}q^2$, p^6q^4, $p^{16}q$, $p^{15}q$, $p^3q^3r^2$, $p^4q^2r^2$, $p^{10}qr$, $p^{11}q^2$, p^8q^3, p^7q^4, p^6q^5, p^5q^5, $p^{17}q$, $p^{18}q$, $p^{11}qr$, $p^{12}q^2$, p^9q^3, p^8q^4, p^7q^5, $p^4q^3r^2$, $p^5q^2r^2$, $p^{12}qr$, $p^{13}q^2$, $p^{10}q^3$, p^9q^4, p^8q^5, p^8q^2r, $p^3q^3r^3$, $p^{19}q$, p^7q^3r, $p^6q^2r^2$, p^9q^2r, p^7q^6, $p^{10}q^4$, $p^{11}q^3$, $p^{14}q^2$, $p^{13}qr$, p^8qrs, $p^{10}q^2r$, p^8q^3r, $p^5q^3r^2$, p^7q^2r, $p^{14}qr$, $p^{12}q^3$, and $p^{15}q^2$. For $1 \leq n \leq 8,000,000$, the respective sample sizes are 46, 56, 163, 107, 38, 73, 301, 26, 14, 165, 92, 37, 51, 19, 19, 26, 12, 29, 52, 167, 252, 1,547, 19, 14, 10, 8, 6, 17, 9, 773, 14, 11, 8, 6, 165, 143, 387, 11, 9, 5, 4, 1,110, 19, 5, 604, 90, 591, 4, 4, 6, 7, 189, 3,047, 313, 313, 92, 56, 90, 5, and 5. For $1 \leq n \leq 8,000,000$, the null hypothesis is rejected by the Lillifors test for z_n values at n values of forms $p^{12}q$, $p^{14}q$, $p^3q^3r^2$, $p^4q^2r^2$, p^8q^4, p^7q^5, $p^4q^3r^2$, $p^5q^2r^2$, p^9q^4, p^8q^5, $p^6q^2r^2$, $p^{10}q^4$, $p^{11}q^3$, $p^5q^3r^2$, $p^7q^2r^2$, and $p^{12}q^3$. The respective test statistics and critical values are (0.0558, 0.0511), (0.0982, 0.0924), (0.0994, 0.0686), (0.0660, 0.0558), (0.2909, 0.2850), (0.3477, 0.3190), (0.1018, 0.0690), (0.0786, 0.0741), (0.3960, 0.3370), (0.4100, 0.3810), (0.0980, 0.0934), (0.4100, 0.3810), (0.3240, 0.3190), (0.1259, 0.0924), (0.1222, 0.1184), and (0.3960, 0.3370). See Figures (80) through (95) for the quantile-quantile plots of these sorted z_n values and the corresponding F^{-1} values. The respective slopes of the reference lines are 0.9796, 0.9459, 0.968, 0.9777, 0.7616, 0.6847, 0.971, 0.9707, 0.6219, 0.5832, 0.9571, 0.5832, 0.7091, 0.9531, 0.9384, and 0.6219. The respective y-intercepts of the reference lines are 0.003147, 0.01042, 0.00308, 0.001663, 0.0005612, −0.009831, 0.004, 0.0037, 0.02662, 0.06879, 0.006284, 0.06879, 0.04205, 0.007199, 0.01304, and 0.02662. The respective R-squared values are 0.9885, 0.9642, 0.9822, 0.9889, 0.8905, 0.7853, 0.9888, 0.9939, 0.6898, 0.6603, 0.9886, 0.6603, 0.8425, 0.9791, 0.9821, and 0.6898. The respective sample sizes are 301, 92, 167, 252, 8, 6, 165, 143, 5, 4, 90, 4, 6, 92, 56, and 5. The fail/pass ratio is about 0.1387 (1,396/10,068).

5 Minima and Maxima in the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ Distribution

Let l_n denote the minimum of the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ distribution. l_n is a slowly decreasing step function with an initial value of $\gamma_2 - \gamma_1$. For n less than 500,000, l_n decreases in value at $n = 2$, 3, 6, 15, 195, 435, 615, 1590, 4305, 4,935, 7,995, 17,355, 32,595, 72,615, 228,165, and 261,555. The prime factorizations of these values are 2, 3, 2·3, $3 \cdot 5$, $3 \cdot 5 \cdot 13$, $3 \cdot 5 \cdot 29$, $3 \cdot 5 \cdot 41$, $2 \cdot 3 \cdot 5 \cdot 53$, $3 \cdot 5 \cdot 7 \cdot 41$, $3 \cdot 5 \cdot 7 \cdot 47$, $3 \cdot 5 \cdot 13 \cdot 41$, $3 \cdot 5 \cdot 13 \cdot 89$, $3 \cdot 5 \cdot 41 \cdot 53$, $3 \cdot 5 \cdot 47 \cdot 103$, $3 \cdot 5 \cdot 7 \cdot 41 \cdot 53$, and $3 \cdot 5 \cdot 7 \cdot 47 \cdot 53$ respectively. The number of prime factors is non-decreasing. The corresponding l_n values are 2.898497, 1.473297, 0.817020, −1.210552, −2.2385884, −2.584503, −3.612455, −3.717184, −3.834182, −4.450978, −5.171634, −5.675754, −6.993299, −7.754984, −8.439405, and −8.711062. The values of n that are the products of 2, $2 \cdot 3$, $2 \cdot 3 \cdot 5$, $2 \cdot 3 \cdot 5 \cdot 7$, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, ... are 2, 6, 30, 210, 2,310, 30,030, ... The above value of 195 is between 30 and 210, the above values of 435, 615, 1,590 are between 210 and 2,310, the above values of 4,305, 4,935, 7995, and 17,355 are between 2,310, and 30,030, etc. This is useful in determining how frequently l_n will decrease in value.

Let g_n denote the maximum of the $\sum_{i|n}(\gamma_{i+1} - \gamma_i)\mu(i)$ distribution. g_n is a slowly increasing step function with an initial value of $\gamma_2 - \gamma_1$. For n less than 500,000, g_n increases in value at $n = 2$, 1,806, 6,118, 17,822, 25,802, 34,314, 70,518, 131,838, 186,018, 204,078, 213,486, and 364,182. The prime factorizations of these values are 2, $2 \cdot 3 \cdot 7 \cdot 43$, $2 \cdot 7 \cdot 19 \cdot 23$, $2 \cdot 7 \cdot 19 \cdot 67$, $2 \cdot 7 \cdot 19 \cdot 97$, $2 \cdot 3 \cdot 7 \cdot 19 \cdot 43$, $2 \cdot 3 \cdot 7 \cdot 23 \cdot 73$, $2 \cdot 3 \cdot 7 \cdot 43 \cdot 73$, $2 \cdot 3 \cdot 7 \cdot 43 \cdot 107$, $2 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 23$, and $2 \cdot 3 \cdot 7 \cdot 13 \cdot 23 \cdot 29$. The number of prime factors is non-decreasing. The corresponding g_n values are 6.887315, 6.938951, 7.786355, 7.926703, 8.147173, 8.684880, 8.930729, 8.999220, 9.061730, 9.071198, 10.498812, and 10.520155.

See Cox [5] for the software used to determine the above results.

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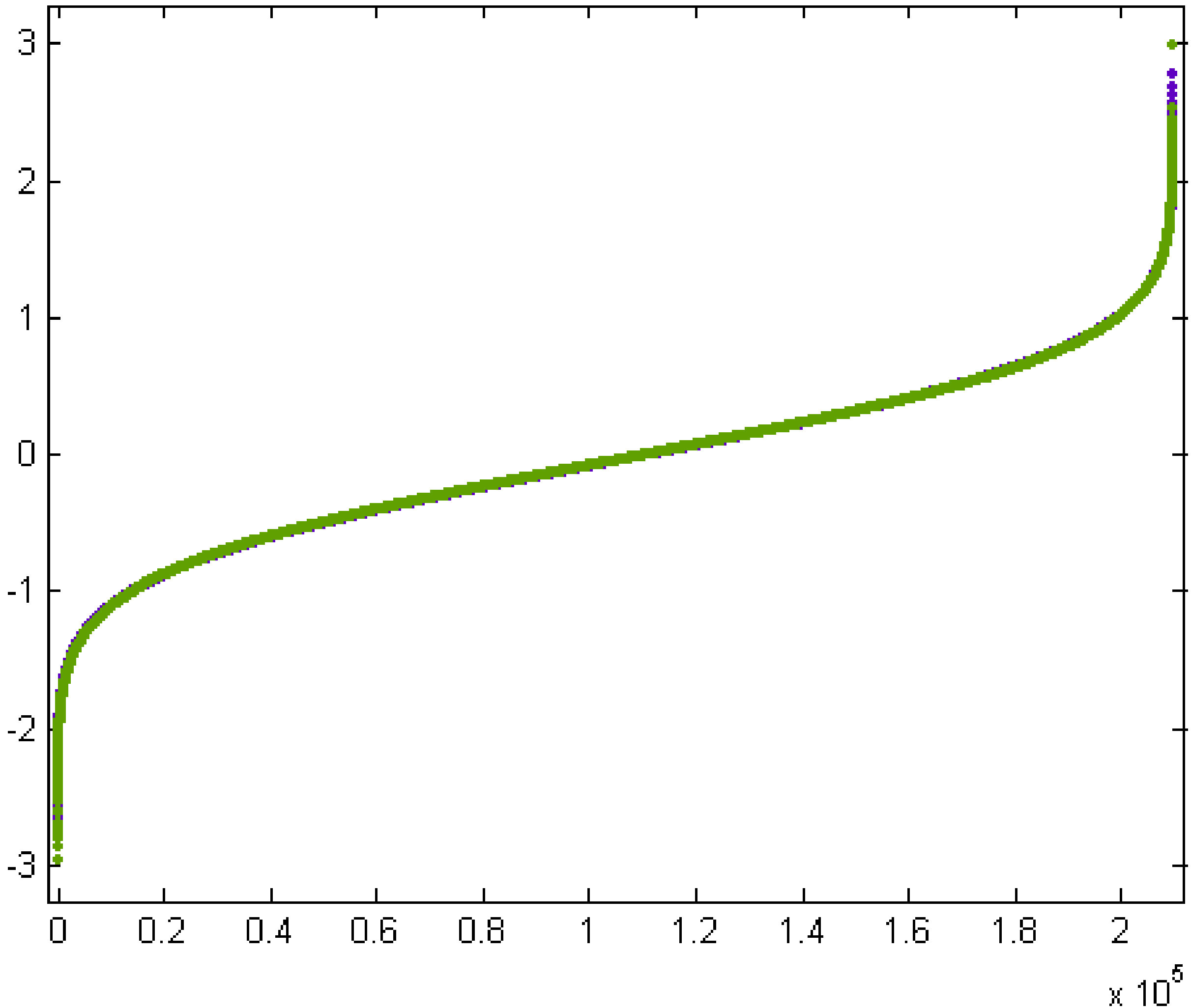
Figure 1 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)



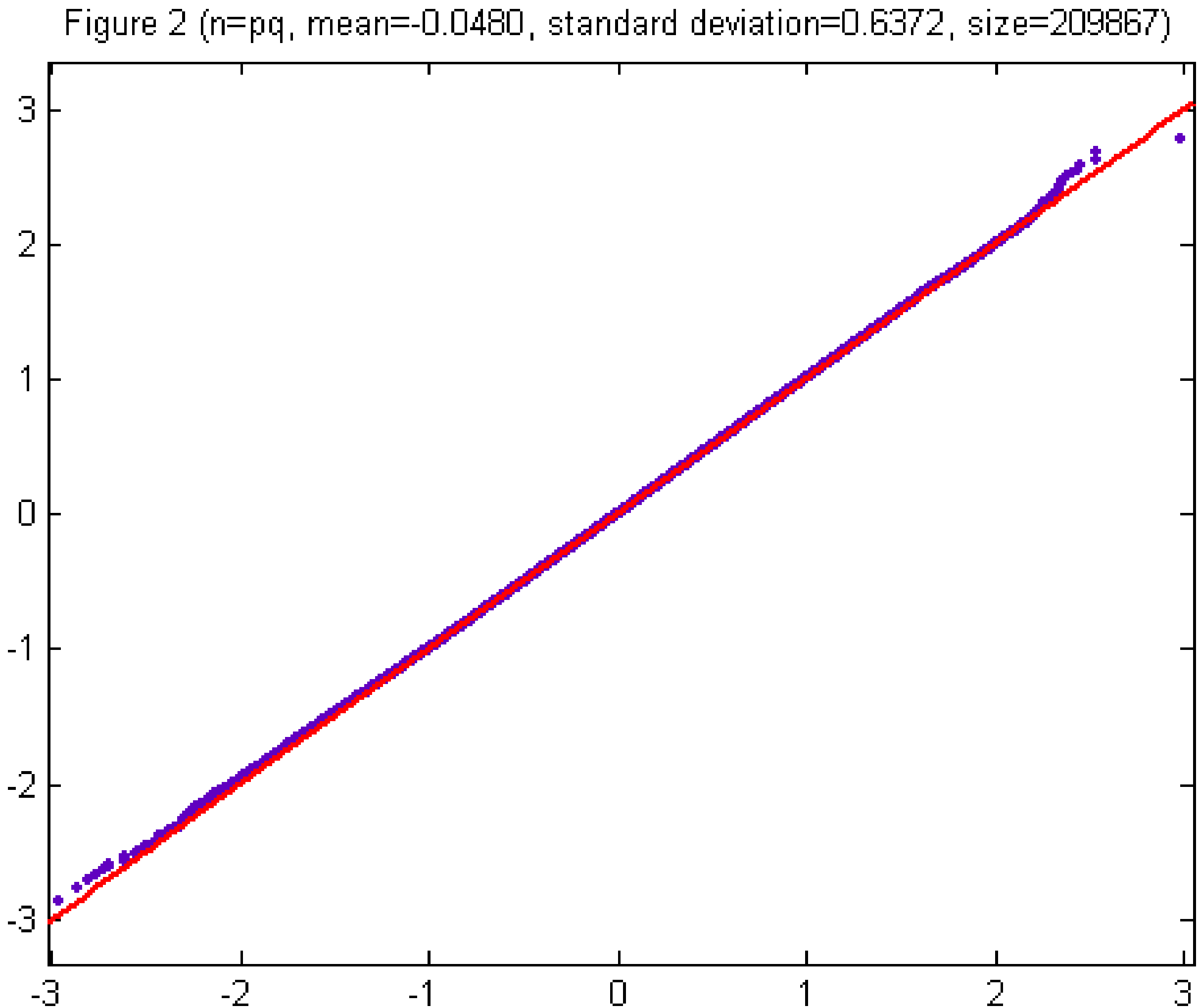


Figure 2 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

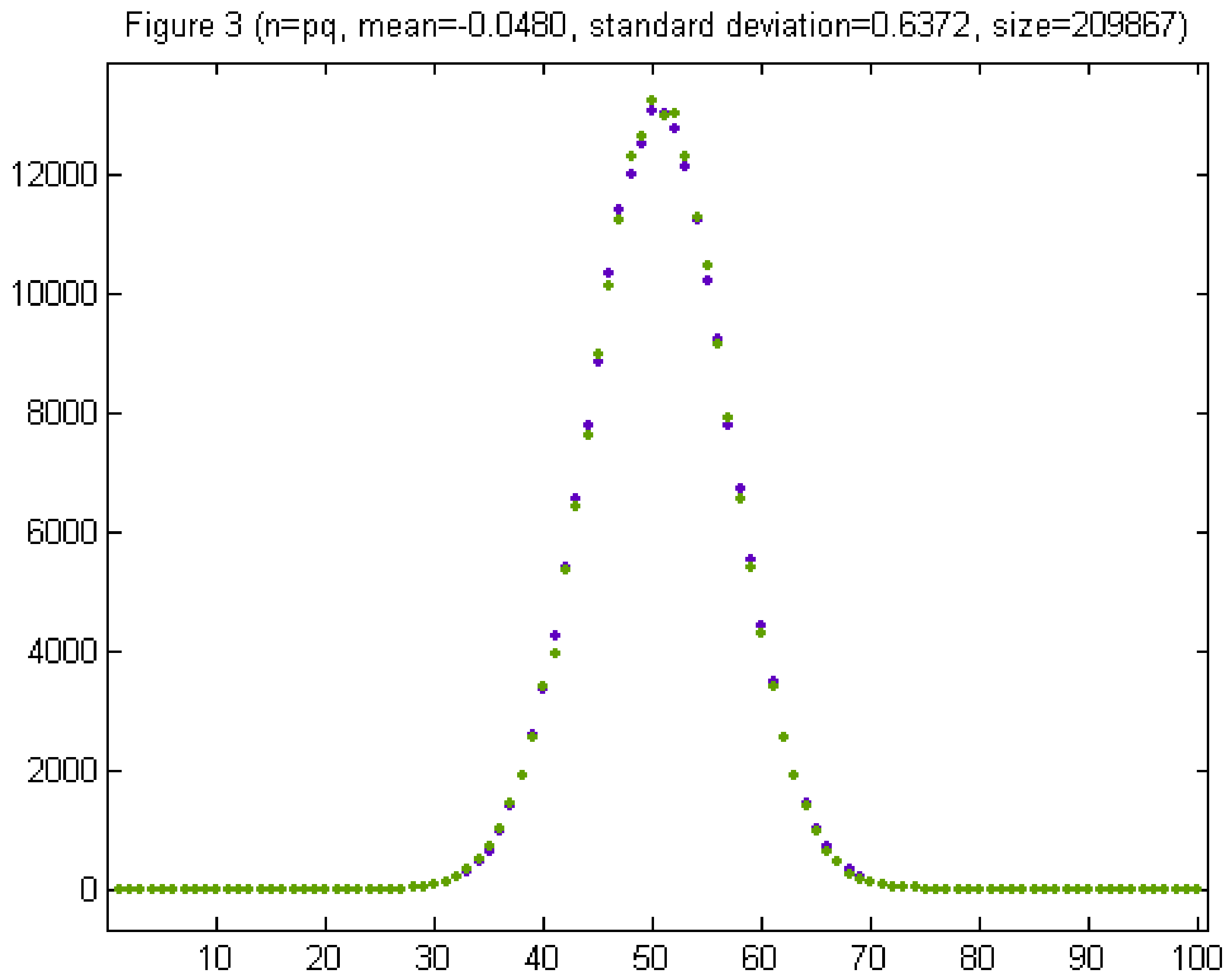


Figure 3 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
12000
10000
8000
6000
4000
2000
0
10
20
30
40
50
60
70
80
90
100

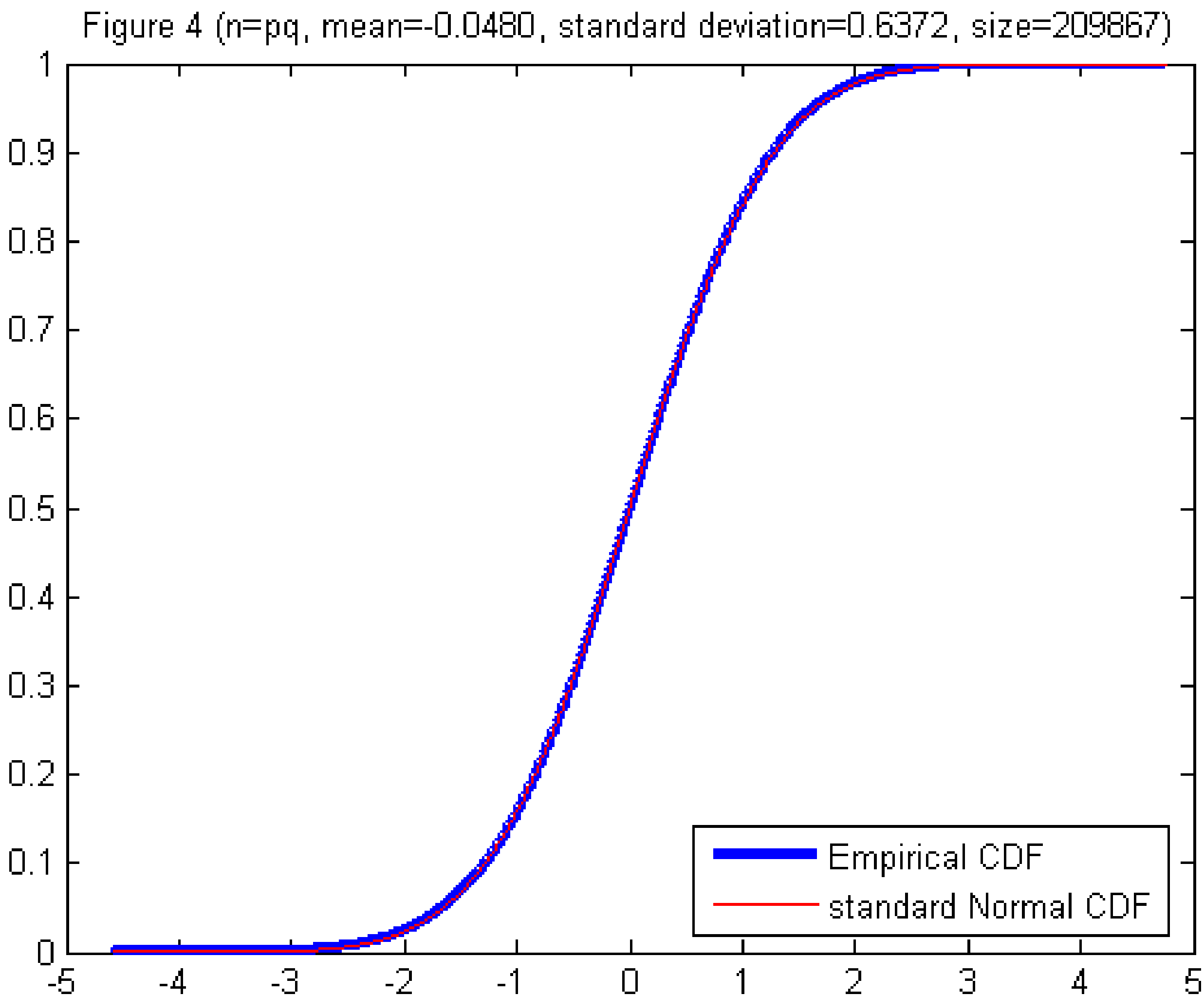


Figure 4 (n=pq, mean=-0.0480, standard deviation=0.6372, size=209867)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
Empirical CDF
standard Normal CDF

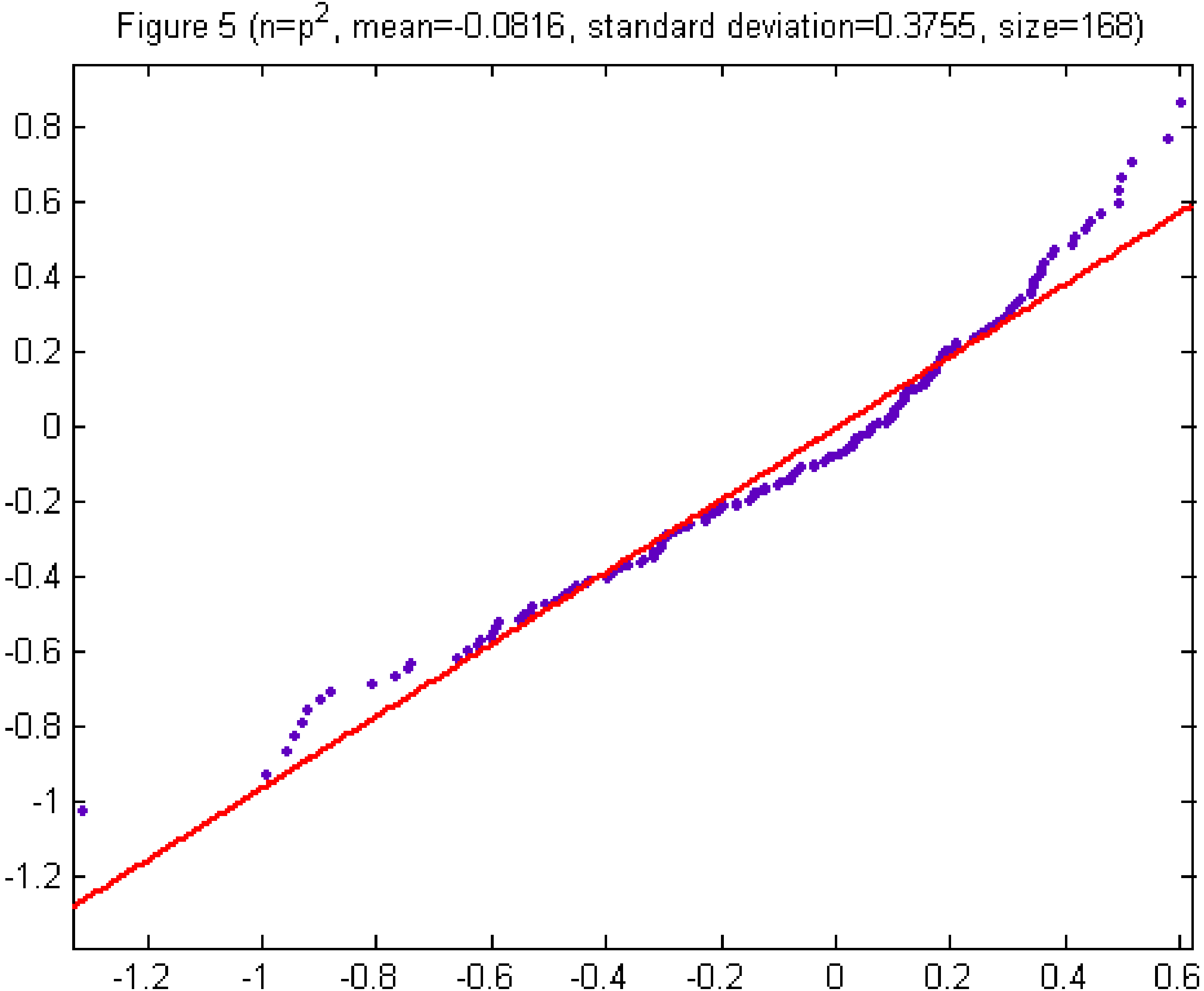


Figure 5 (n=p^2, mean=-0.0816, standard deviation=0.3755, size=168)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

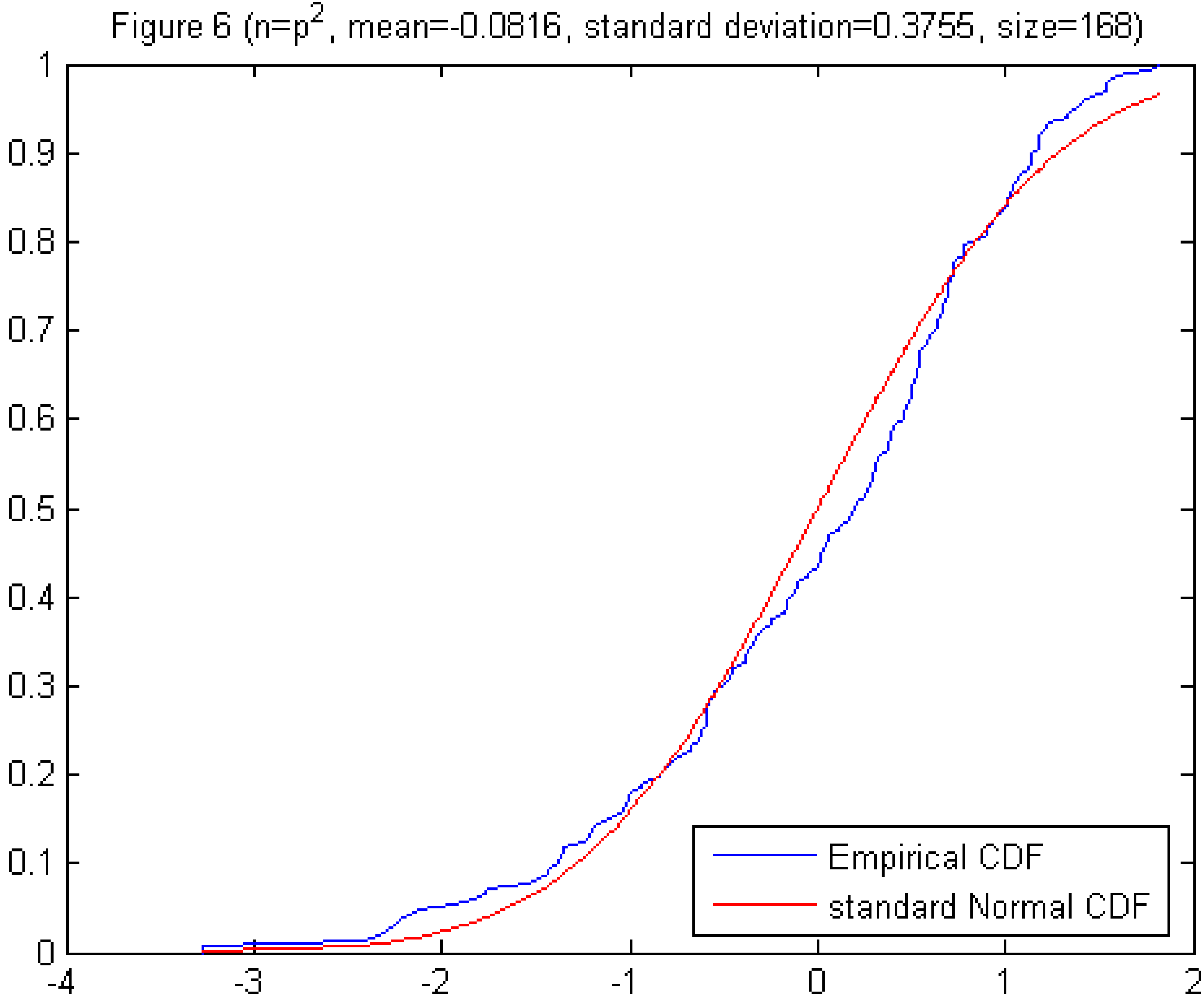


Figure 6 (n=p2, mean=-0.0816, standard deviation=0.3755, size=168)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
Empirical CDF
standard Normal CDF

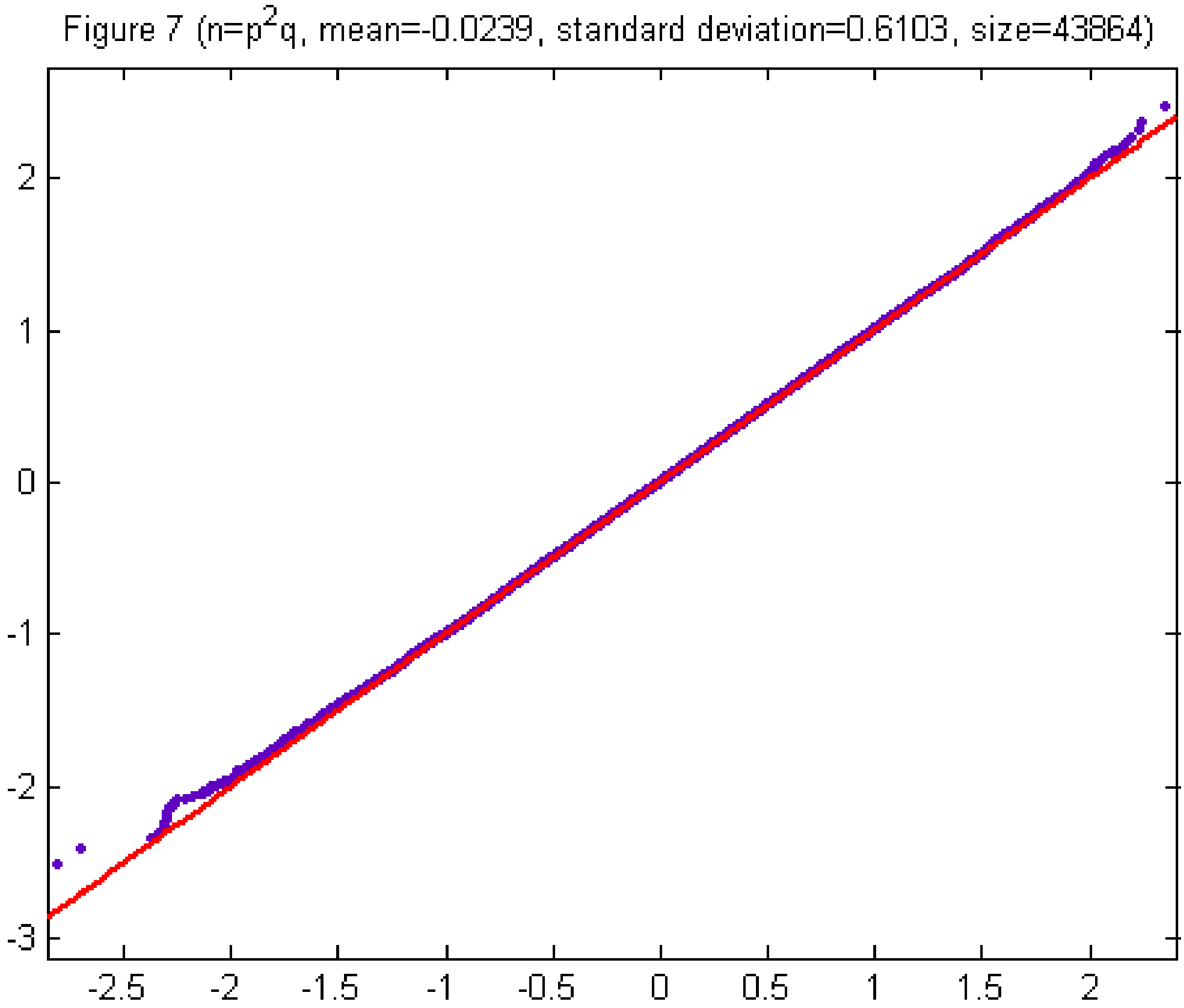


Figure 7 (n=p^2q, mean=-0.0239, standard deviation=0.6103, size=43864)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

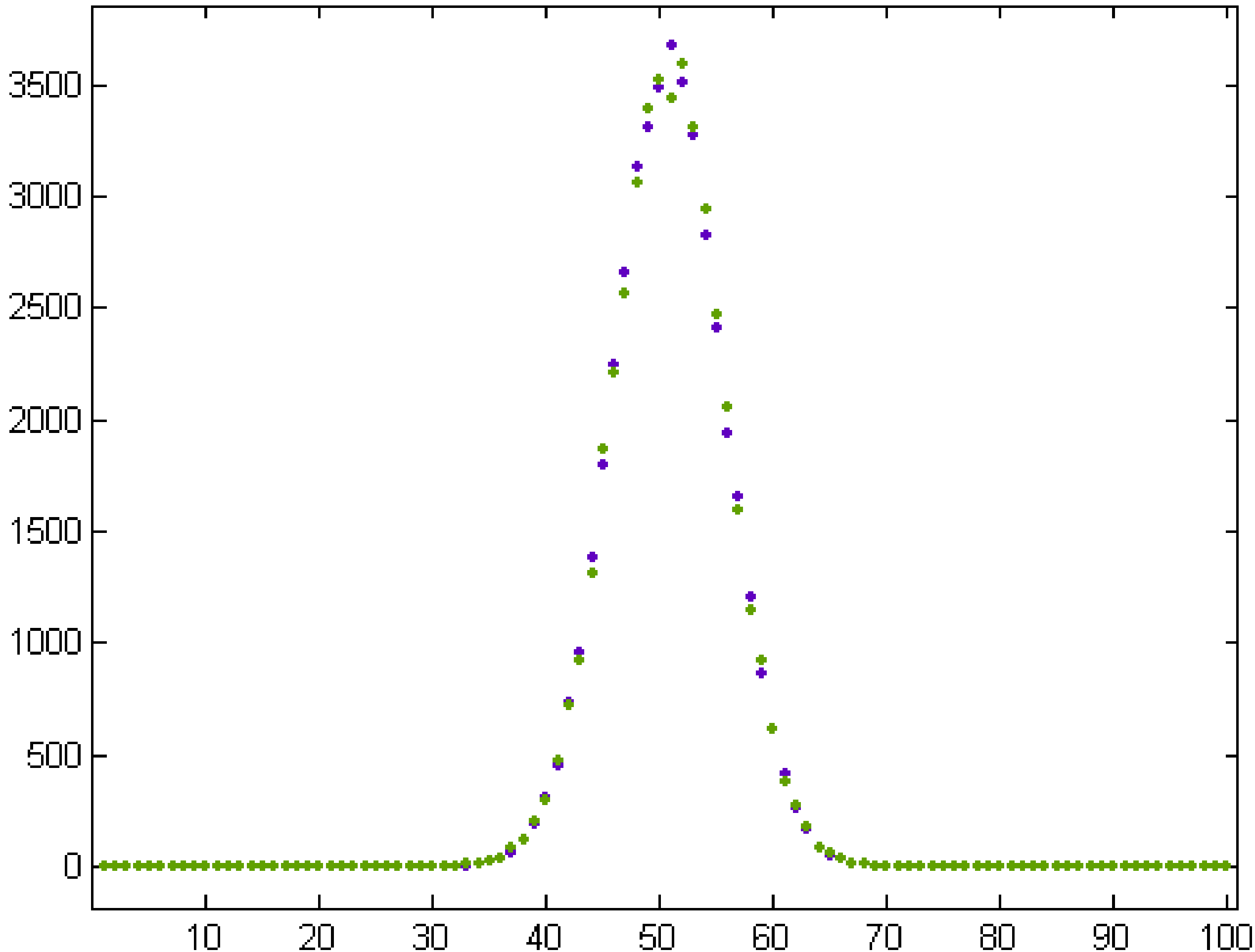


Figure 8 ($n=p^2q$, mean=-0.0239, standard deviation=0.6103, size=43864)

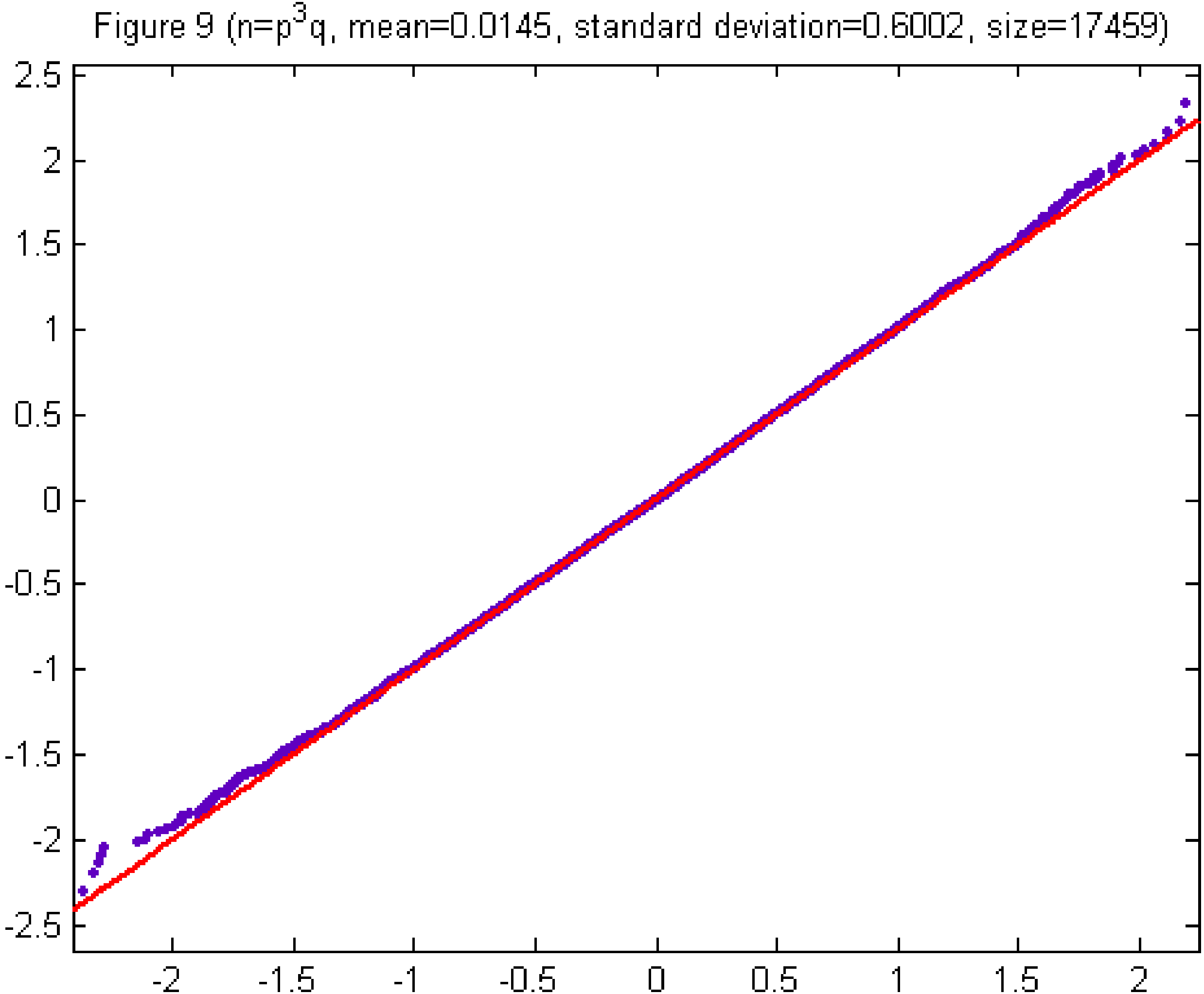


Figure 9 (n=p3q, mean=0.0145, standard deviation=0.6002, size=17459)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

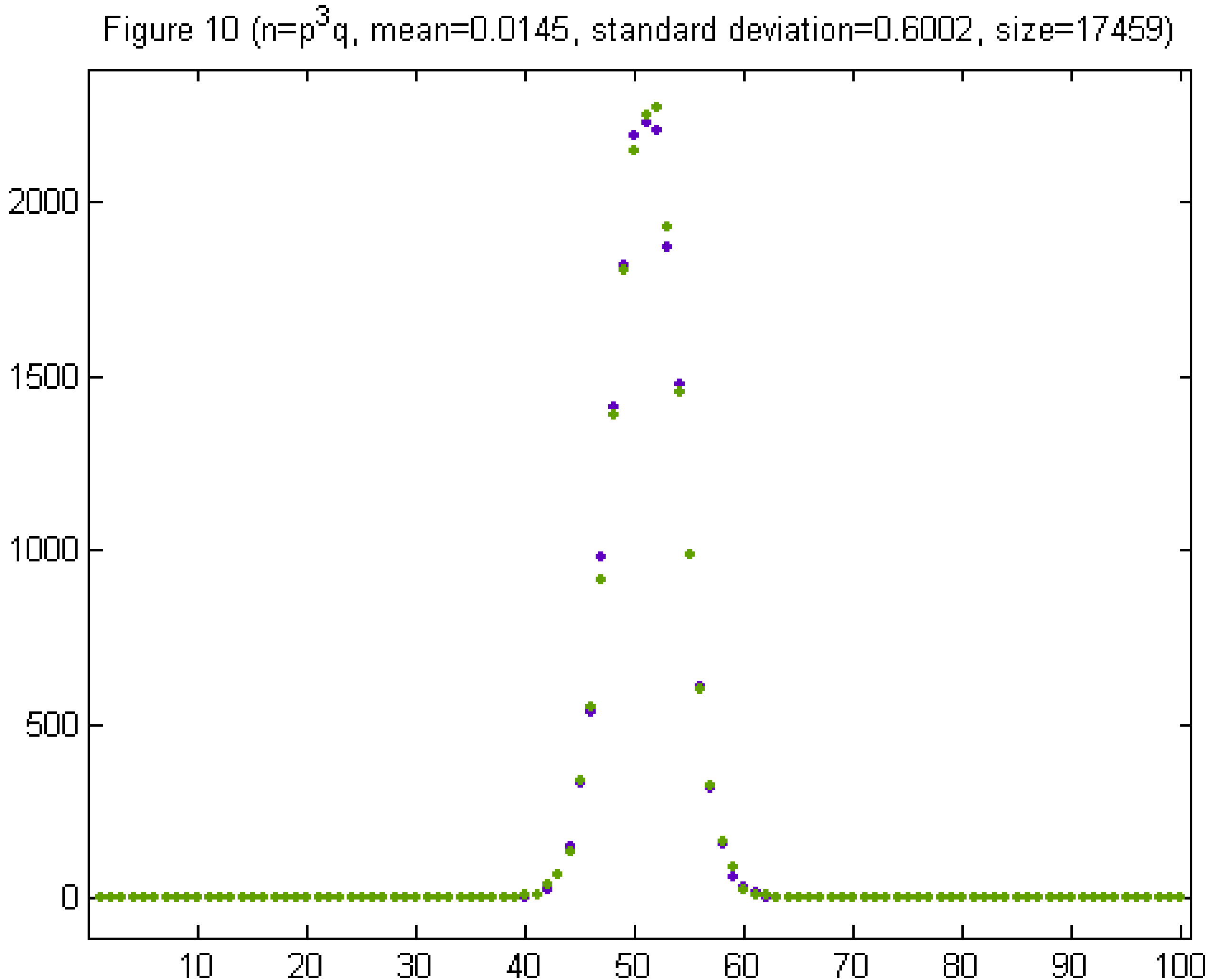


Figure 10 (n=p^3q, mean=0.0145, standard deviation=0.6002, size=17459)
2000
1500
1000
500
0
10
20
30
40
50
60
70
80
90
100

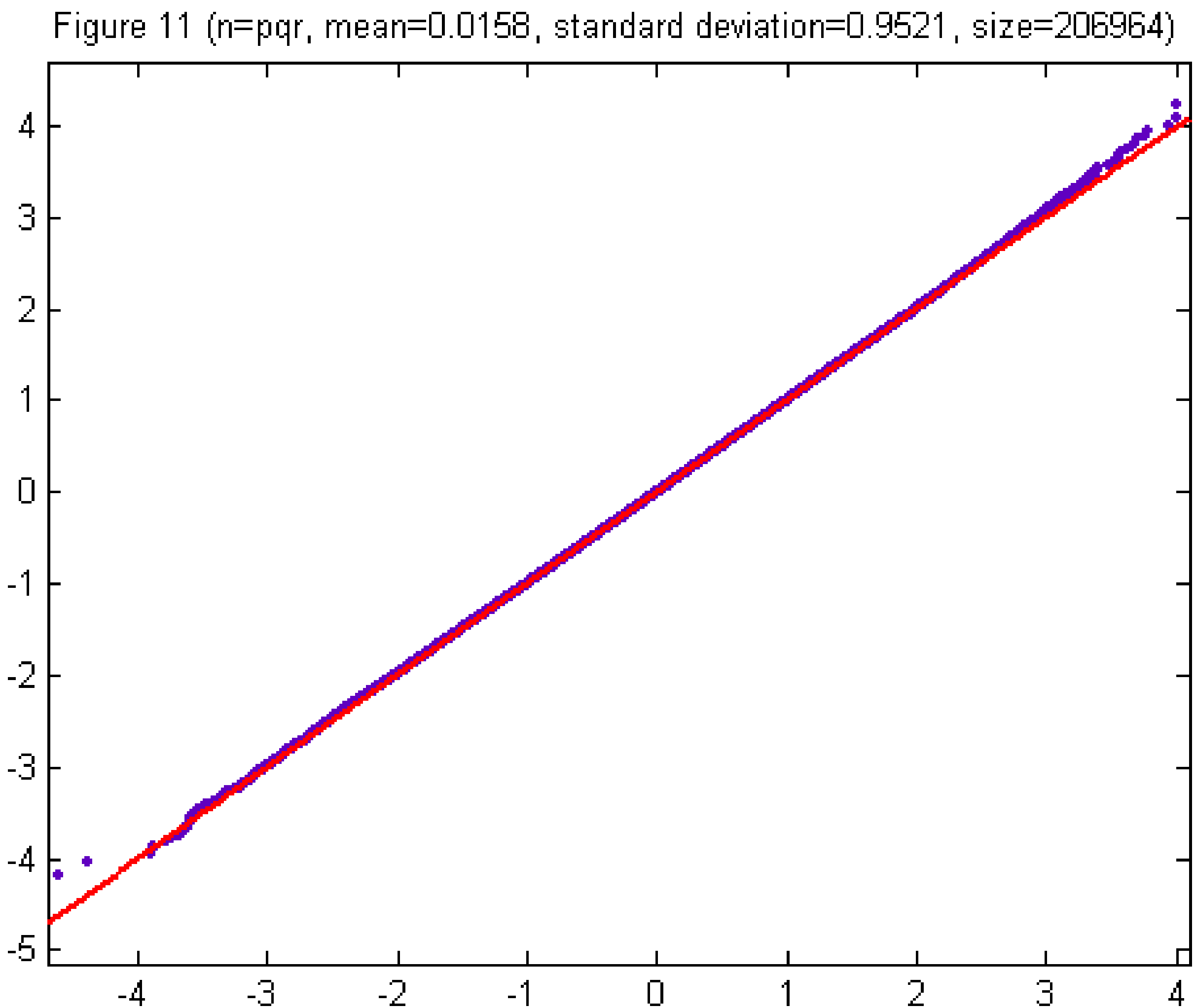


Figure 11 (n=pqr, mean=0.0158, standard deviation=0.9521, size=206964)
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2
3
4

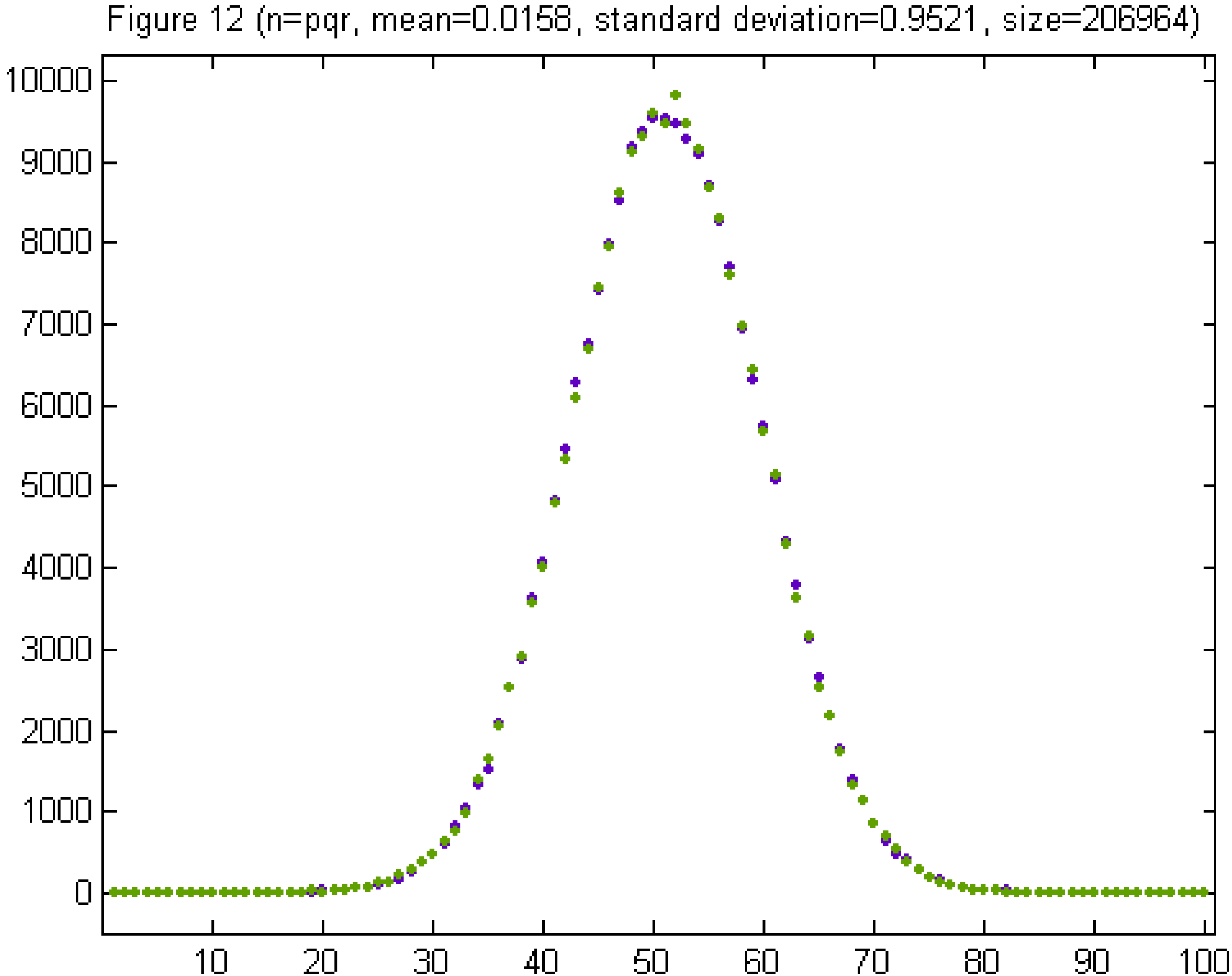


Figure 12 (n=pqr, mean=0.0158, standard deviation=0.9521, size=206964)
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
10
20
30
40
50
60
70
80
90
100

Figure 13 ($n=p^4q$, mean=0.1025, standard deviation=0.8221, size=8114)

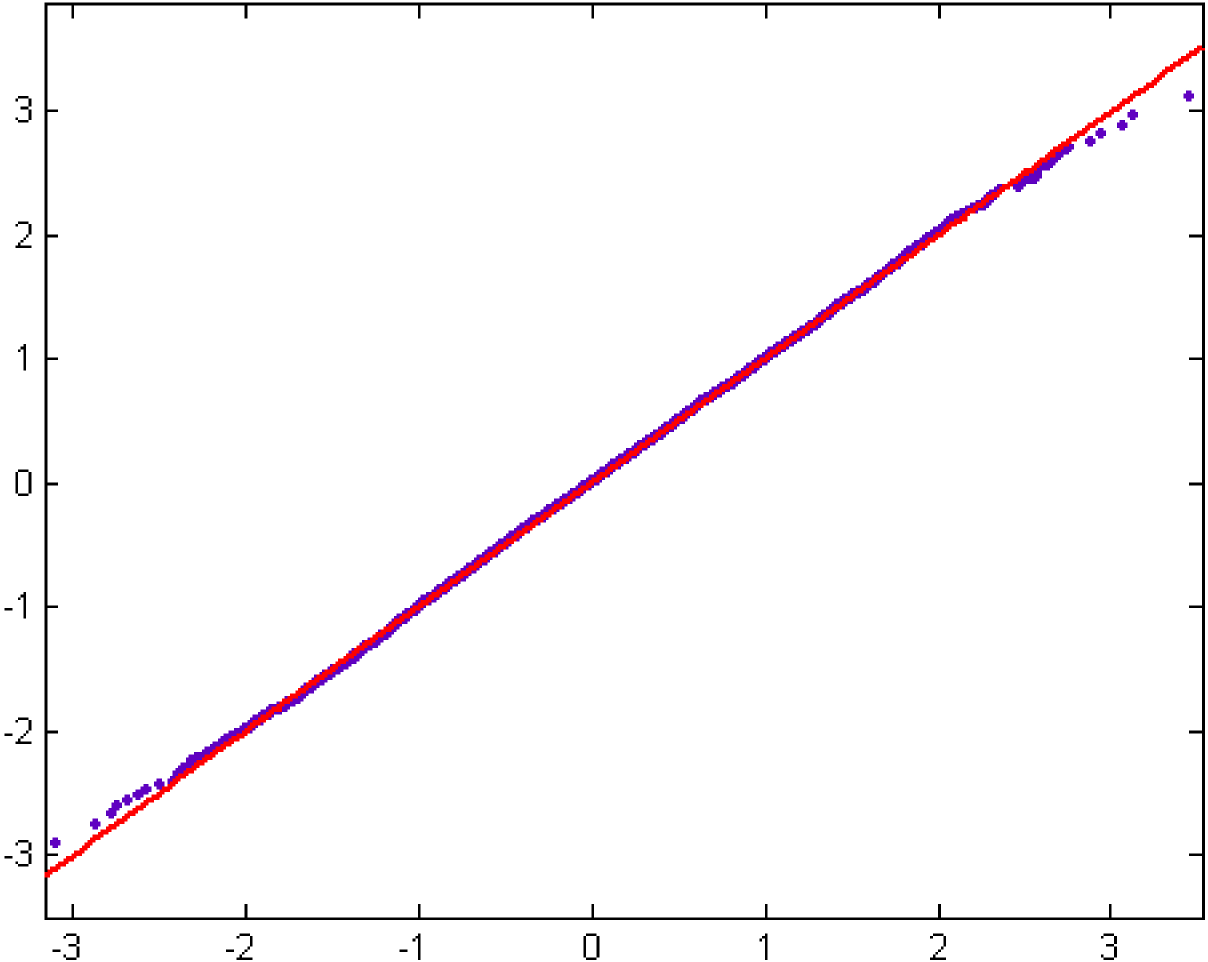


Figure 14 ($n=p^2q^2$, mean=-0.0478, standard deviation=0.7565, size=287)

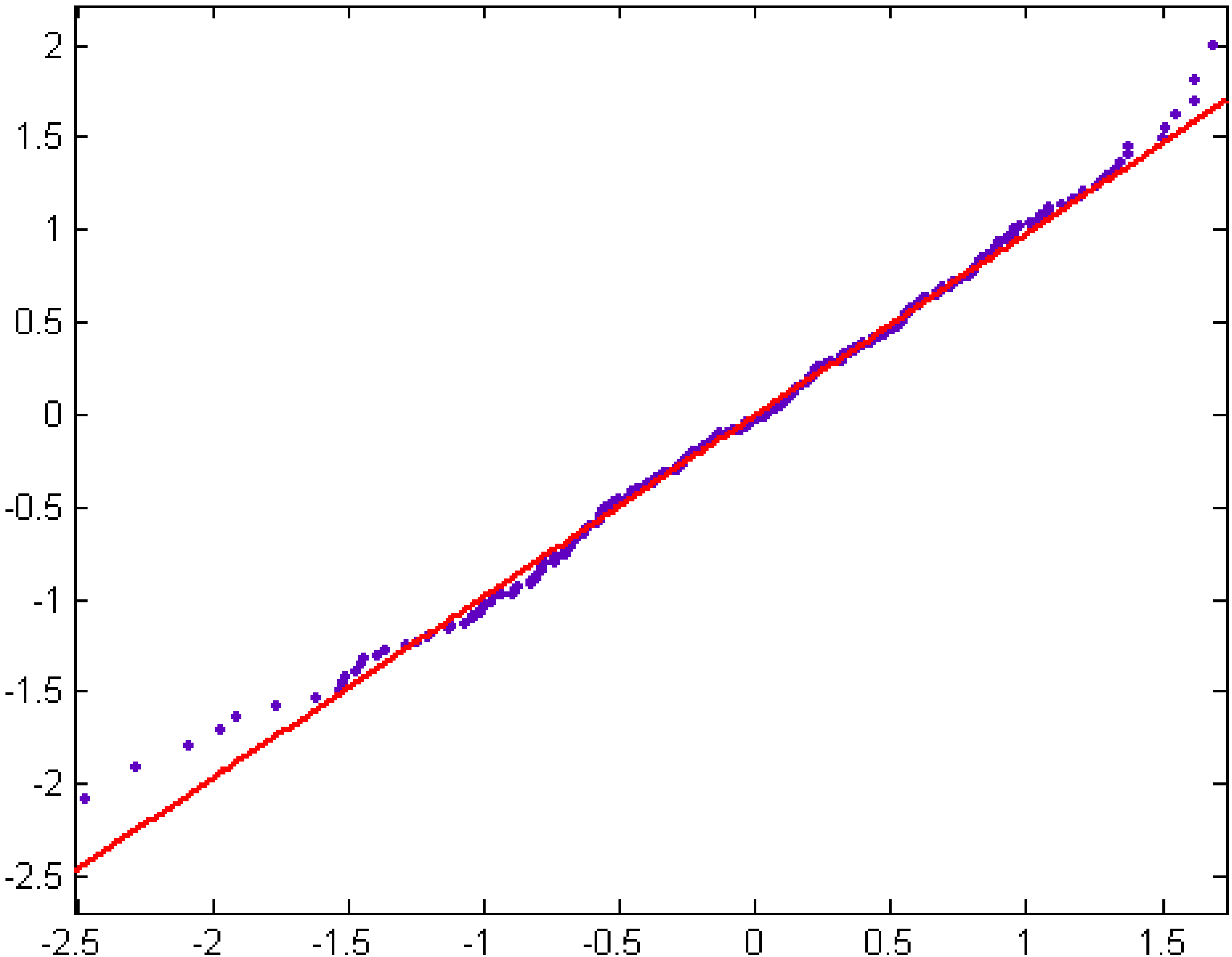


Figure 15 ($n=p^5q$, mean=0.1644, standard deviation=0.8146, size=4016)

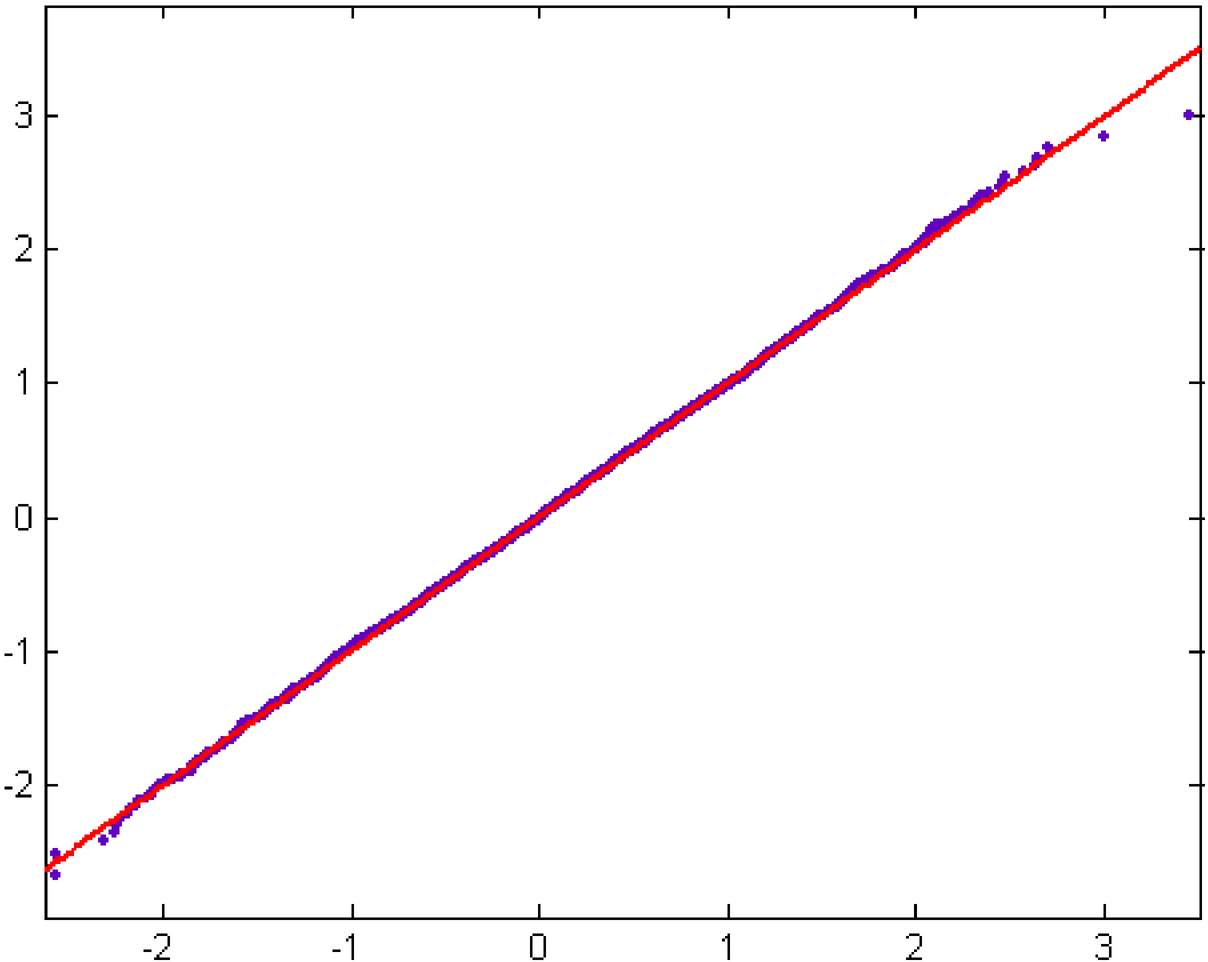


Figure 16 ($n=p^3q^2$, mean=0.0417, standard deviation=0.7650, size=196)

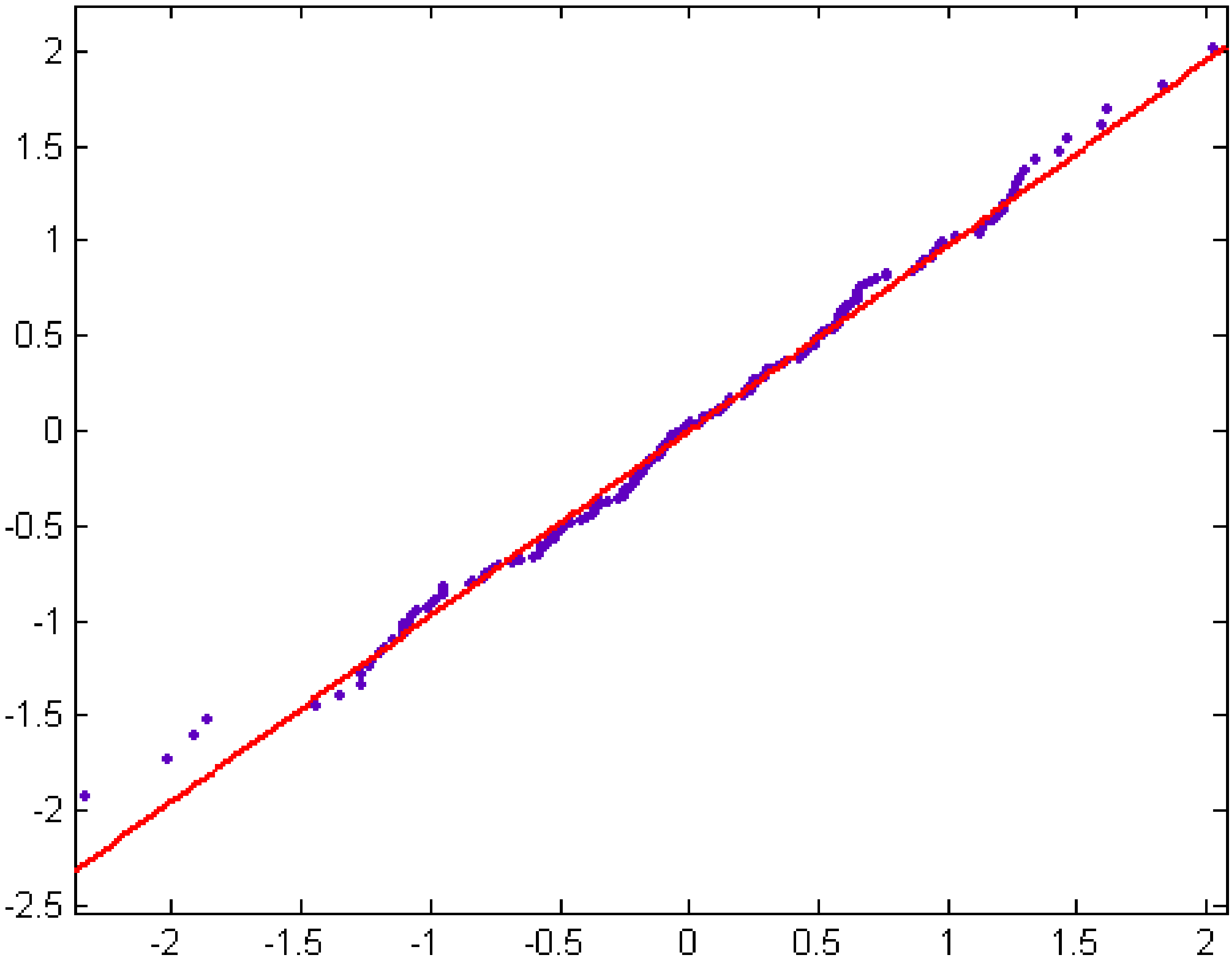
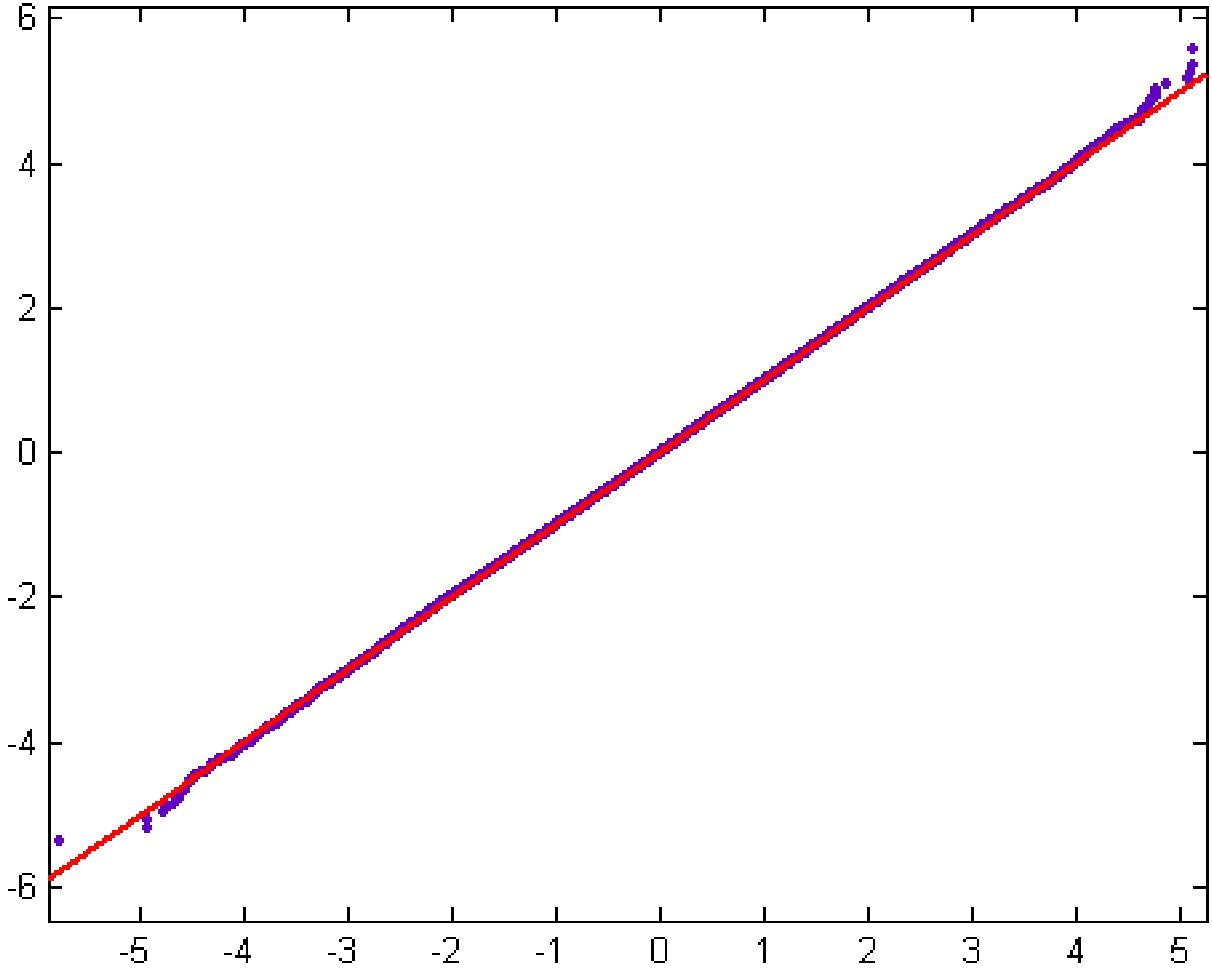


Figure 17 ($n=p^2qr$, mean=0.0886, standard deviation=1.2953, size=87337)



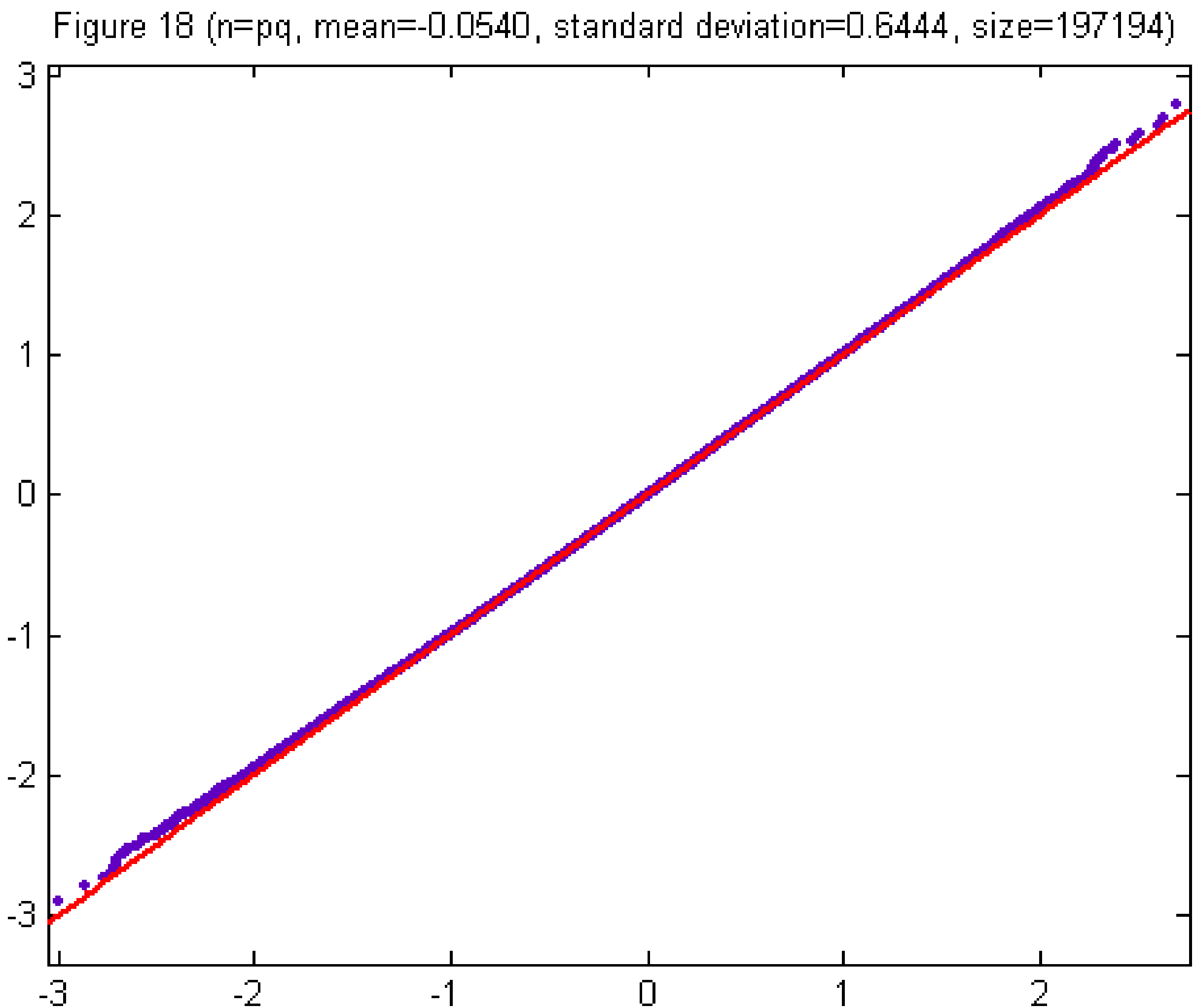


Figure 19 (n=pqrst, mean=0.0807, standard deviation=1.3978, size=104322)

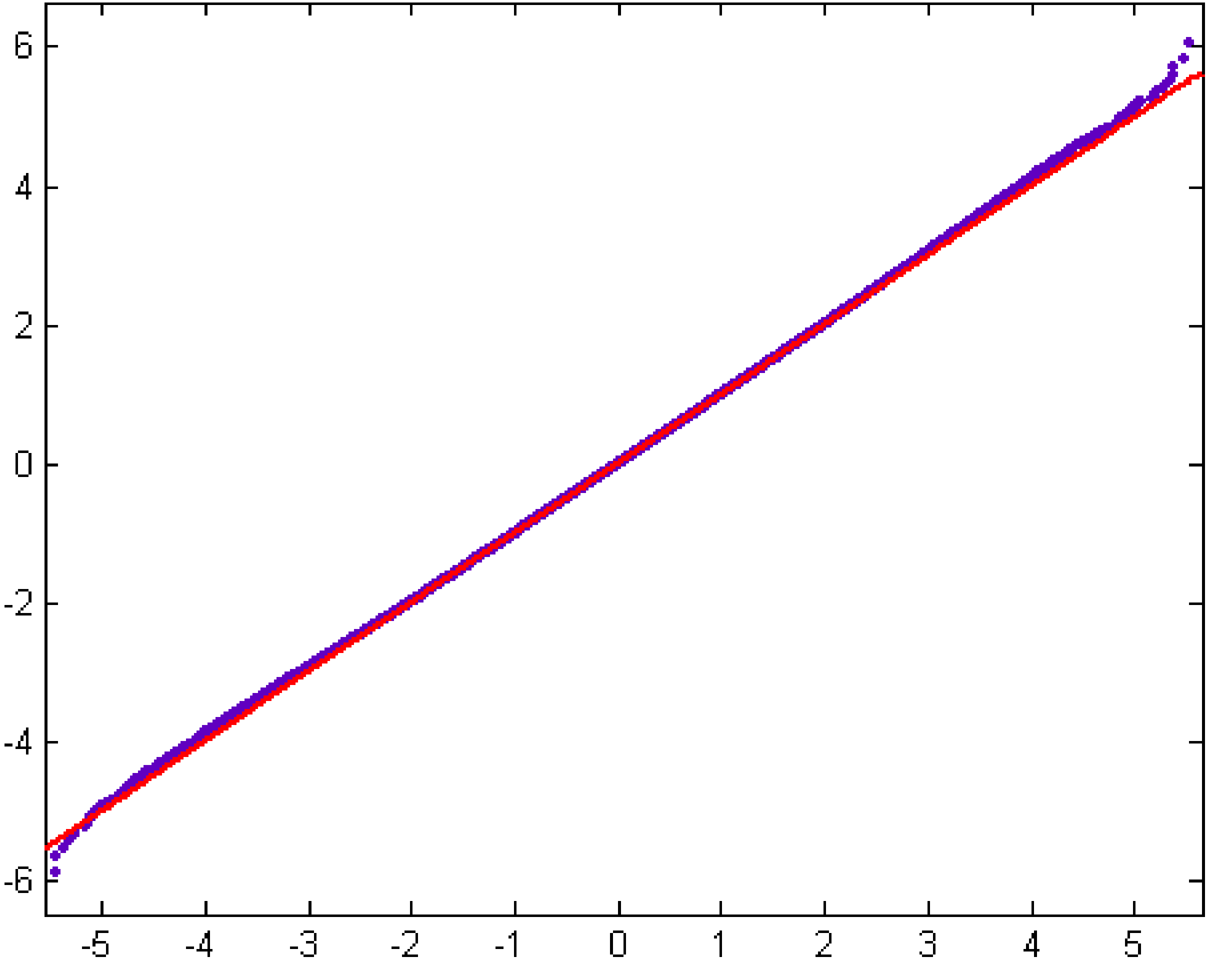


Figure 20 ($n=p^2q^2r$, mean=0.0175, standard deviation=0.9072, size=7322)

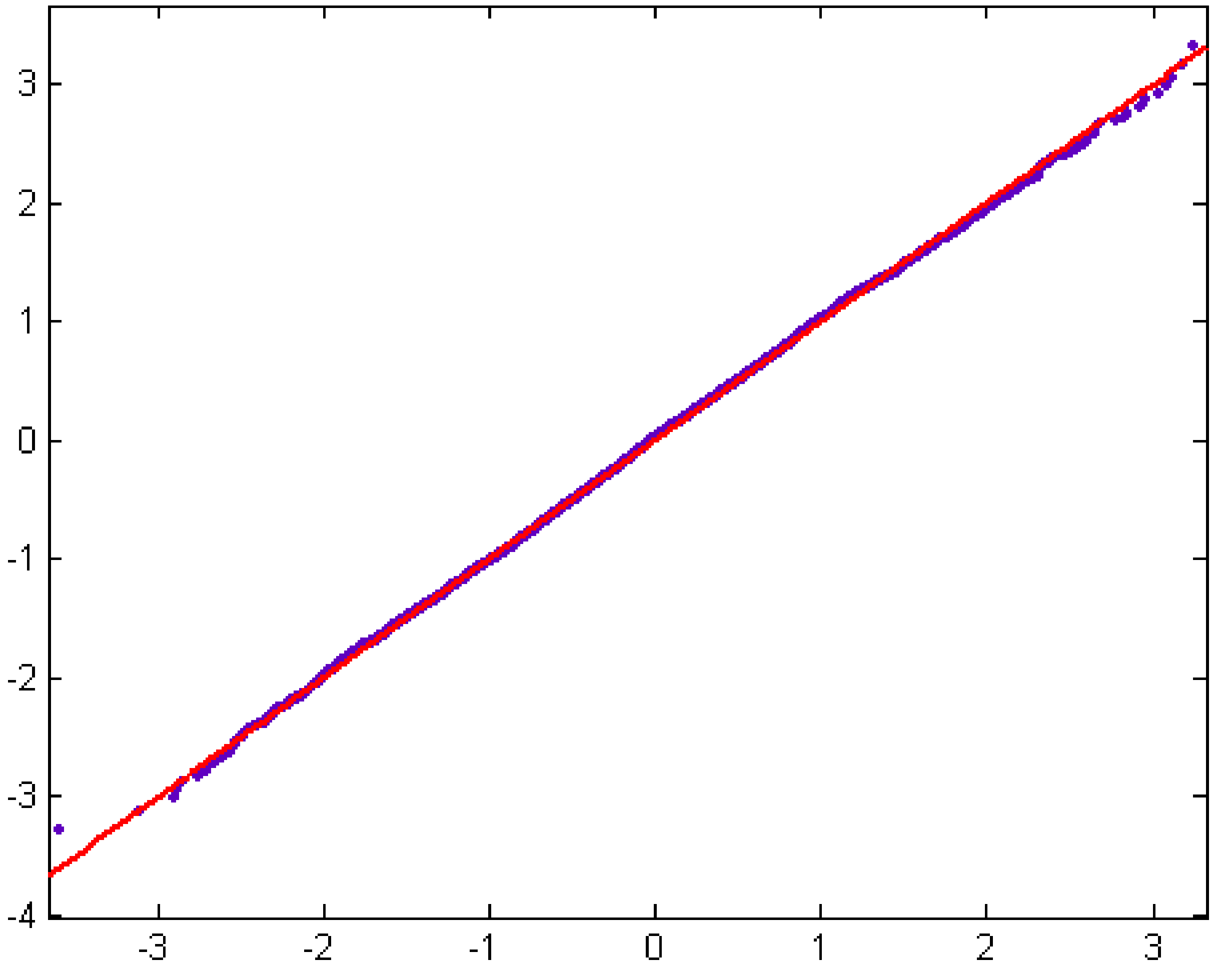


Figure 21 ($n=p^2qrs$, mean=0.0977, standard deviation=1.3717, size=60656)

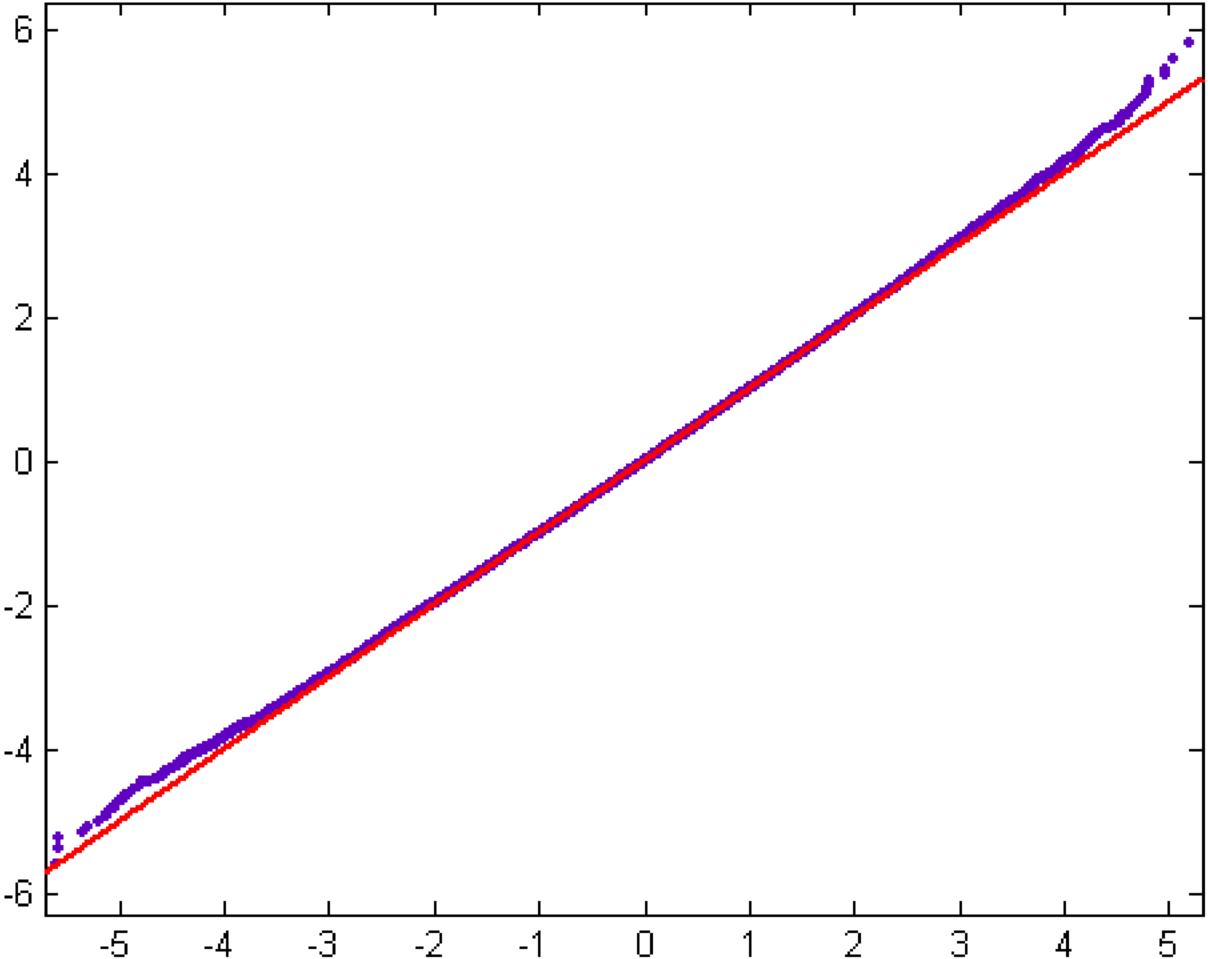


Figure 22 ($n=p^3qrs$, mean=0.1433, standard deviation=1.3539, size=21543)

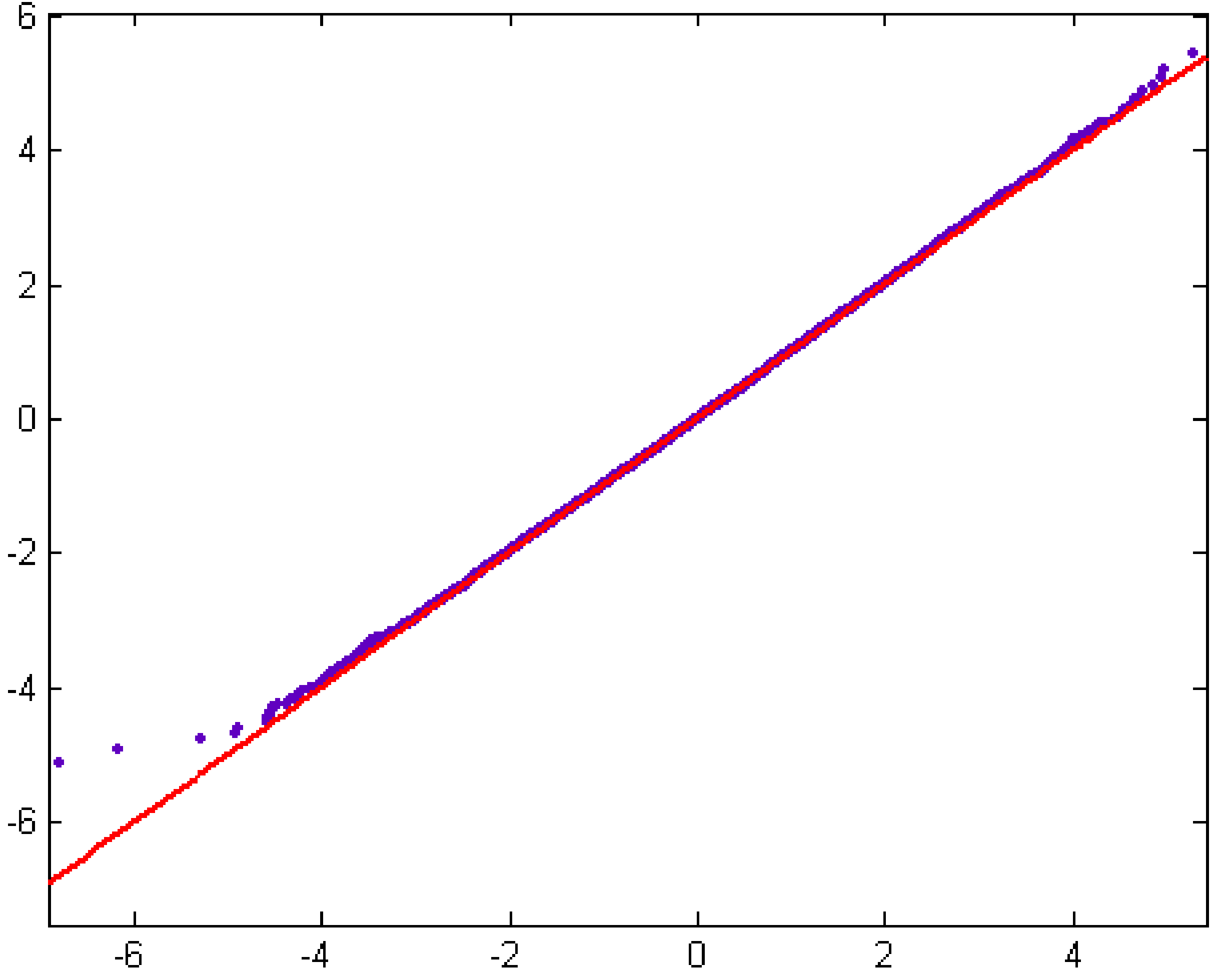
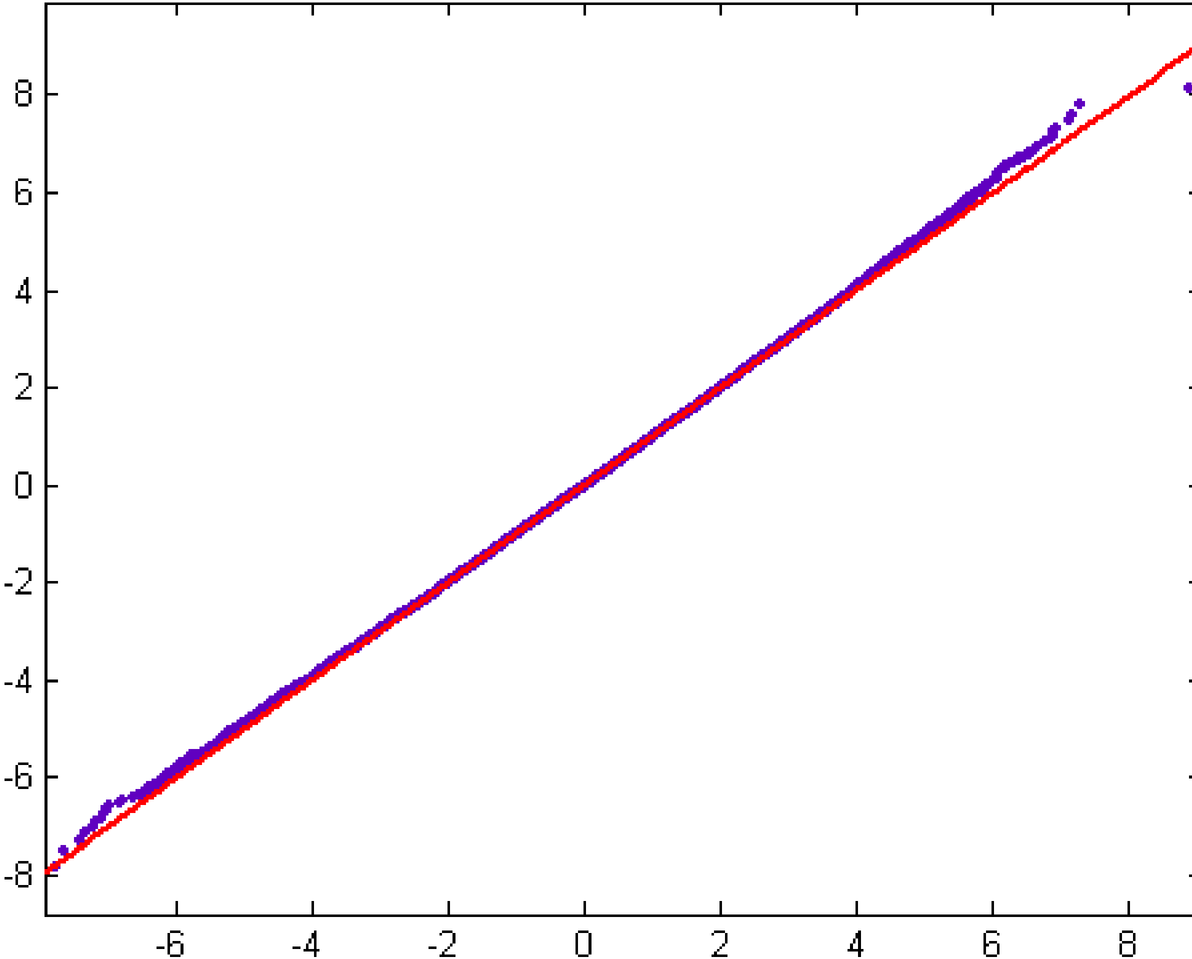


Figure 23 (n=pqrst, mean=0.1548, standard deviation=2.0295, size=24568)



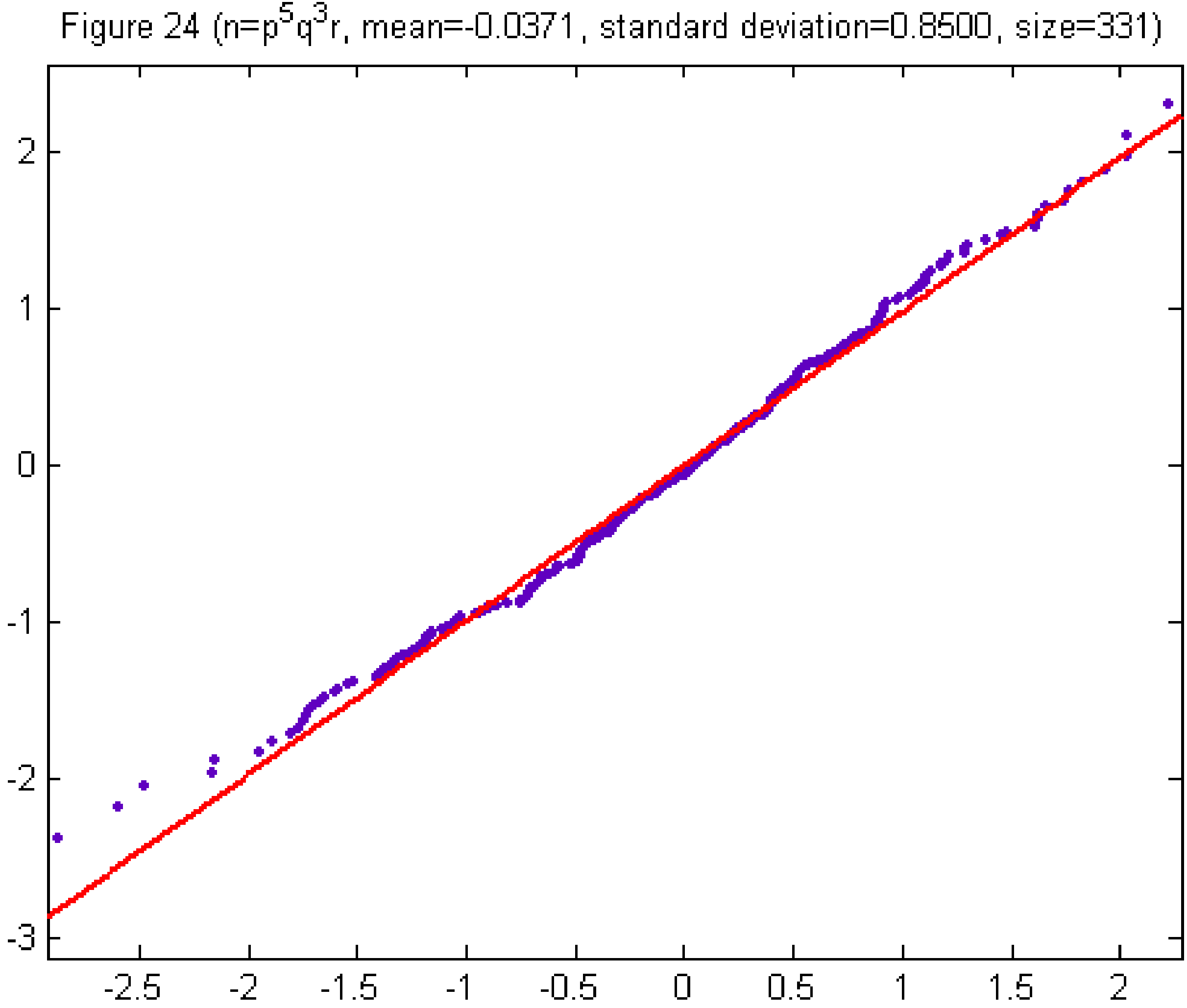
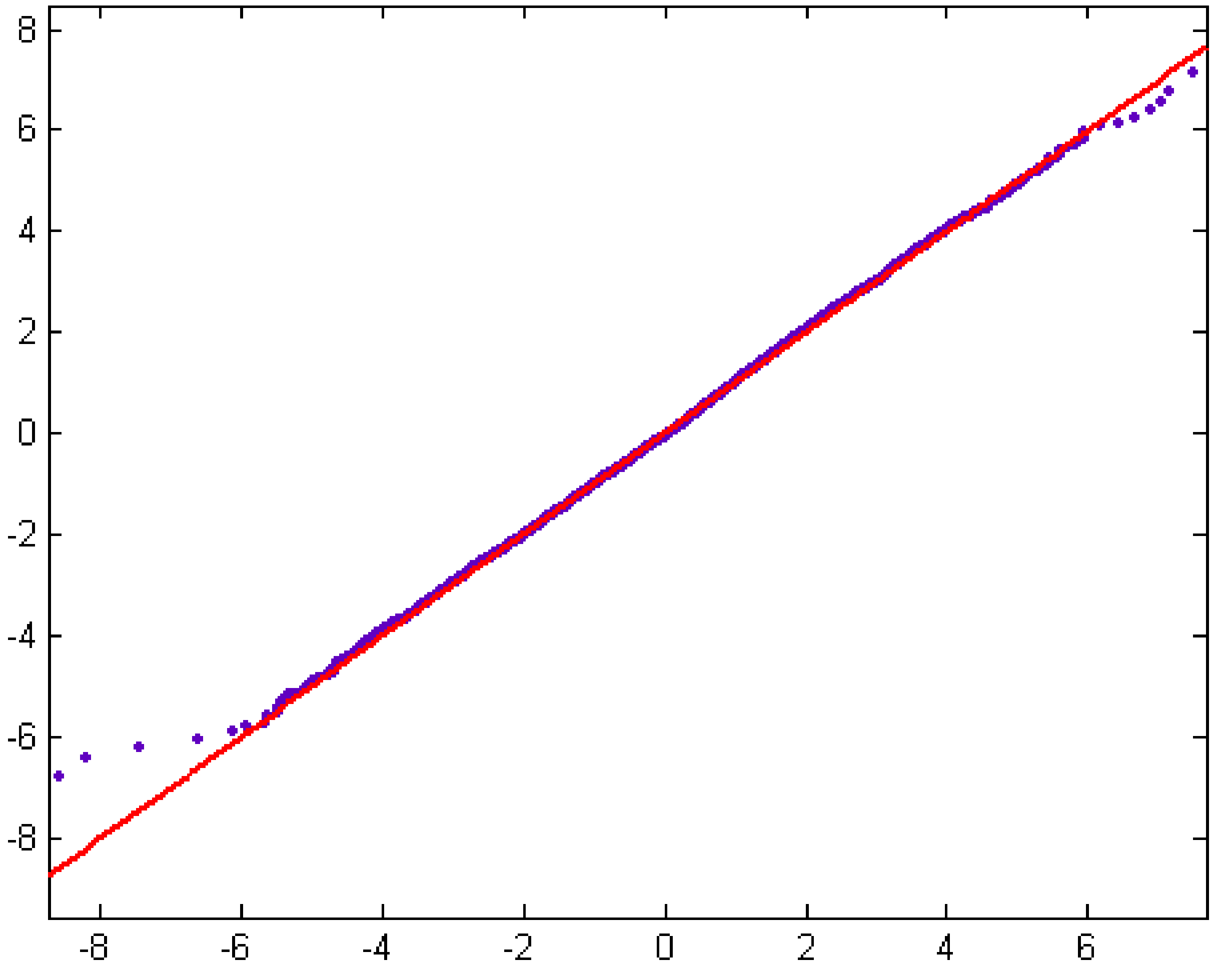


Figure 24 (n=p5q3r, mean=-0.0371, standard deviation=0.8500, size=331)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 25 ($n=p^3qrst$, mean=0.1696, standard deviation=1.9532, size=5398)



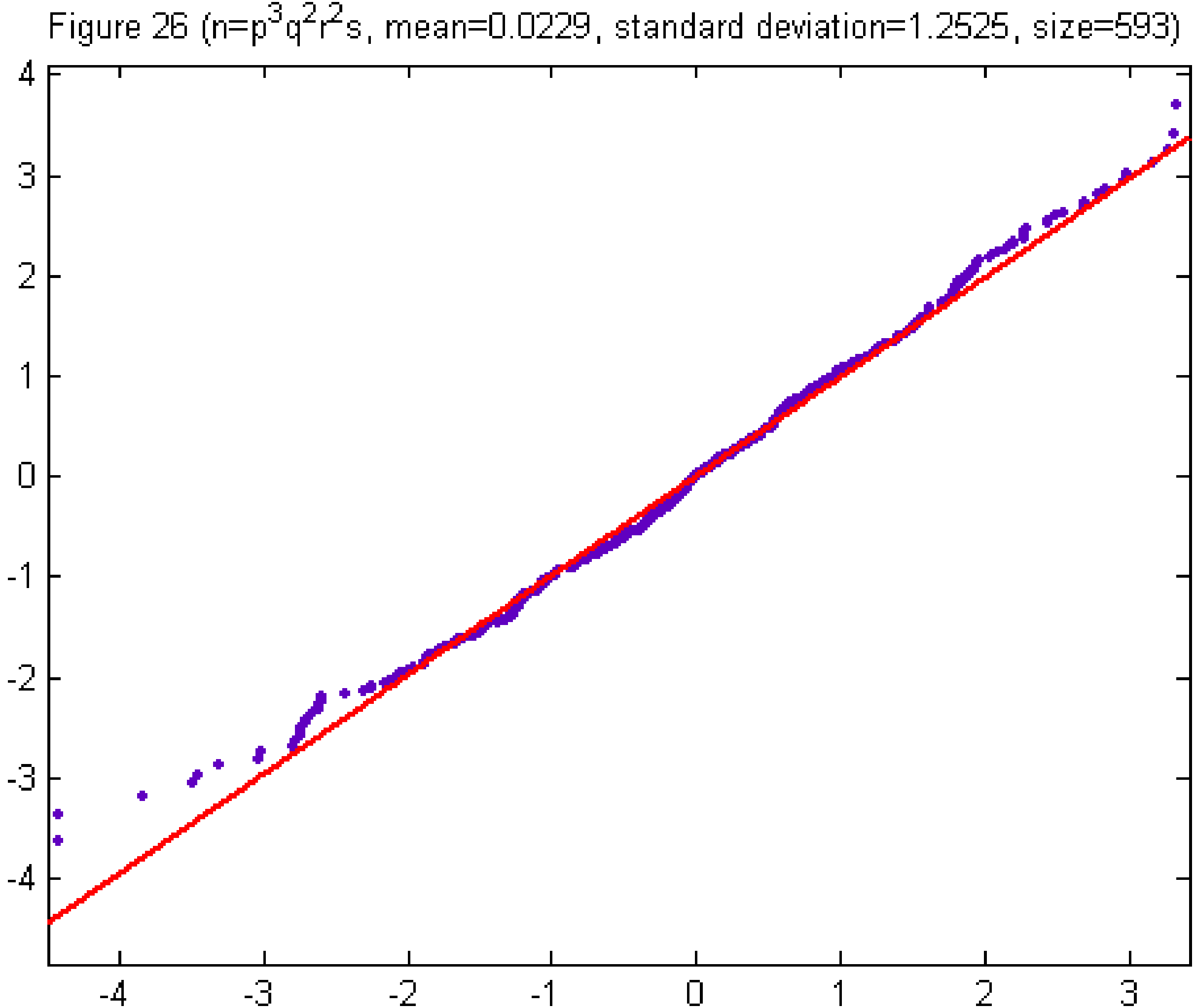


Figure 26 (n=p3q2r2s, mean=0.0229, standard deviation=1.2525, size=593)
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3

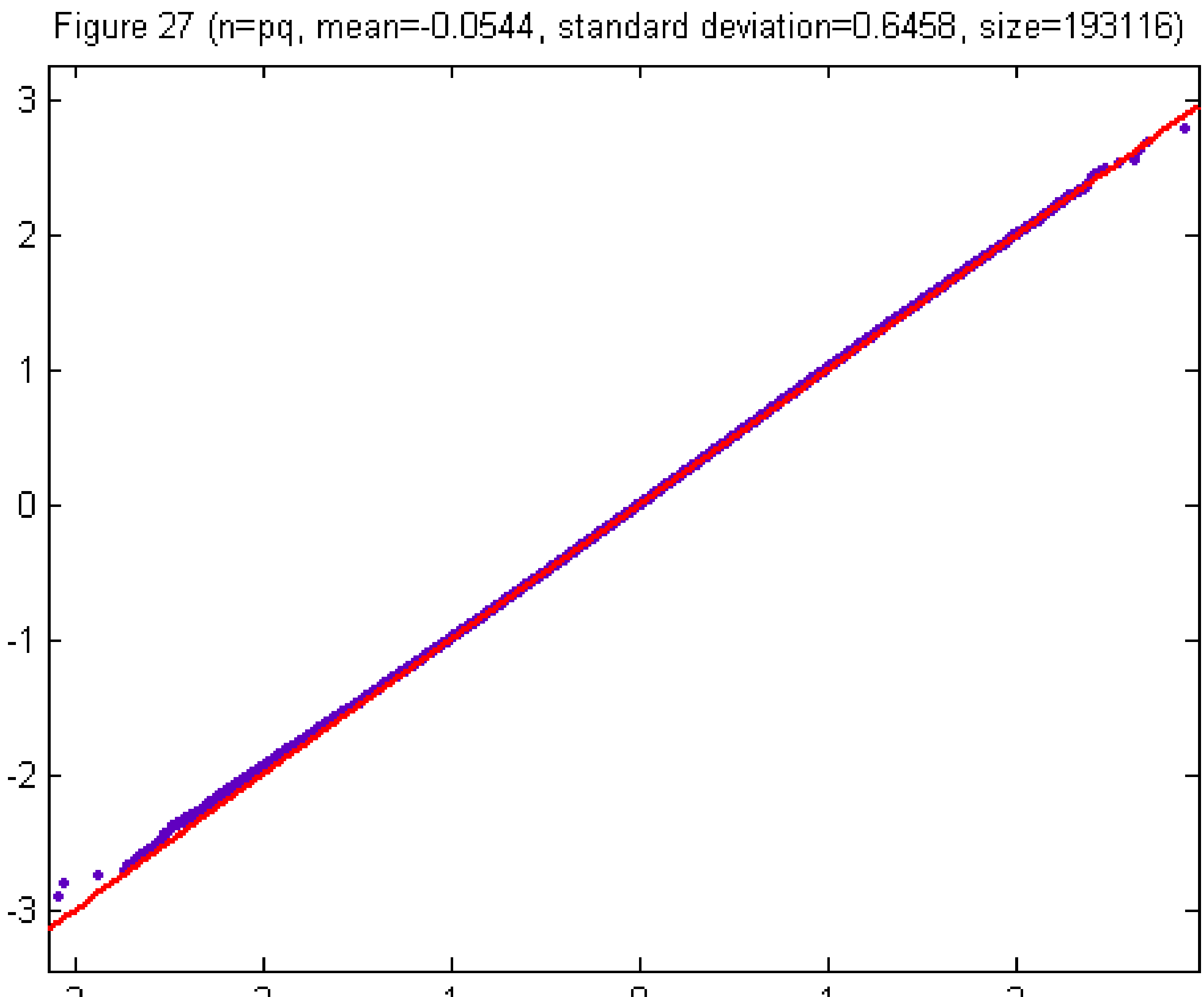


Figure 27 (n=pq, mean=-0.0544, standard deviation=0.6458, size=193116)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

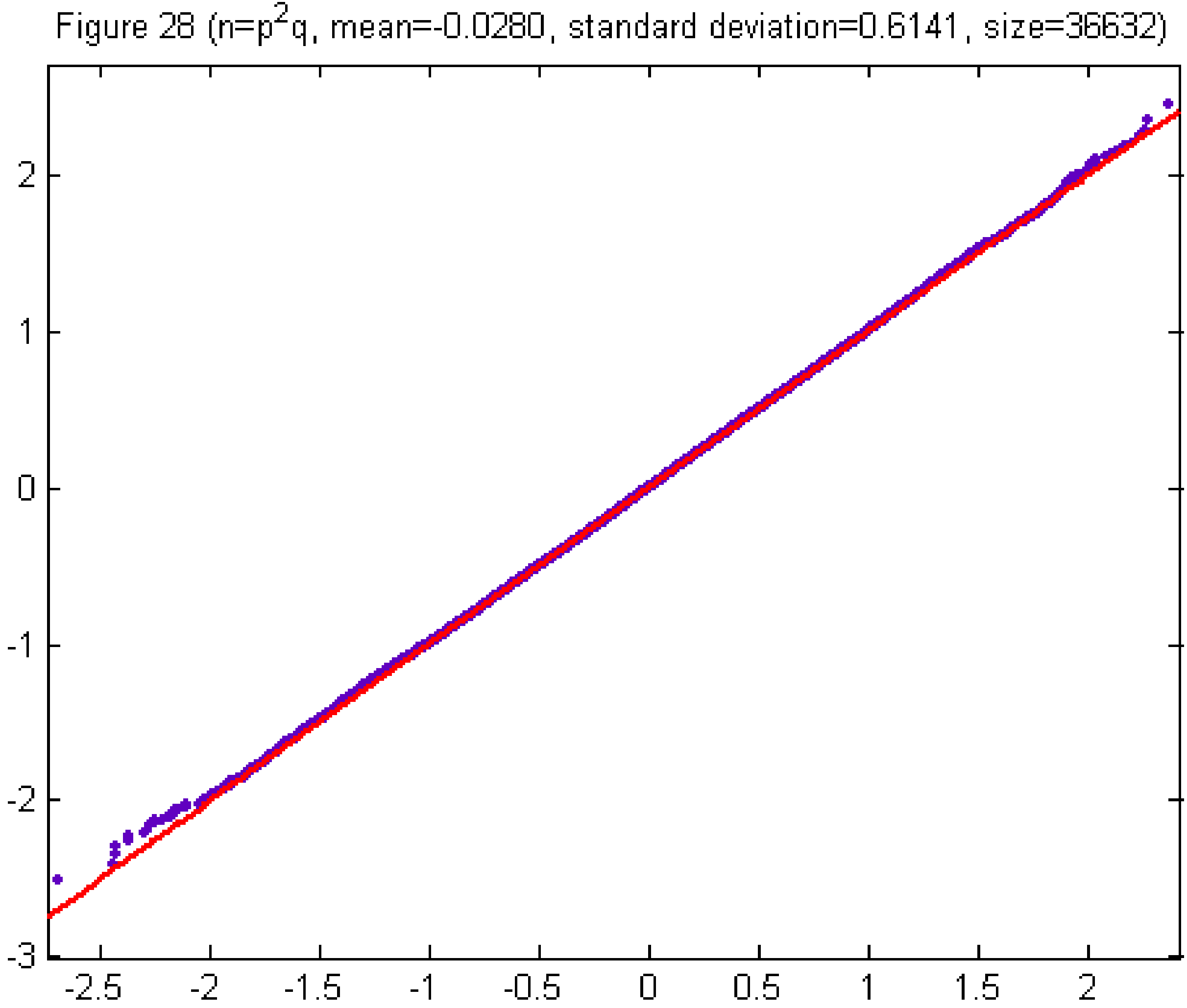


Figure 28 (n=p2q, mean=-0.0280, standard deviation=0.6141, size=36632)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 29 (n=pqrs, mean=0.0802, standard deviation=1.4069, size=108253)

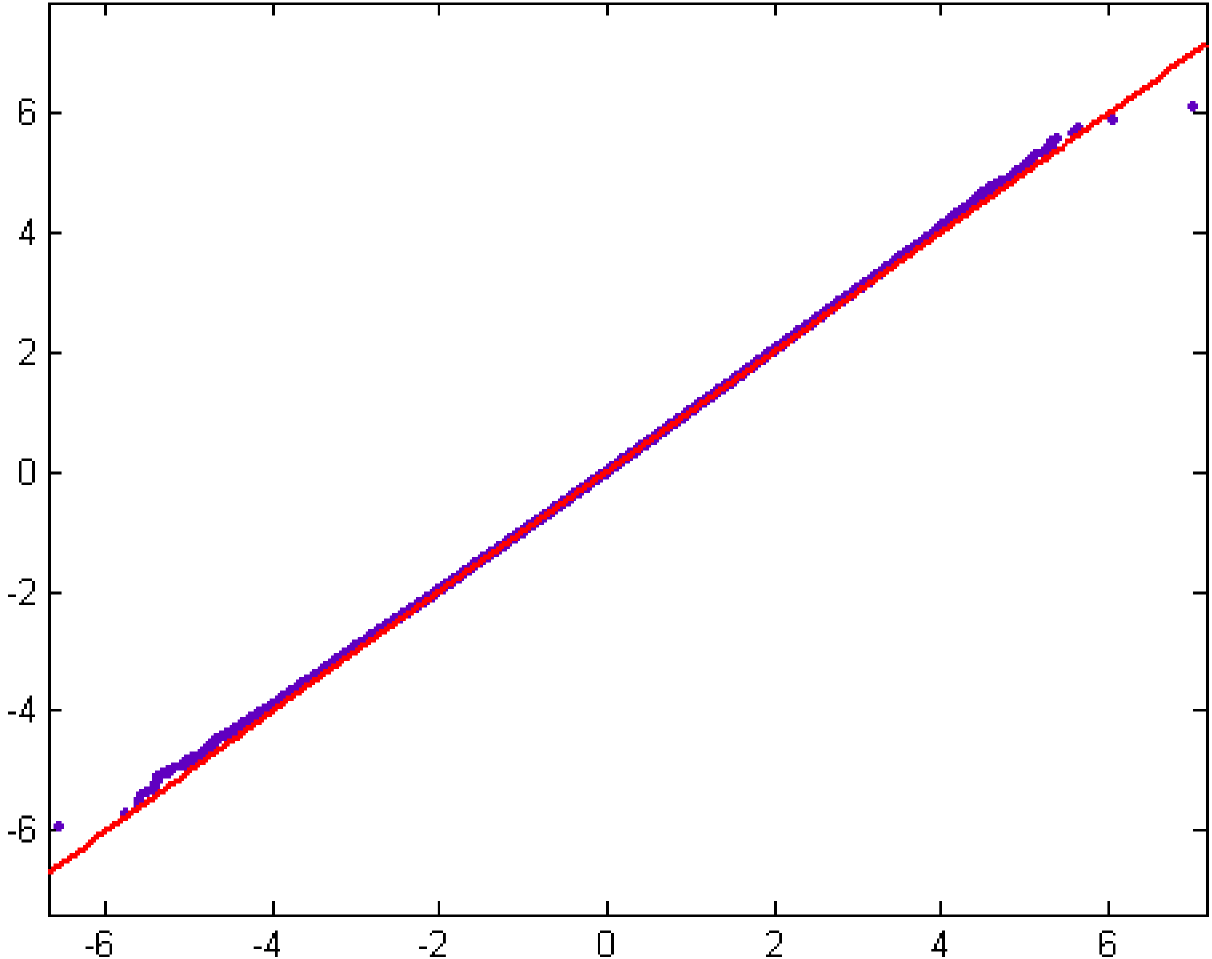


Figure 30 ($n=p^2qrs$, mean=0.0985, standard deviation=1.3805, size=61819)

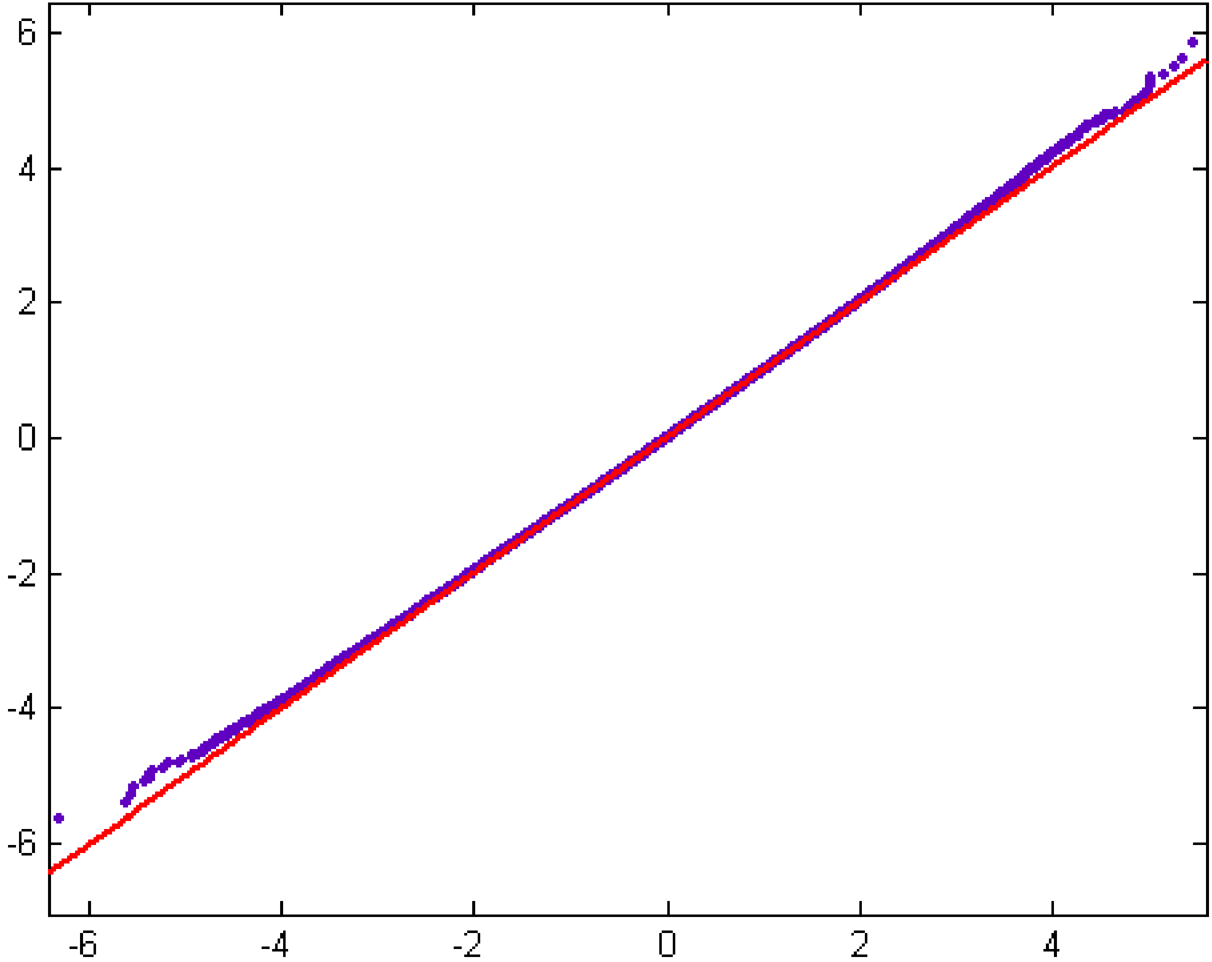


Figure 31 ($n=p^3qrs$, mean=0.1259, standard deviation=1.3582, size=22100)

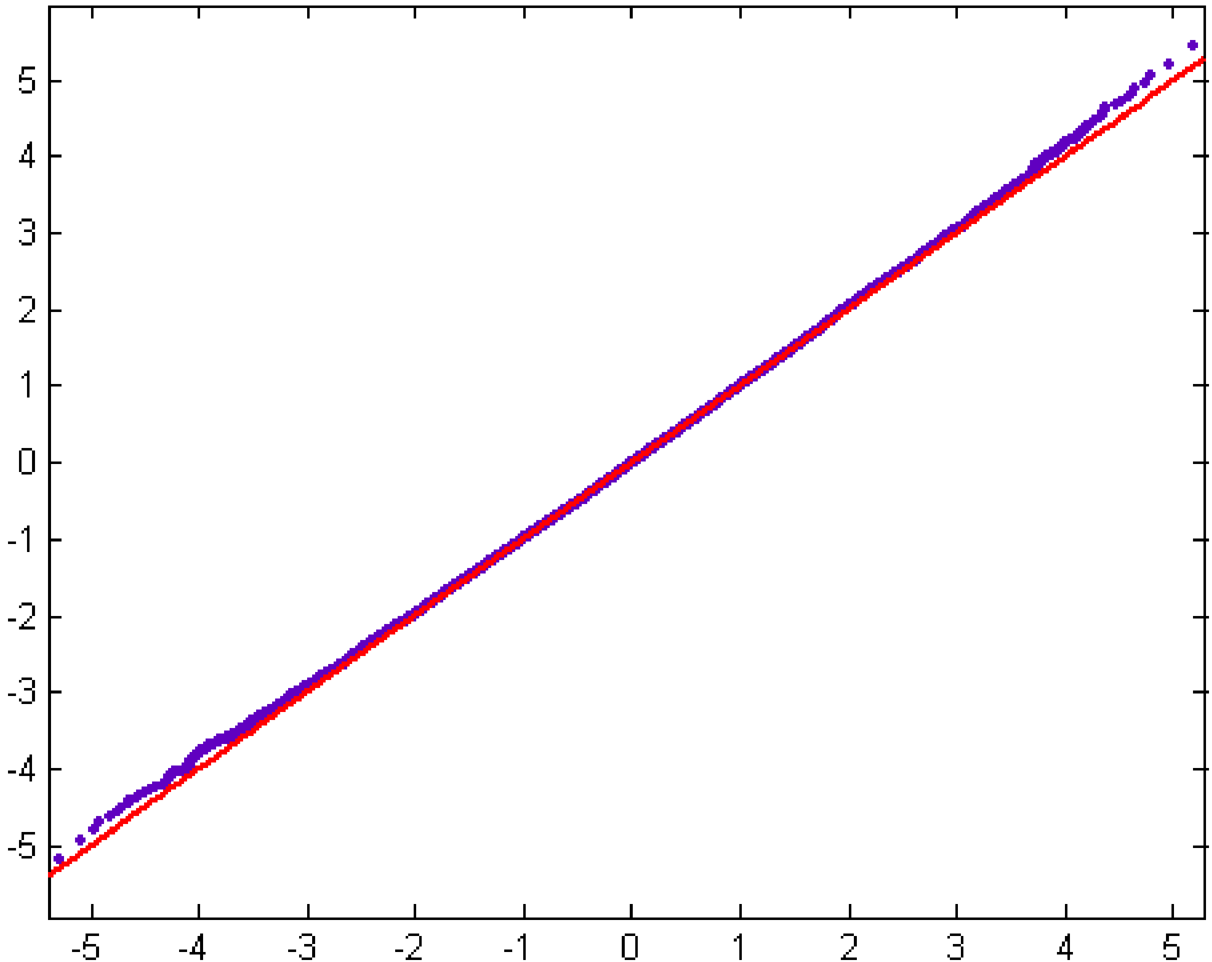


Figure 32 ($n=p^2q^2rs$, mean=0.0895, standard deviation=1.3317, size=10485)

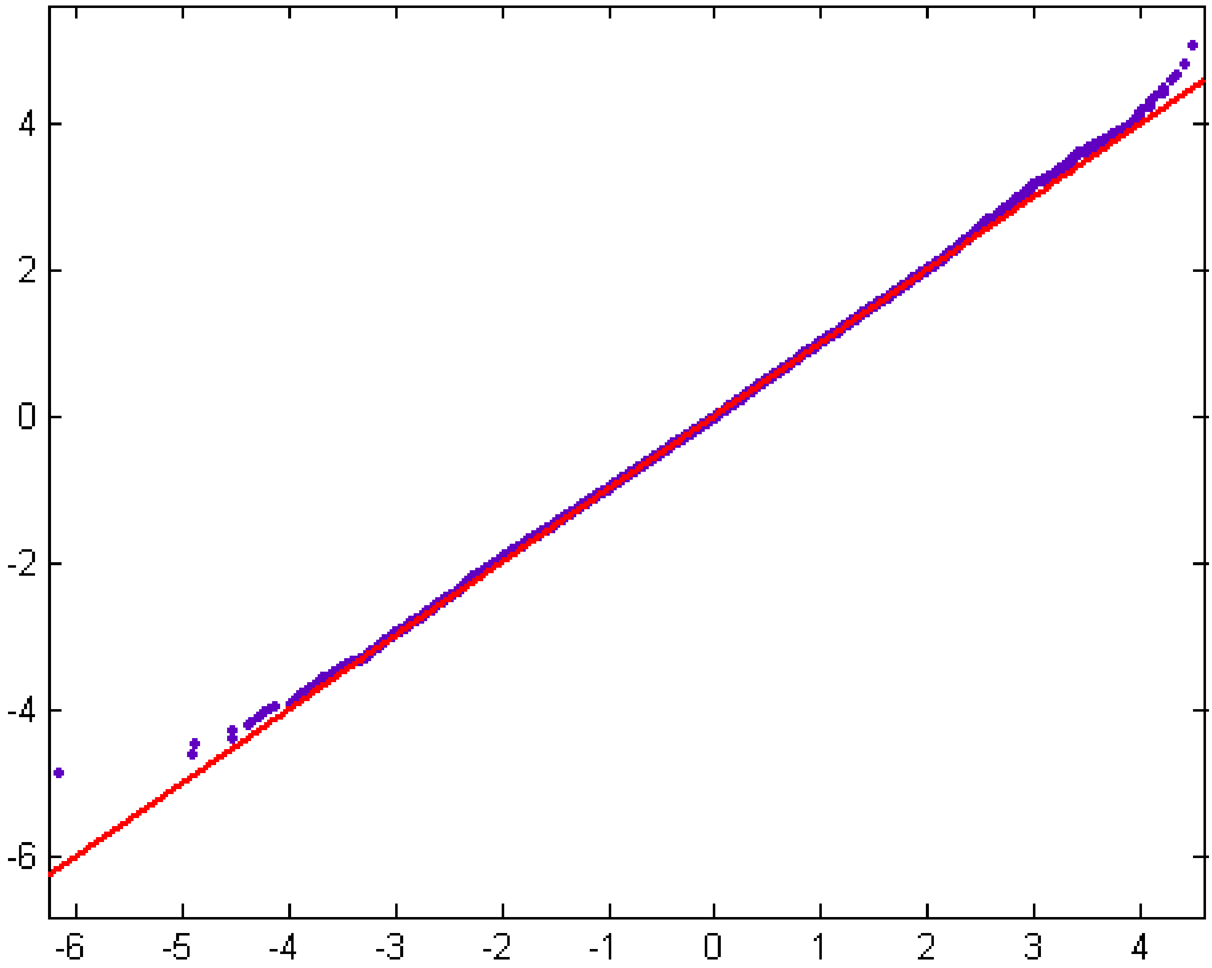
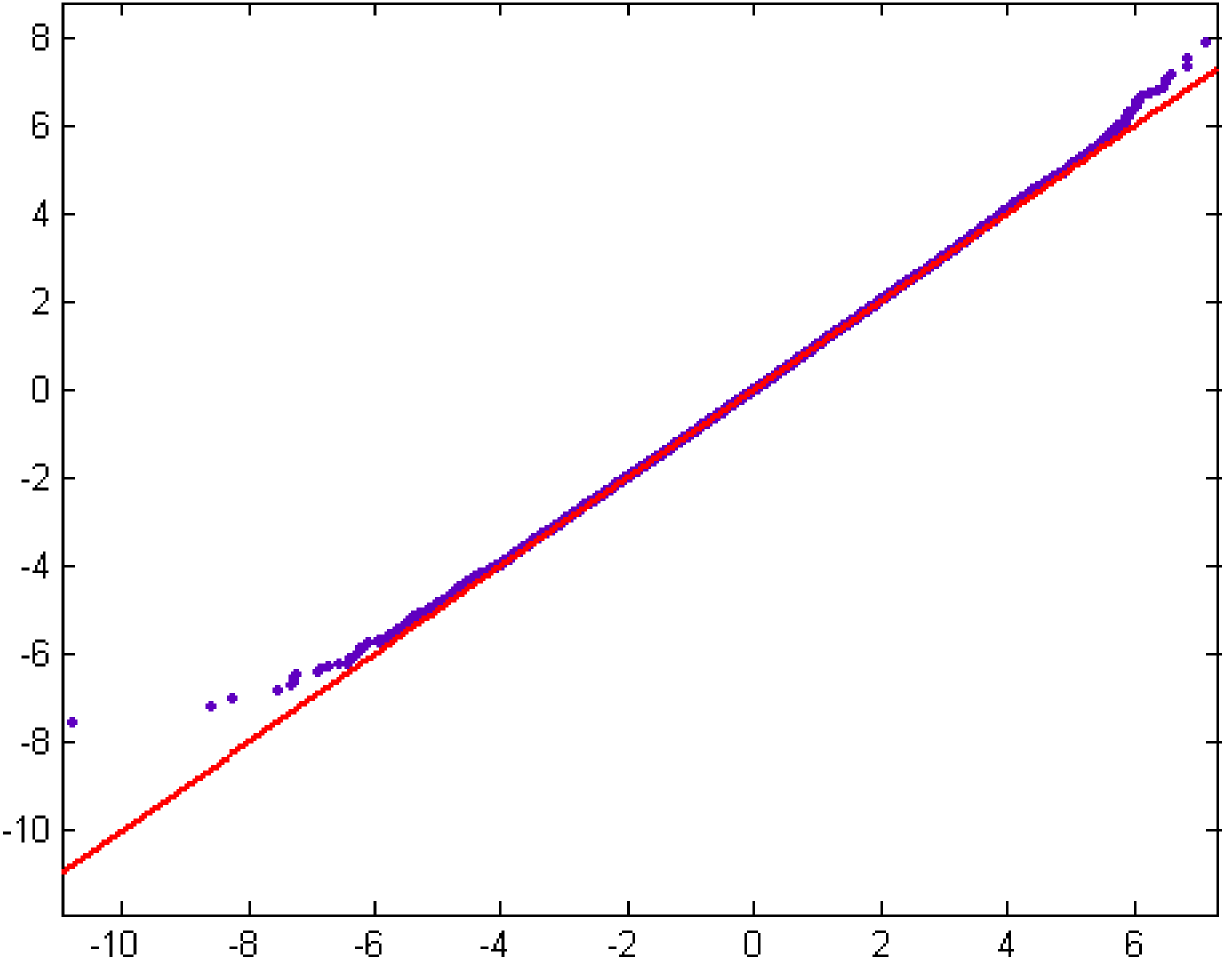


Figure 33 ($n=p^2qrst$, mean=0.1613, standard deviation=1.9899, size=18761)



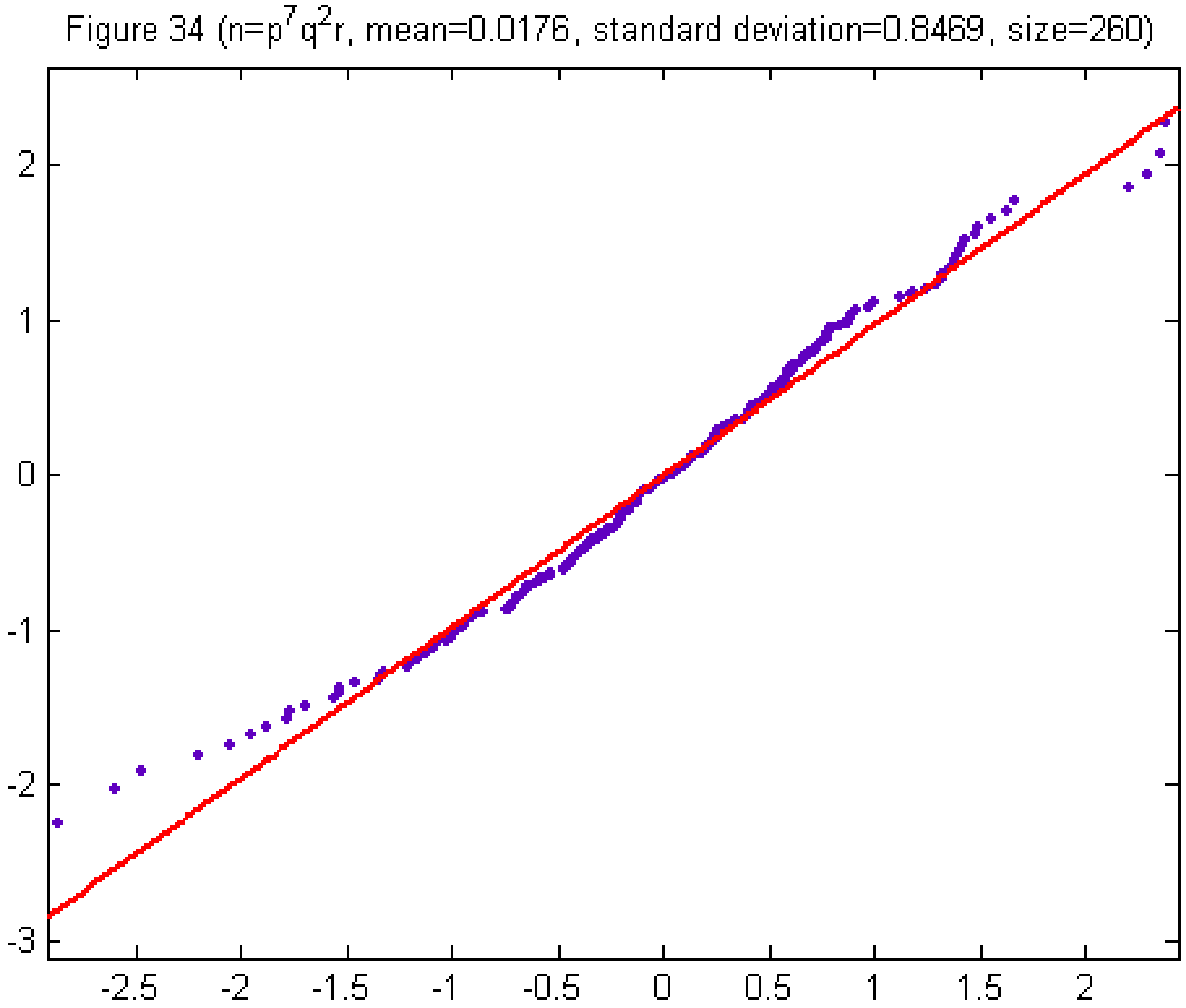
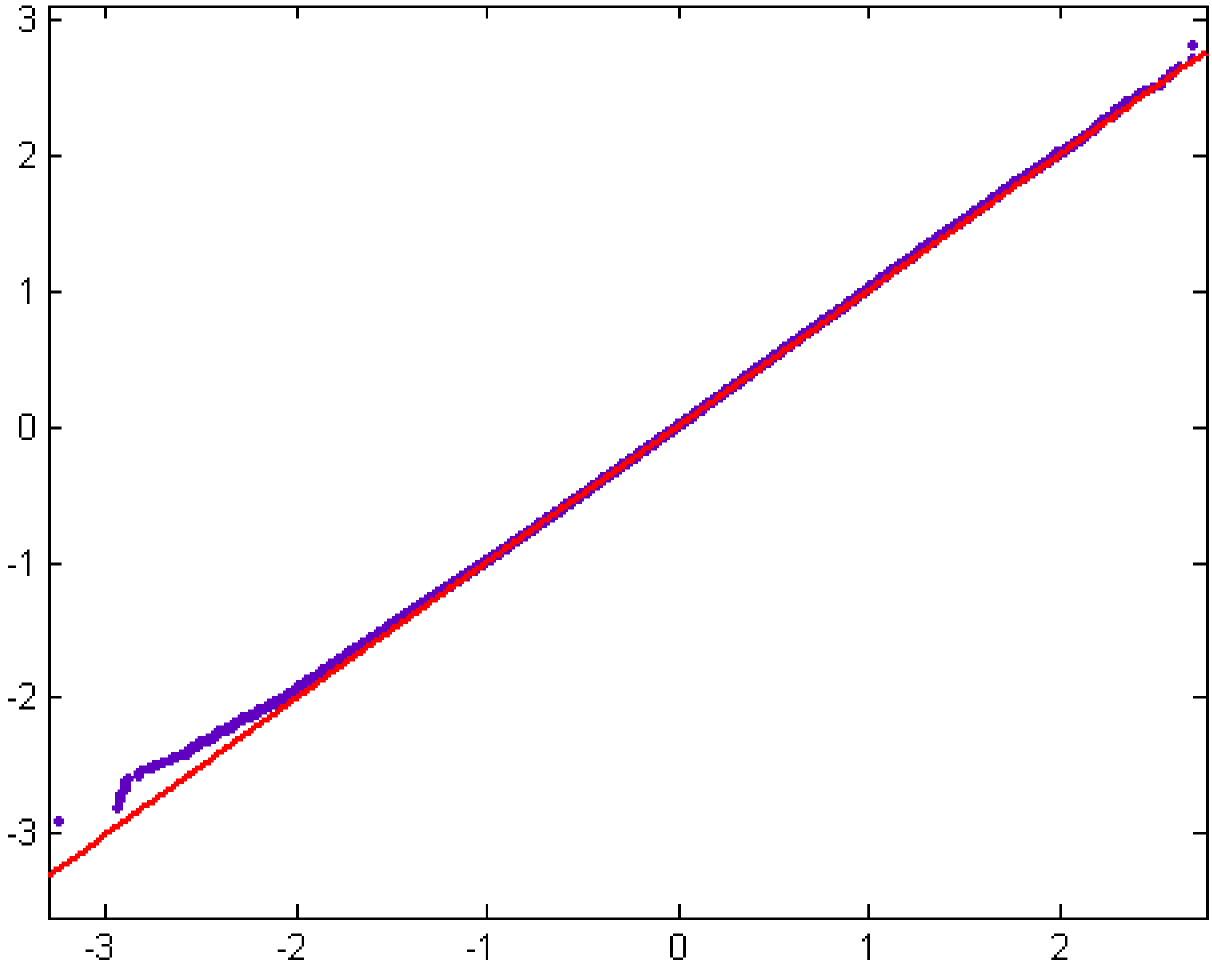


Figure 34 (n=p7q2r, mean=0.0176, standard deviation=0.8469, size=260)
2
1
0
-1
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 35 (n=pq, mean=-0.0546, standard deviation=0.6485, size=190506)



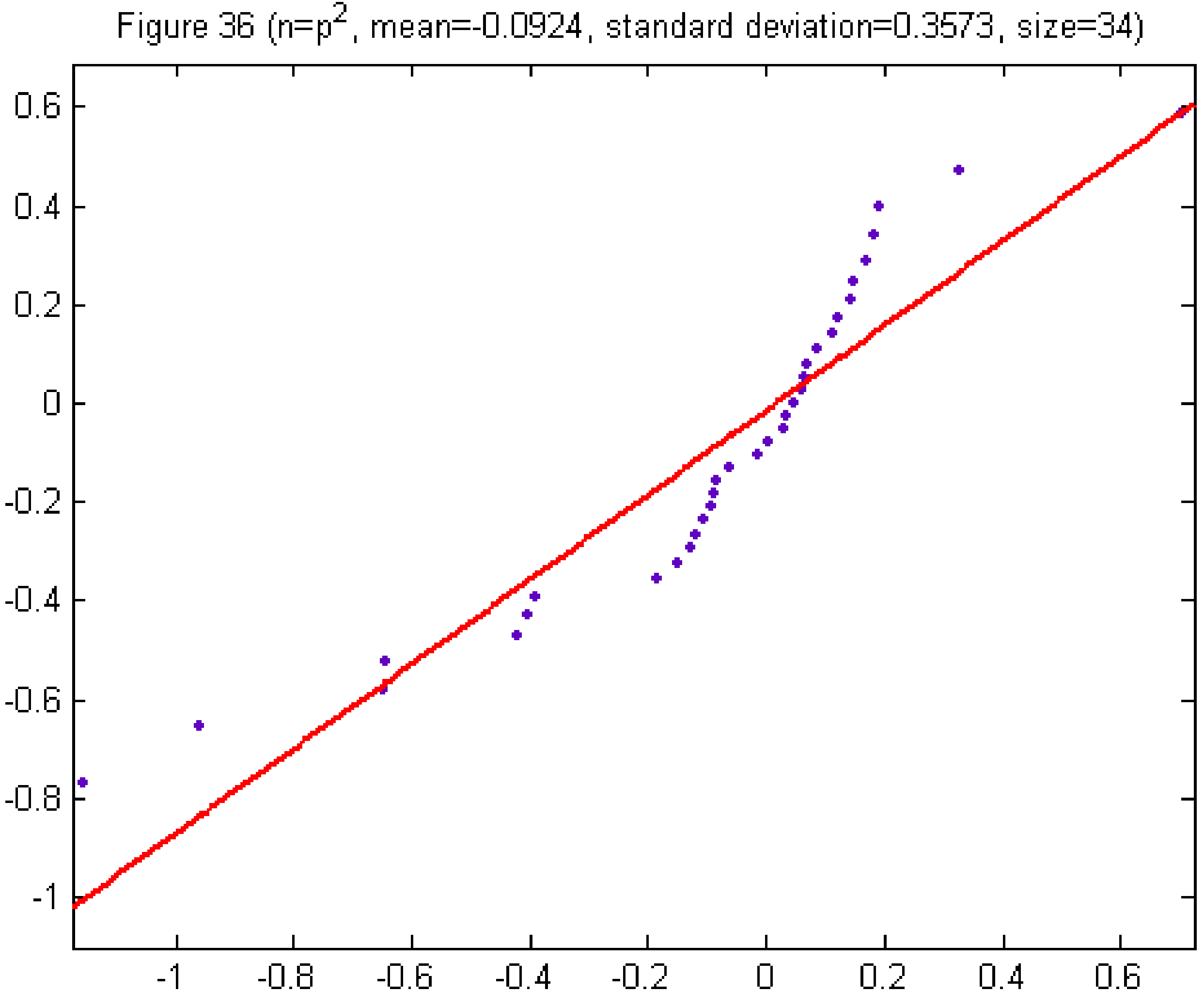


Figure 36 ($n=p^2$, mean=-0.0924, standard deviation=0.3573, size=34)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

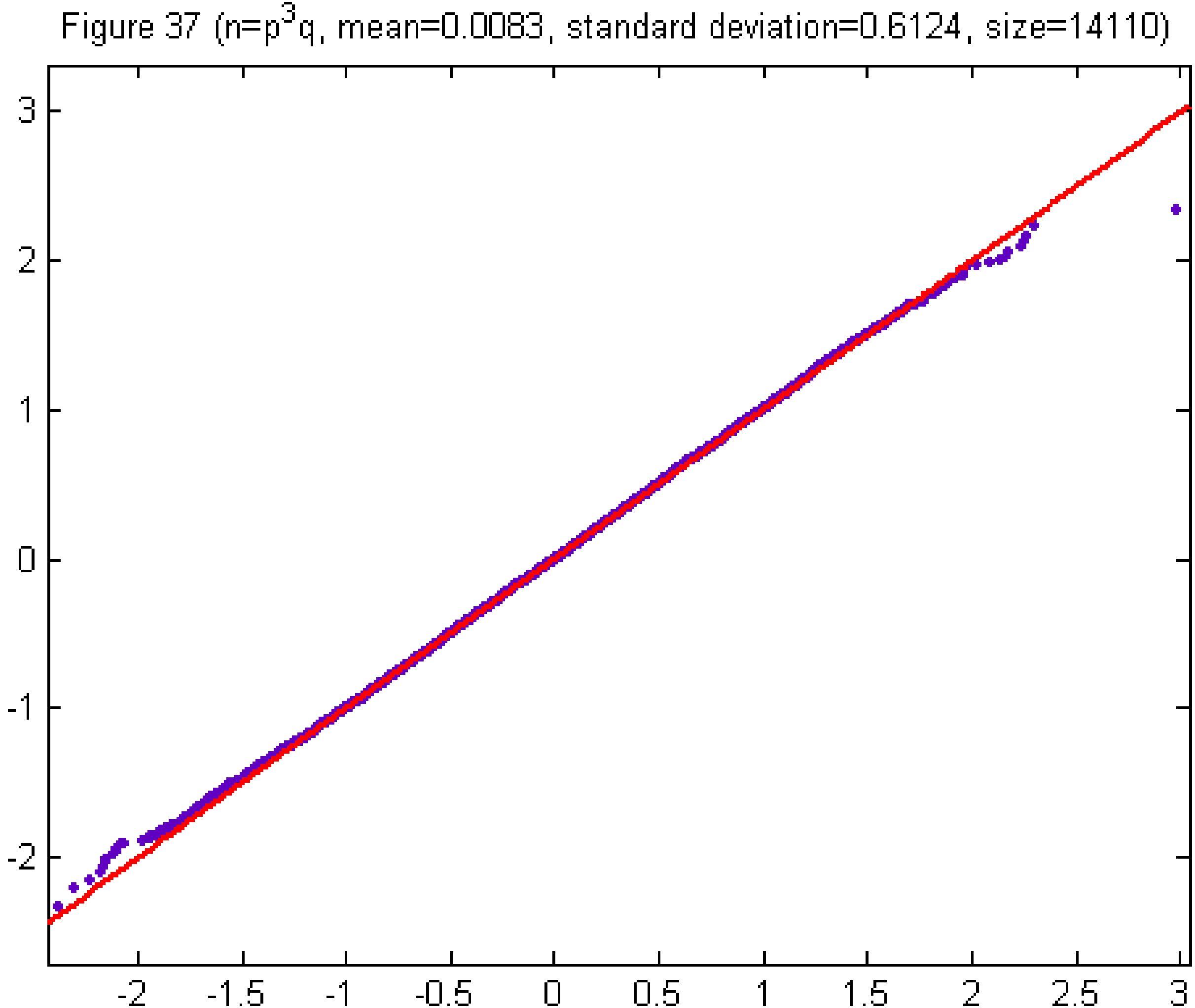
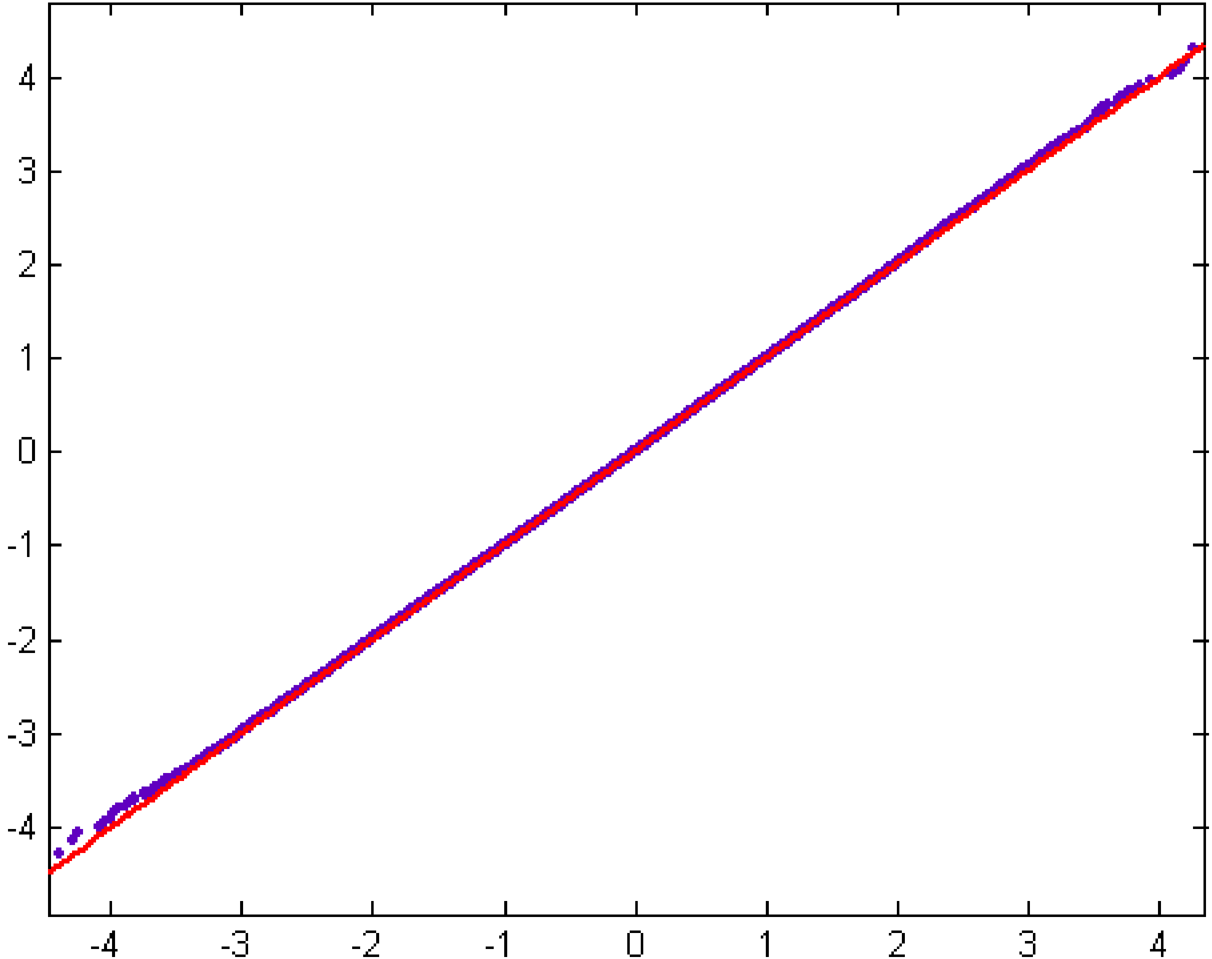


Figure 37 (n=p3q, mean=0.0083, standard deviation=0.6124, size=14110)
3
2
1
0
-1
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

Figure 38 (n=pqr, mean=0.0067 , standard deviation=0.9721 , size=209072)



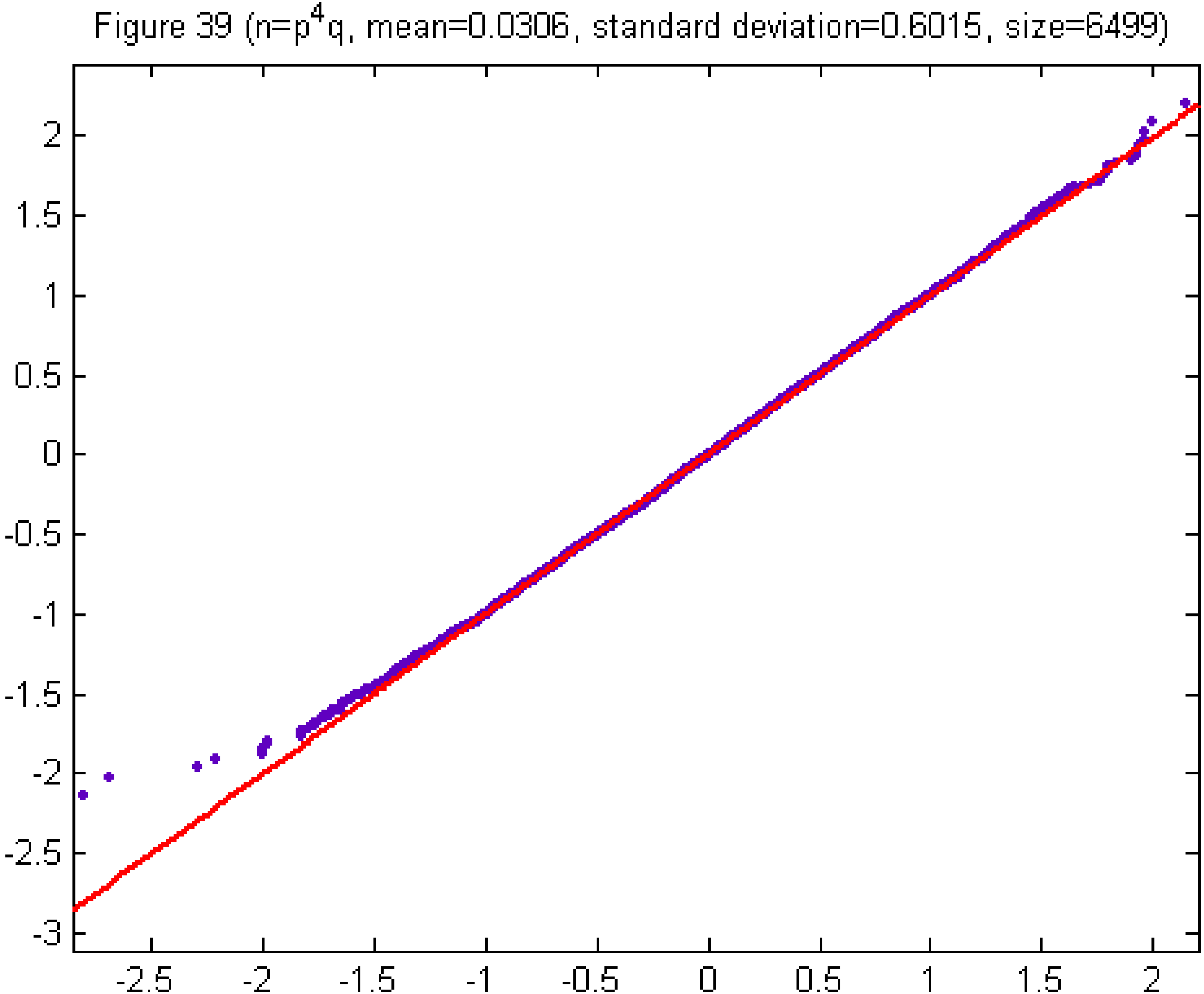


Figure 39 (n=p^4q, mean=0.0306, standard deviation=0.6015, size=6499)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

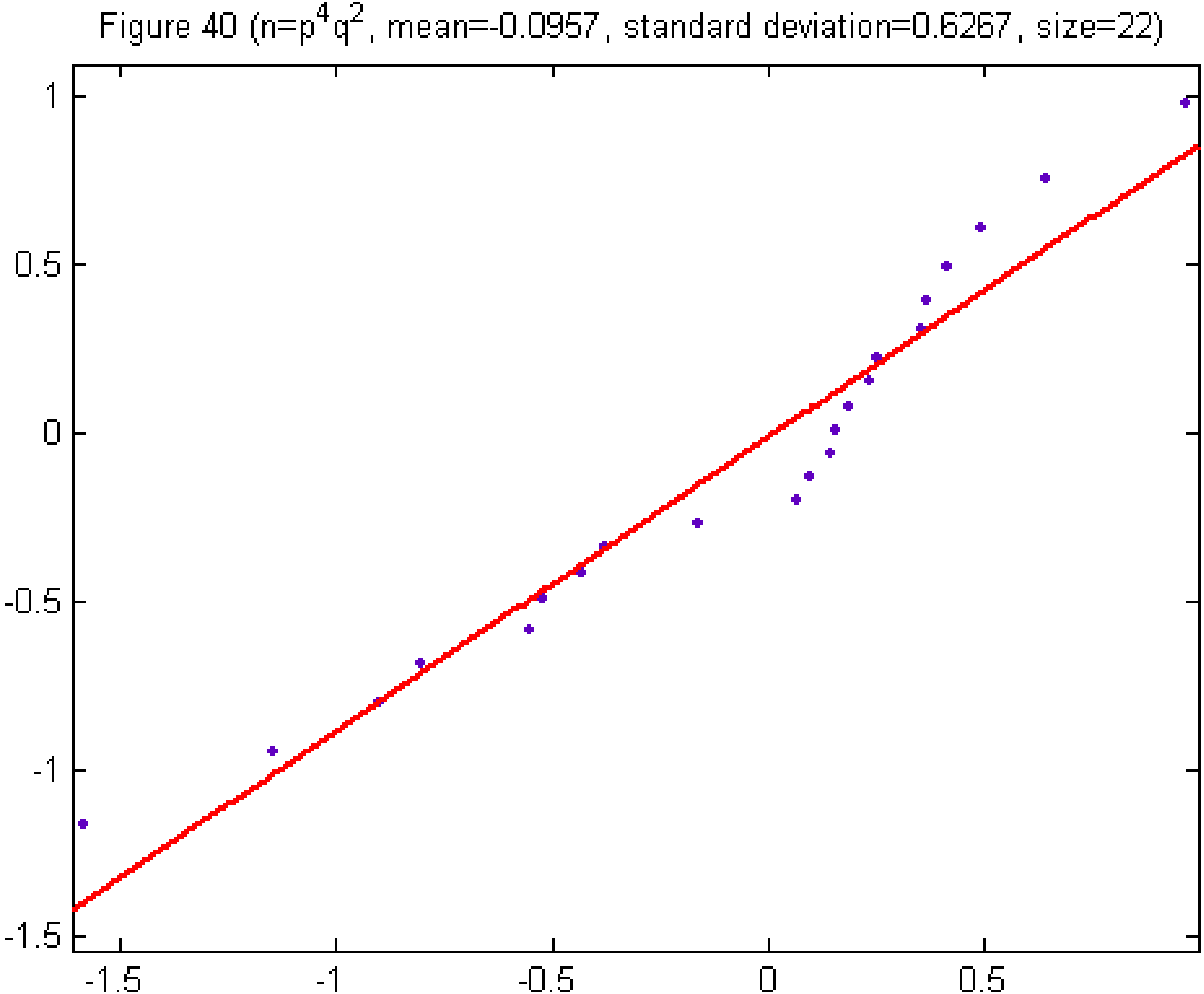


Figure 40 (n=p4q2, mean=-0.0957, standard deviation=0.6267, size=22)
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5

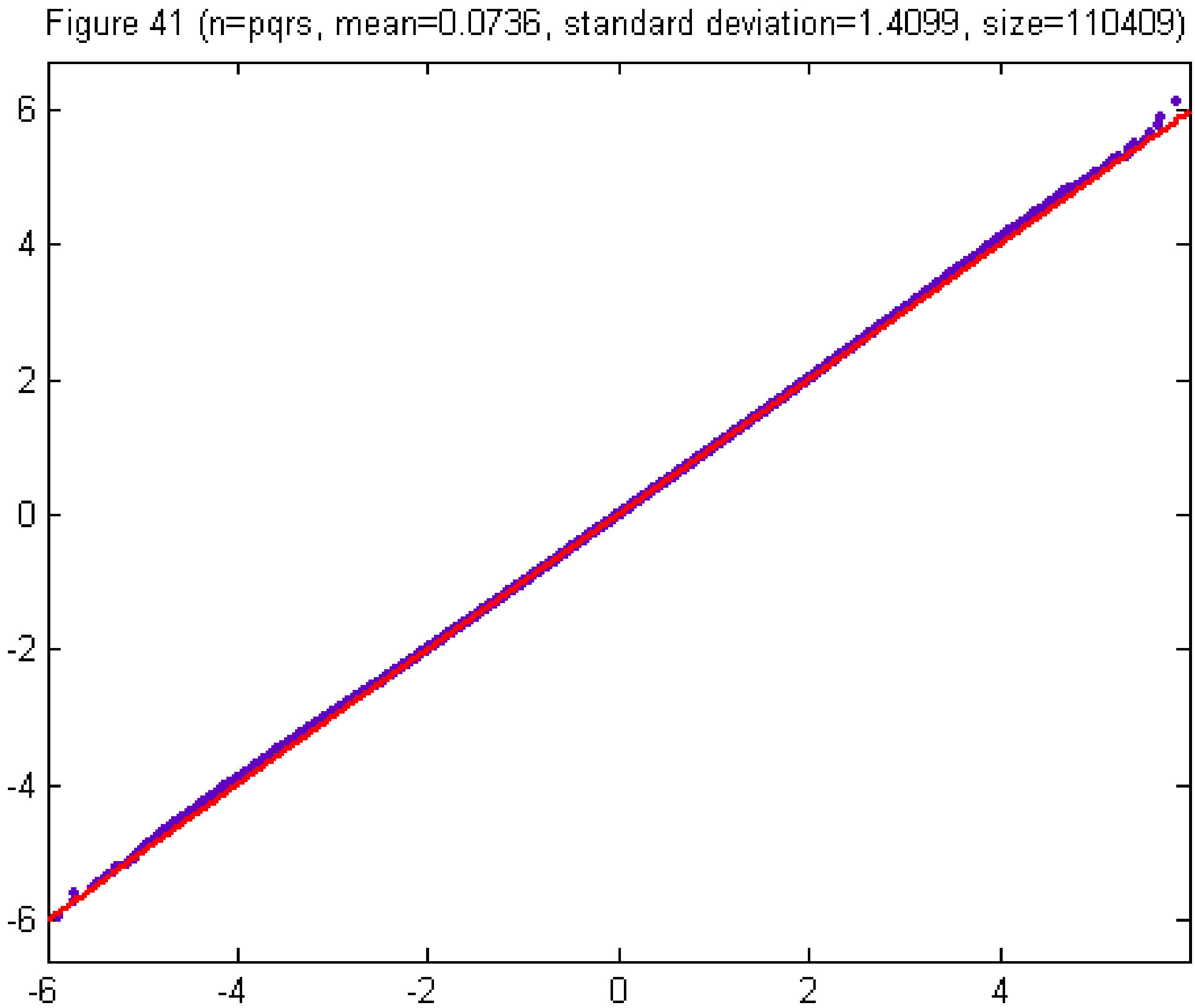


Figure 41 (n=pqrs, mean=0.0736, standard deviation=1.4099, size=110409)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

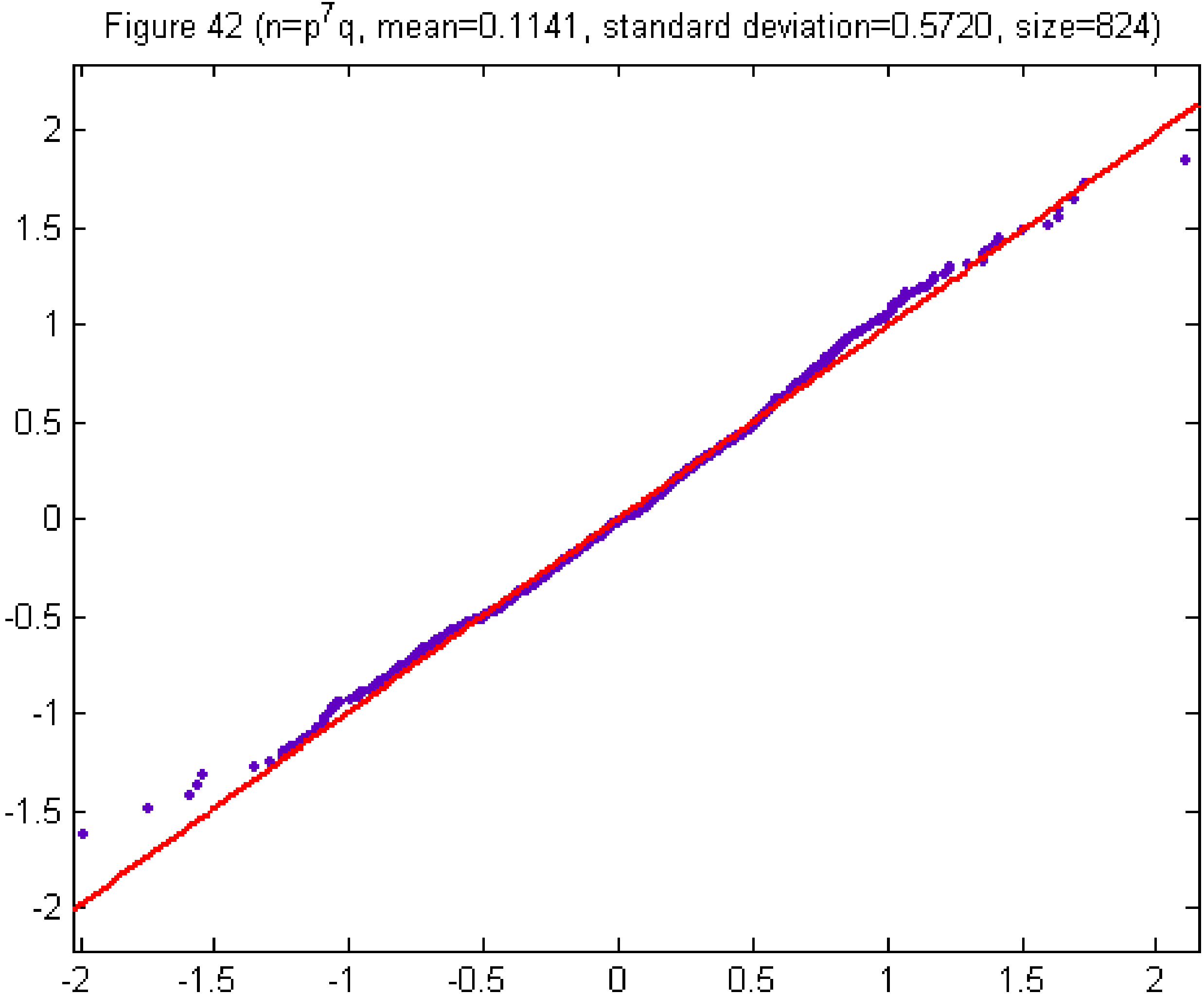


Figure 42 (n=p^7q, mean=0.1141, standard deviation=0.5720, size=824)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 43 ($n=p^2q^2r$, mean=0.0167, standard deviation=0.9110, size=6695)

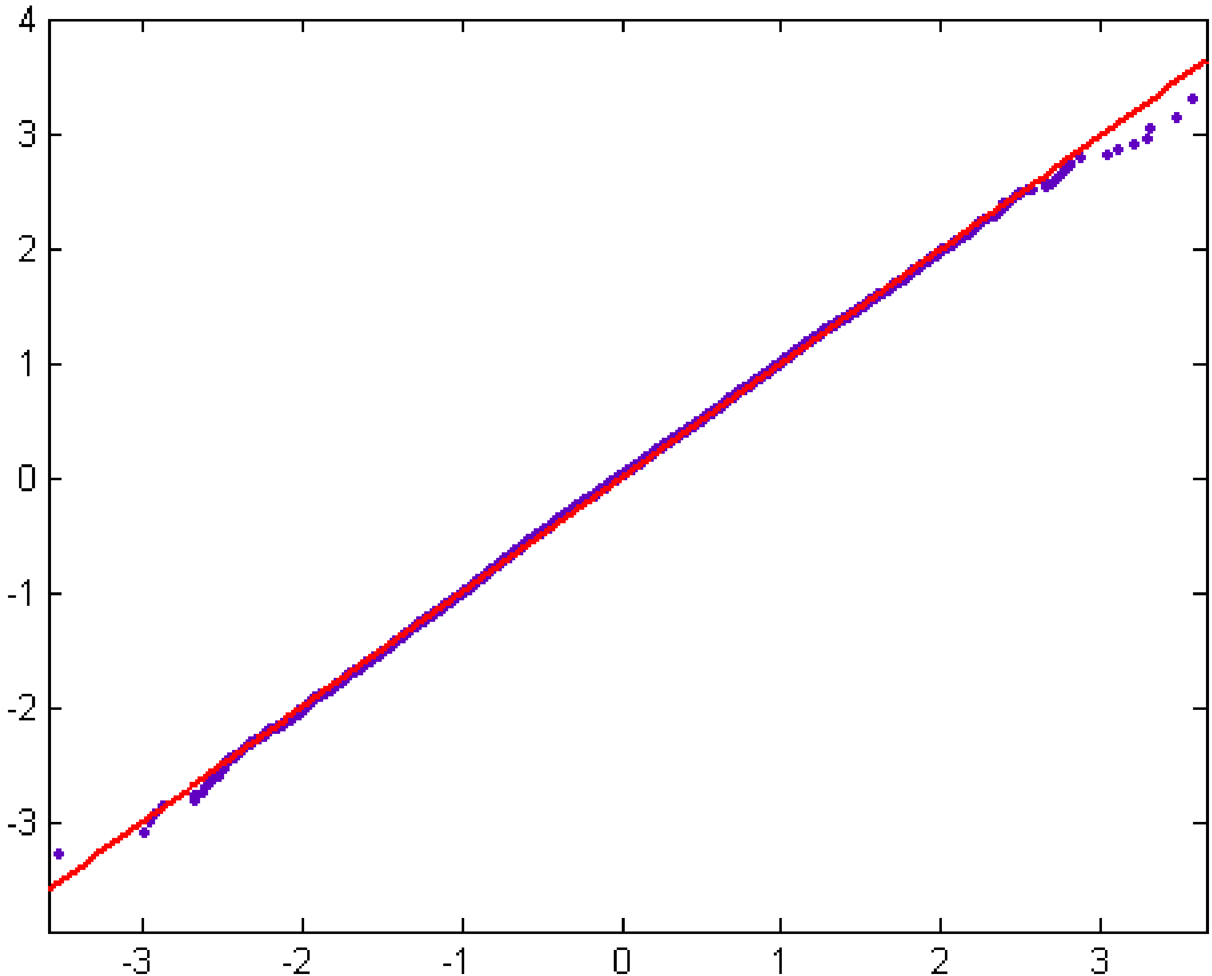


Figure 44 ($n=p^3q^2r$, mean=0.0170, standard deviation=0.8952, size=4753)

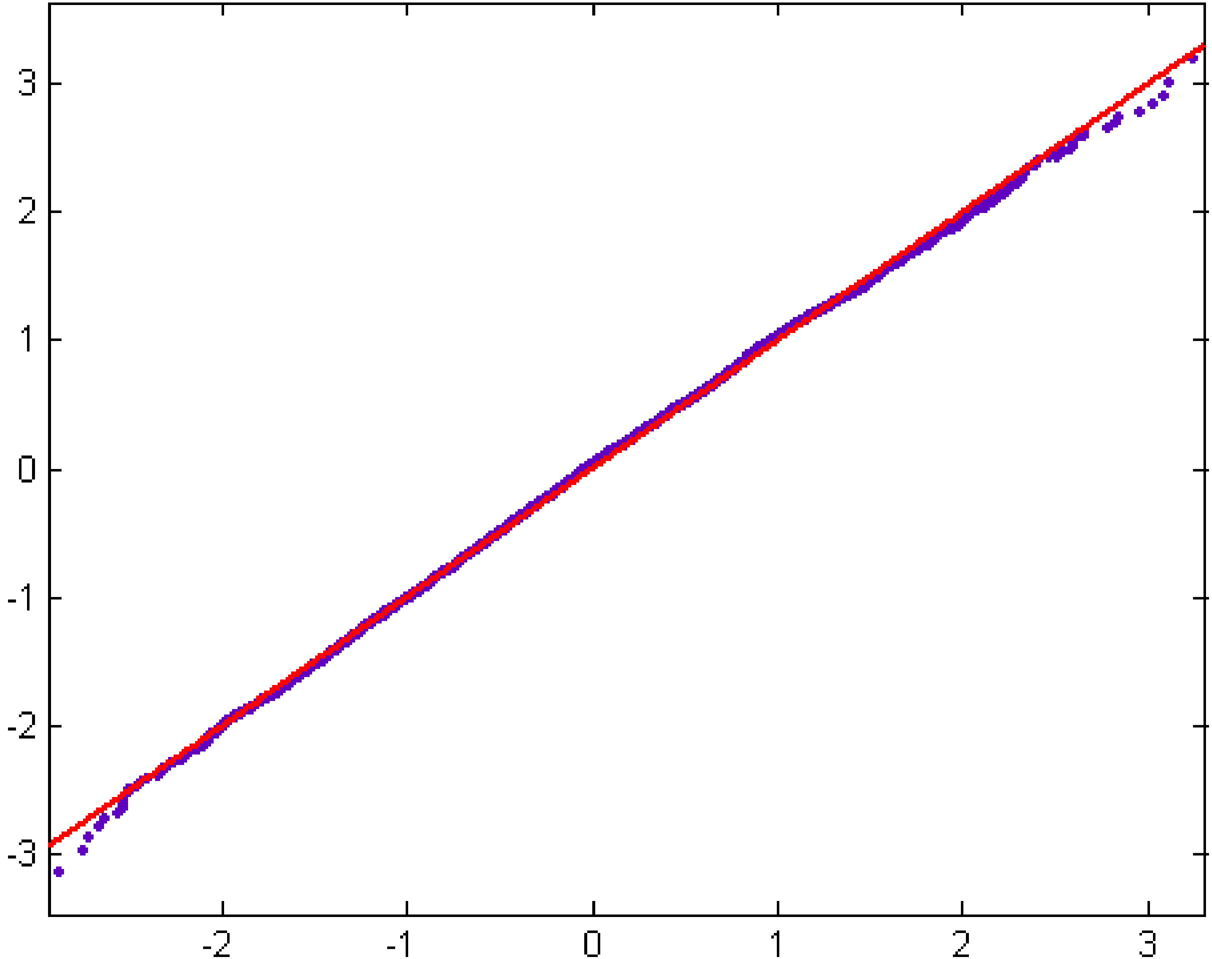


Figure 45 ($n=p^2qrs$, mean=0.0991, standard deviation=1.3829, size=62558)

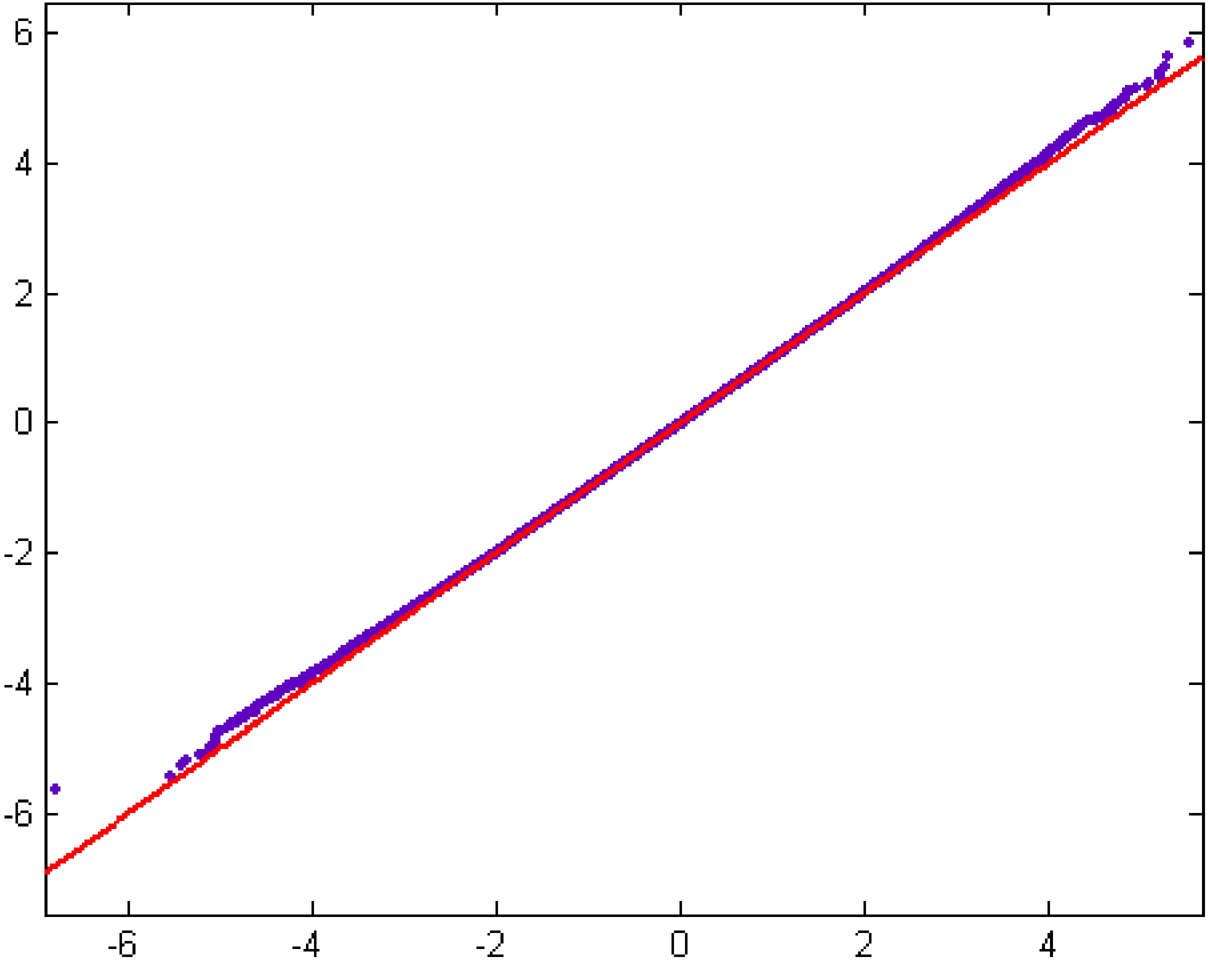


Figure 46 (n=pqrst, mean==0.1435, standard deviation=2.0335, size=28451)

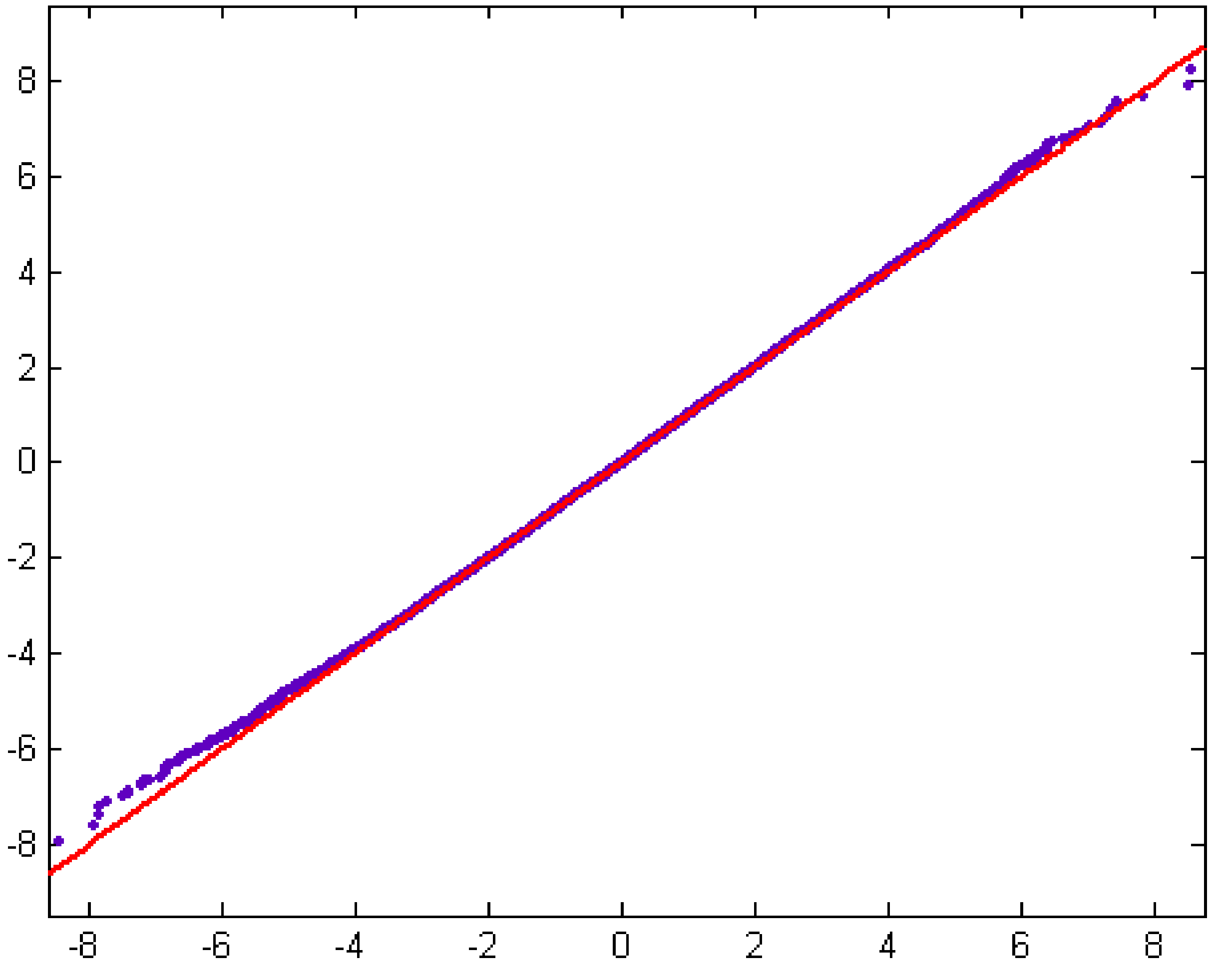


Figure 47 ($n=p^3q^2rs$, mean=0.1015, standard deviation=1.3283, size=6913)

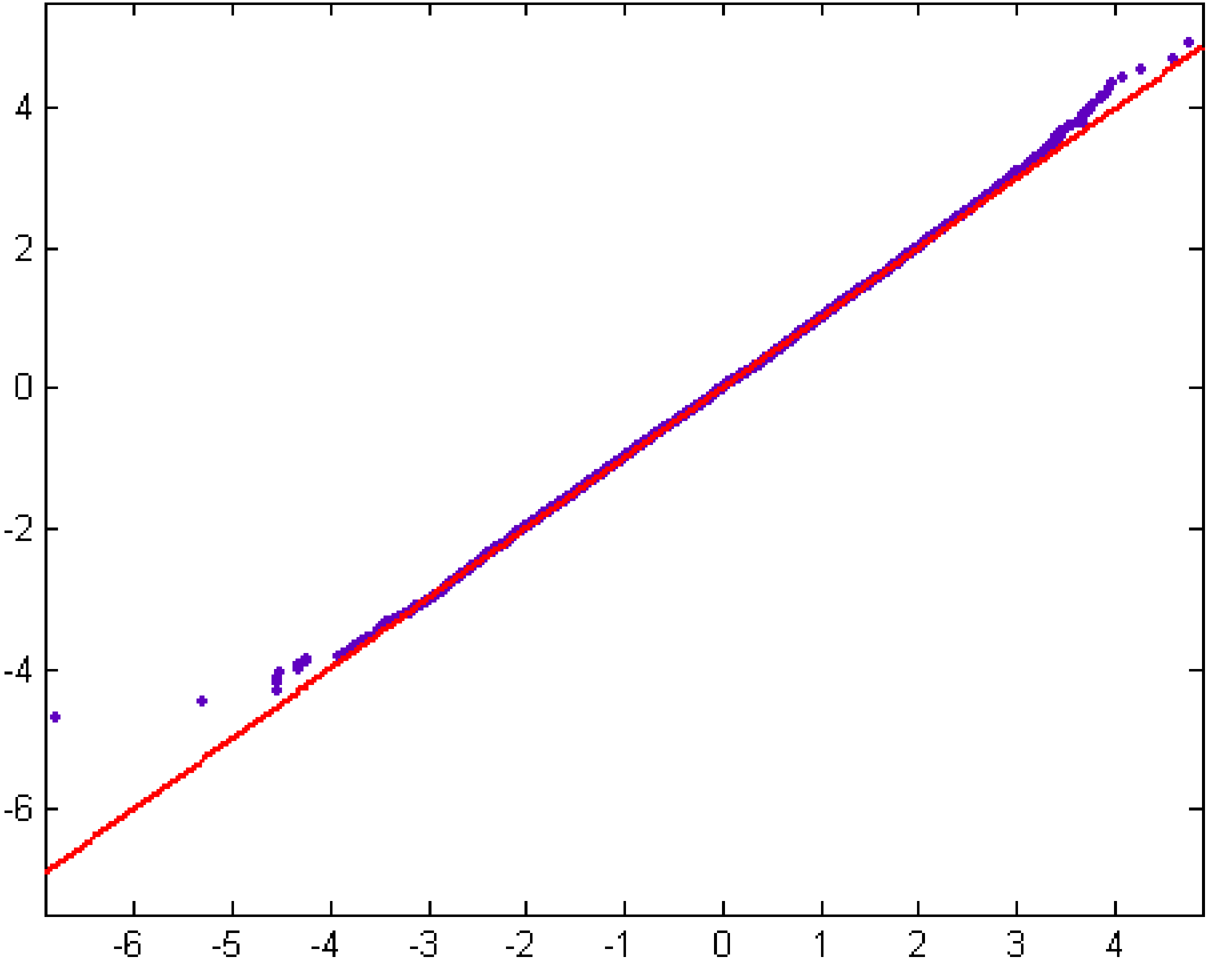


Figure 48 (n=pq, mean=-0.0562, standard deviation=0.6487 , size=188260)

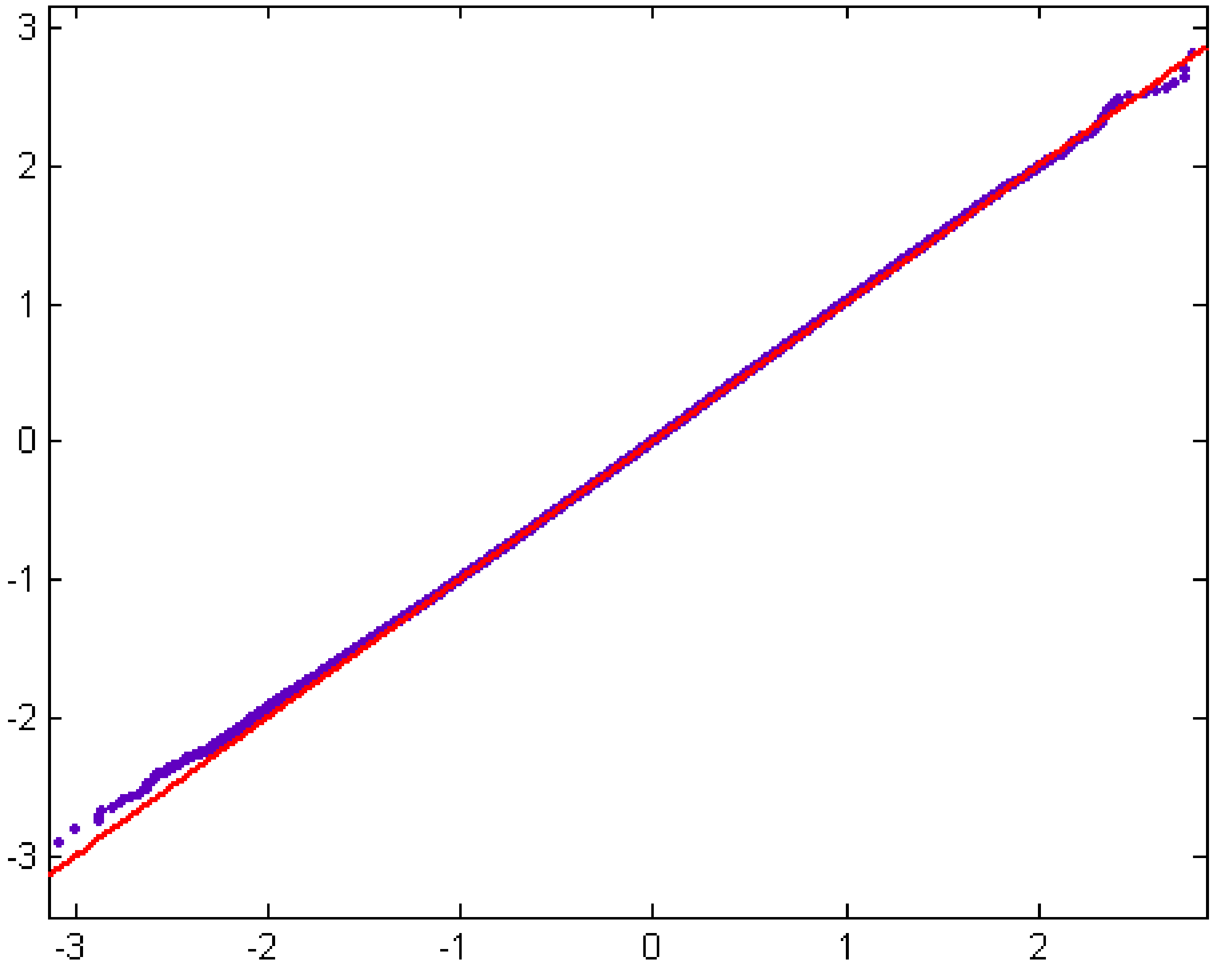


Figure 49 (n=pqrs, mean=0.0729, standard deviation=1.4168, size=111897)

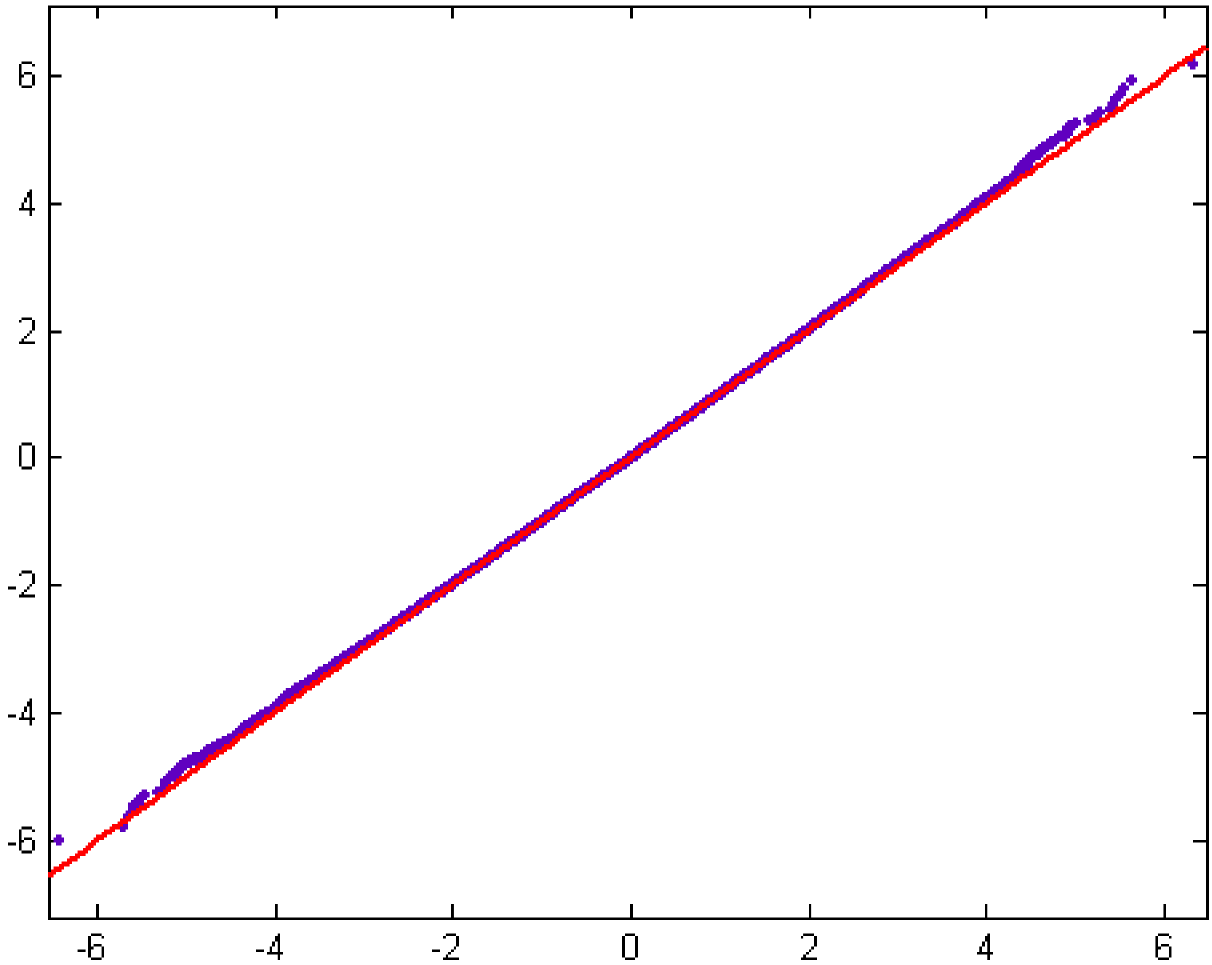


Figure 50 ($n=p^4qr$, mean=0.1020, standard deviation=0.9257 , size=13546)

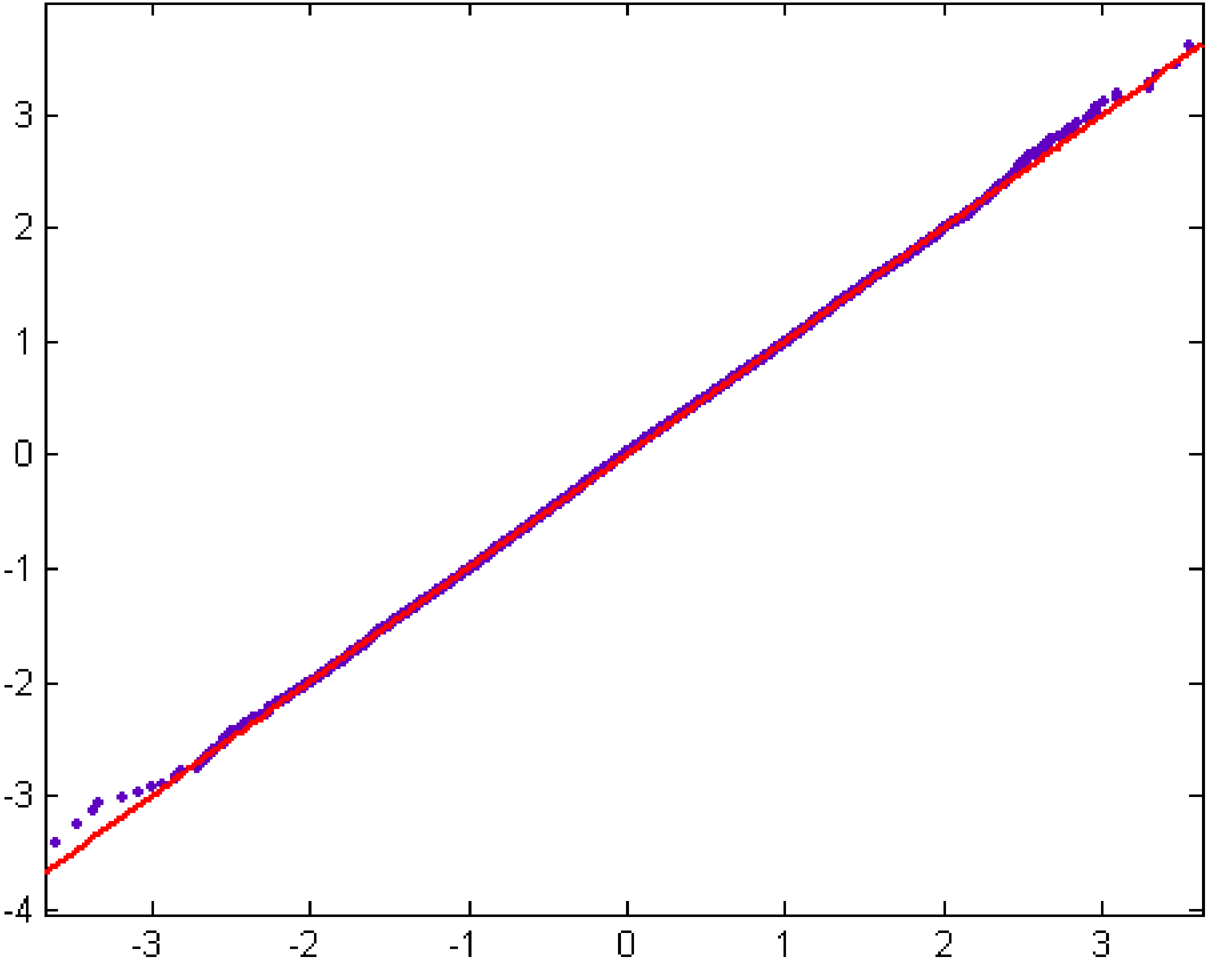
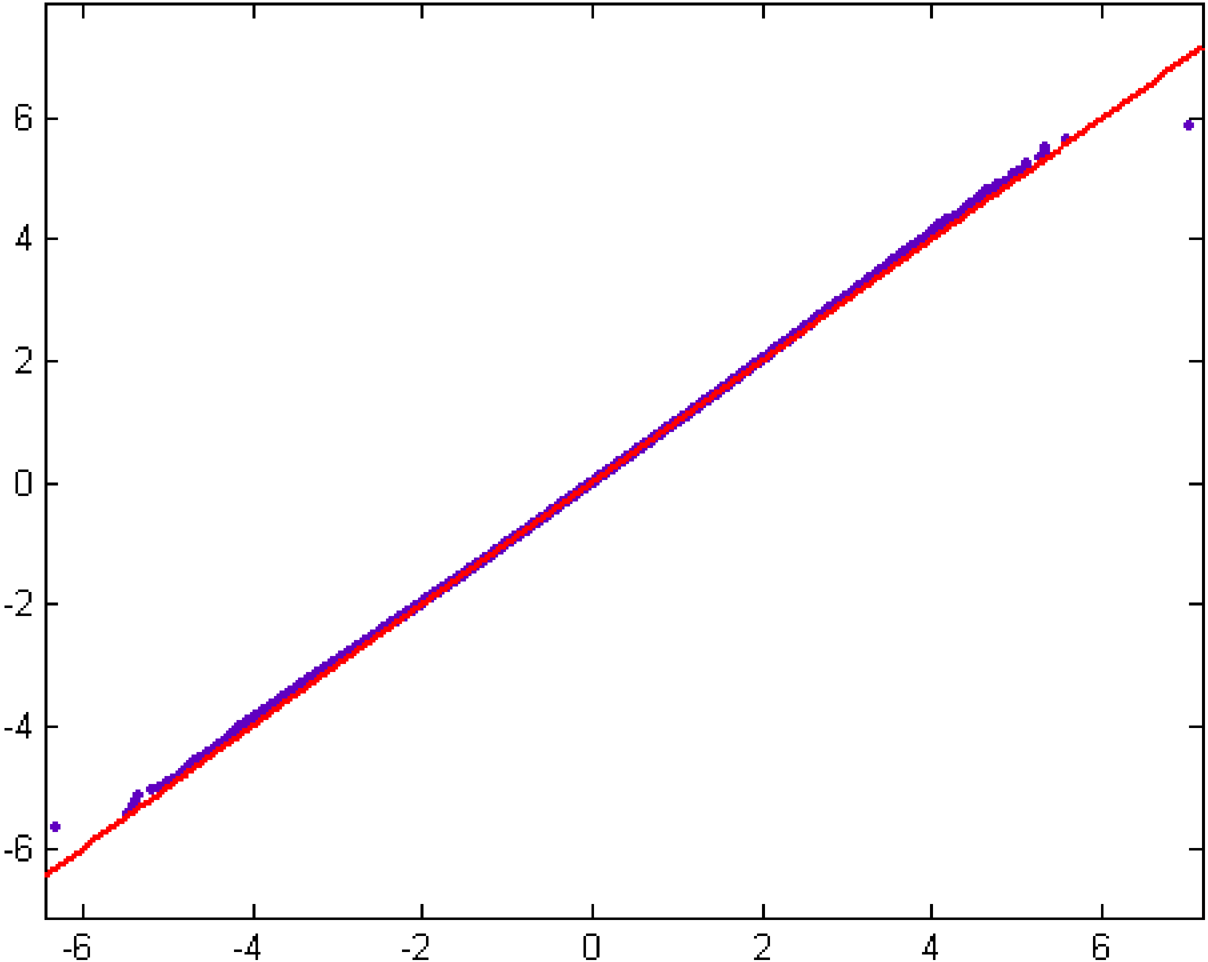


Figure 51 ($n=p^2qrs$, mean=0.0948, standard deviation=1.3852, size=63148)



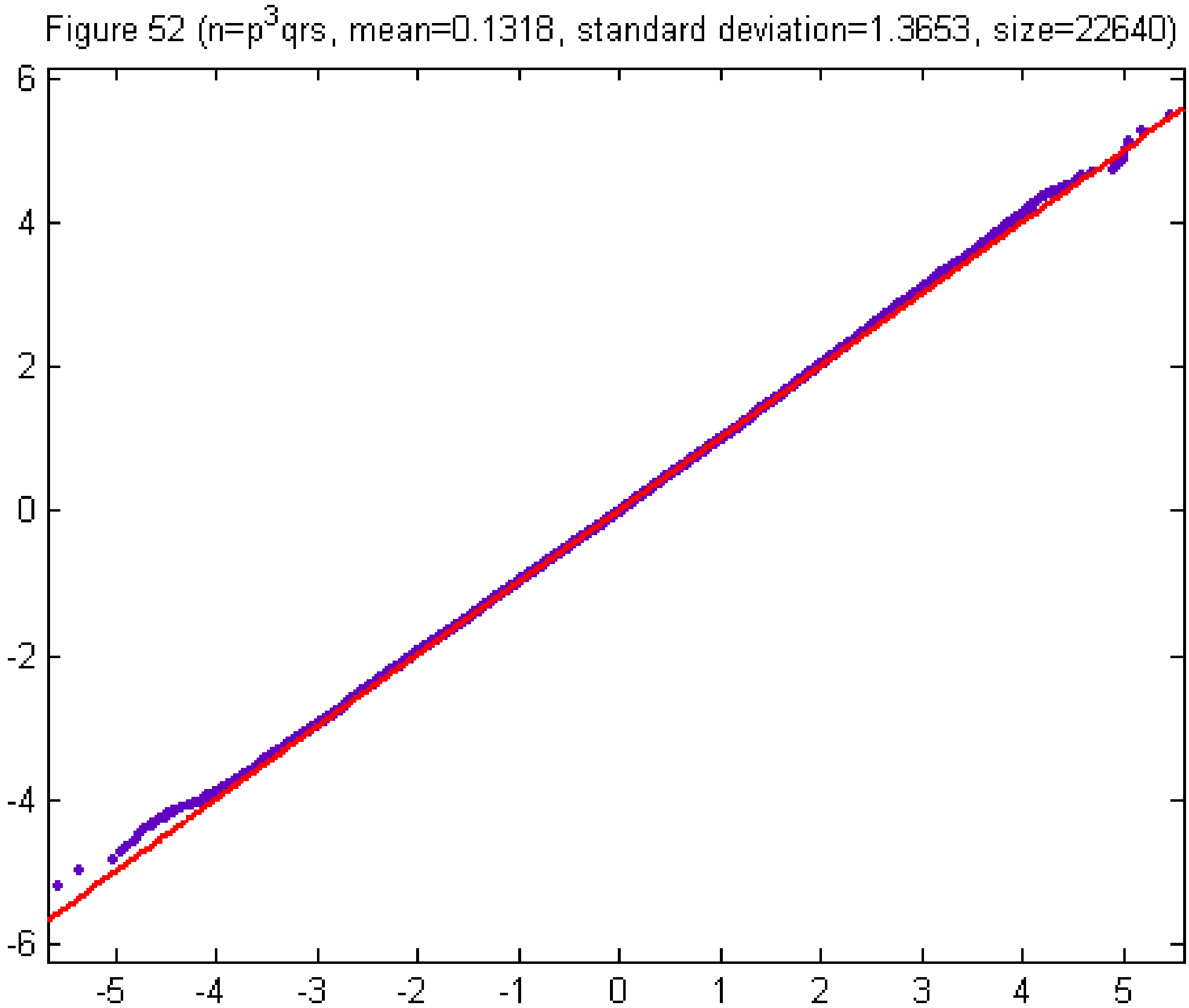


Figure 52 (n=p3qrs, mean=0.1318, standard deviation=1.3653, size=22640)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

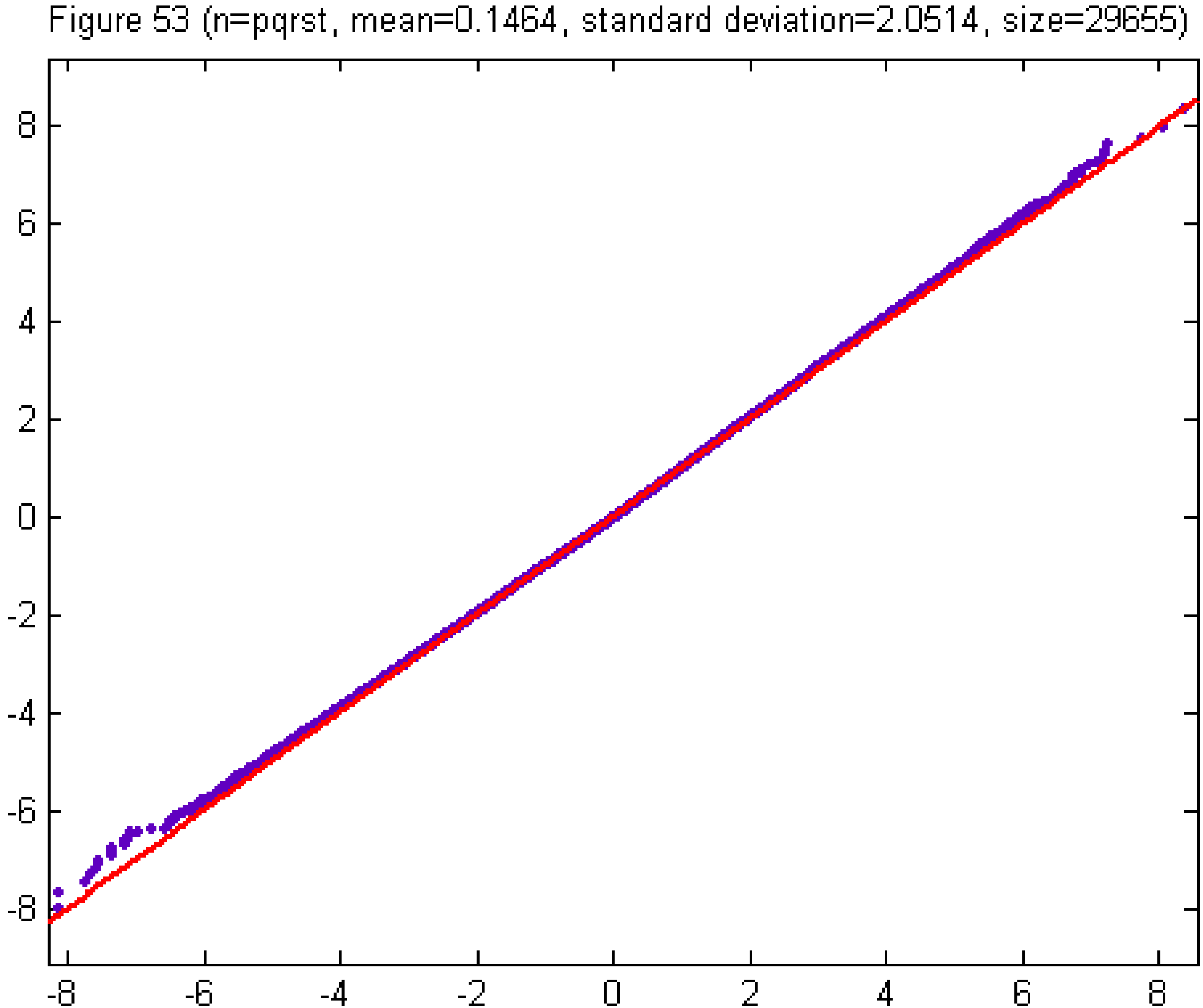


Figure 53 (n=pqrst, mean=0.1464, standard deviation=2.0514, size=29655)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

Figure 54 ($n=p^2q^2rs$, mean=0.0992, standard deviation=1.3539, size=10342)

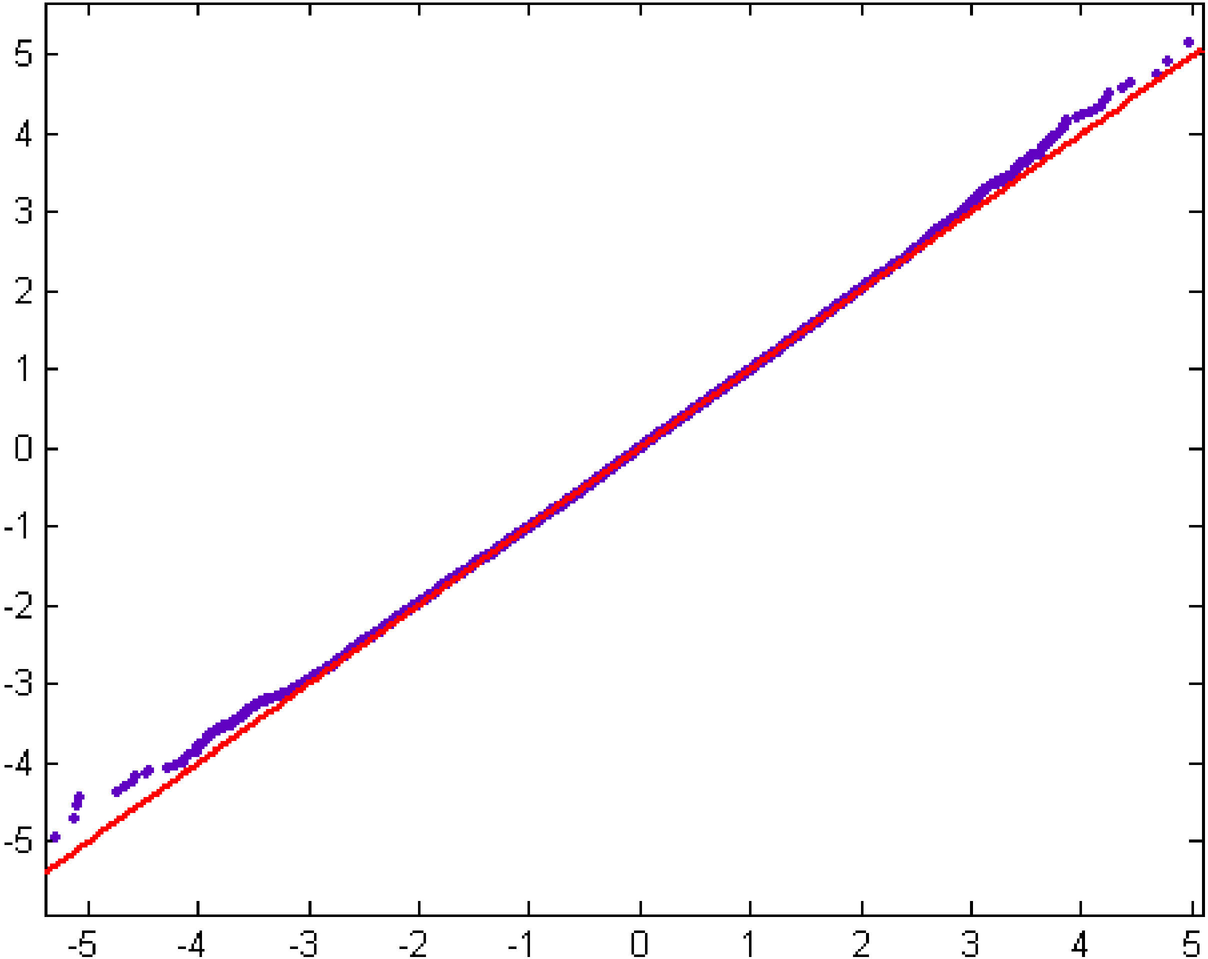
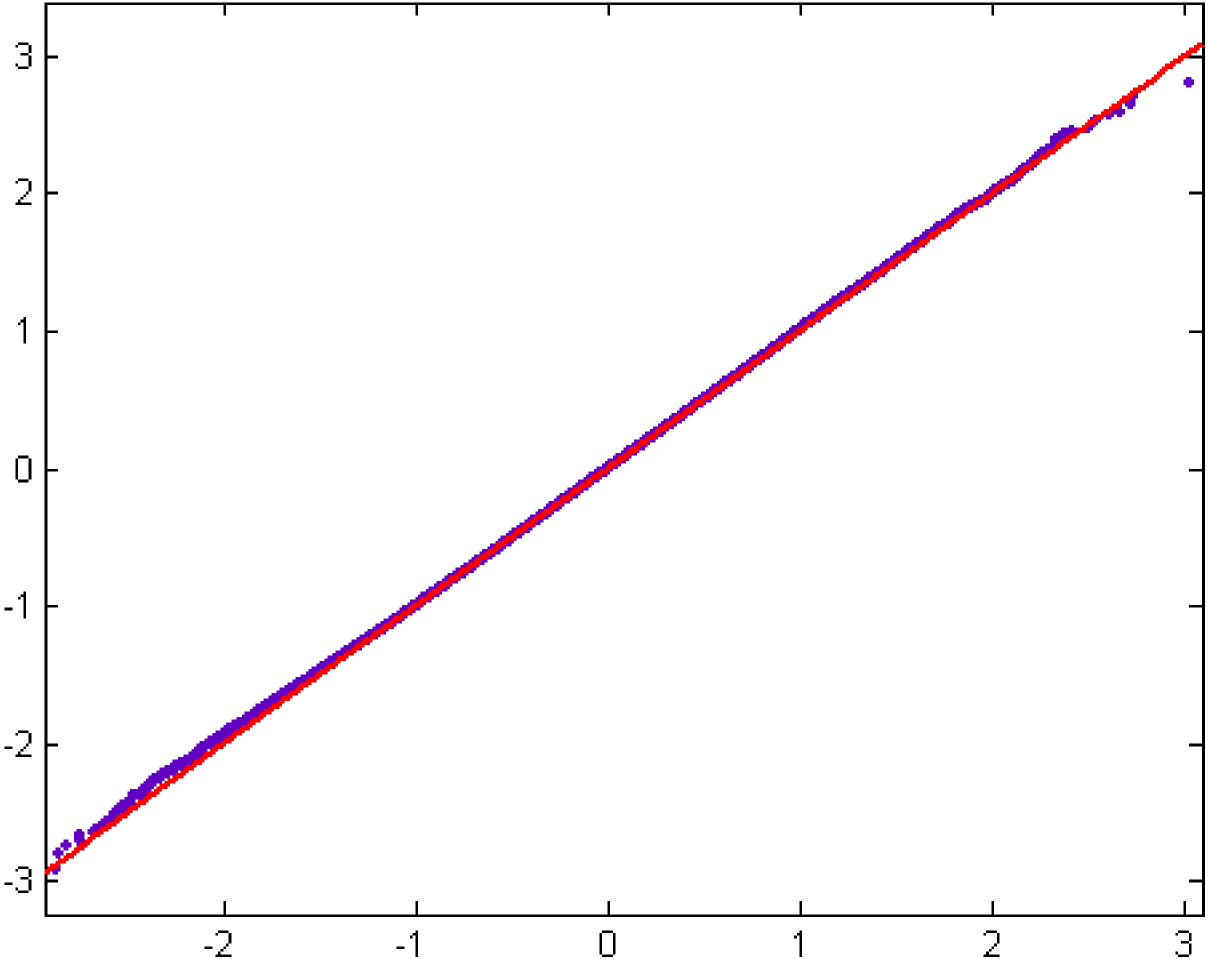


Figure 55 (n=pq, mean=-0.0551, standard deviation=0.6490, size=187276)



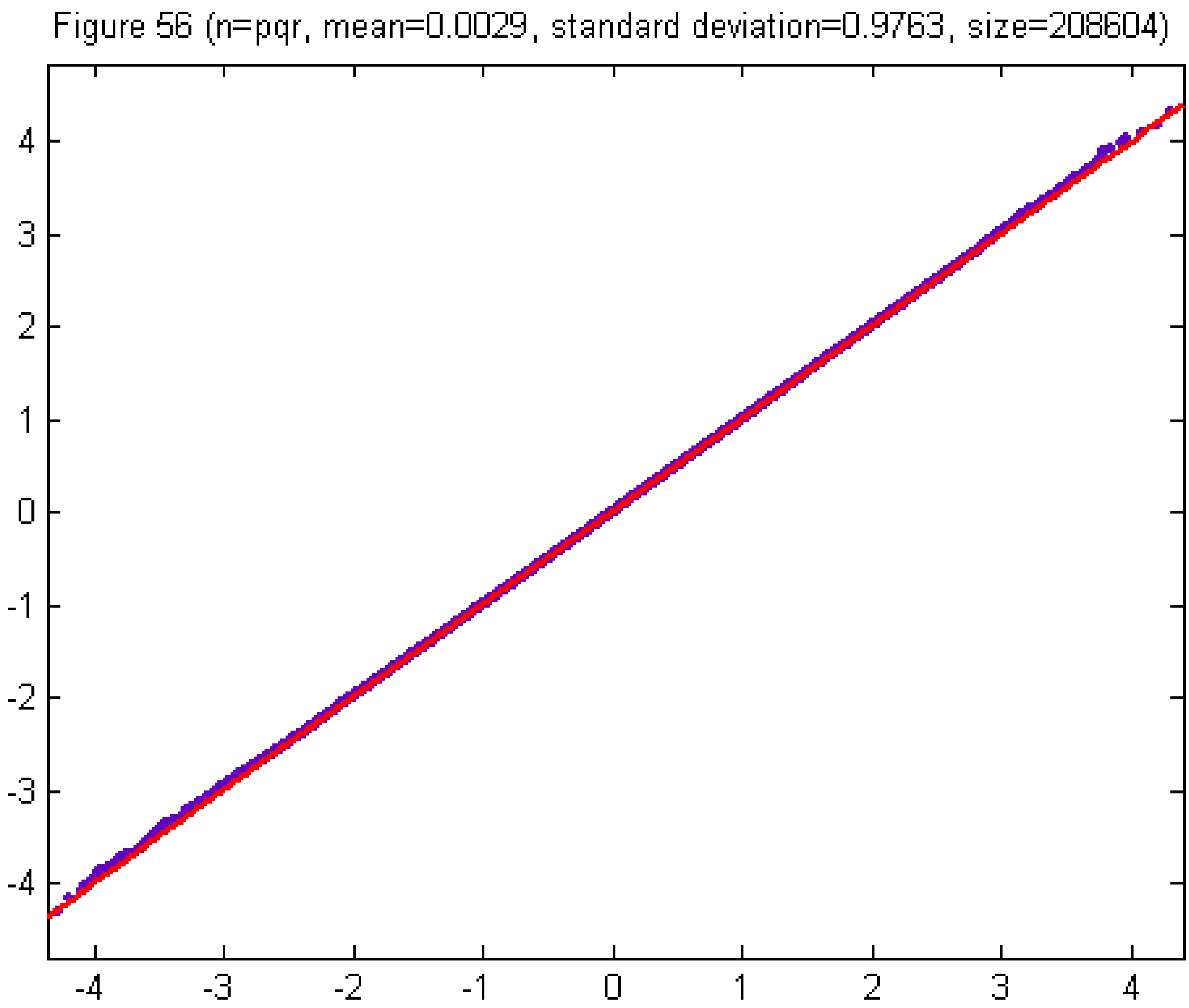


Figure 56 (n=pqr, mean=0.0029, standard deviation=0.9763, size=208604)
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4

Figure 57 (n=pqrs, mean=0.0730, standard deviation=1.4160, size=113555)

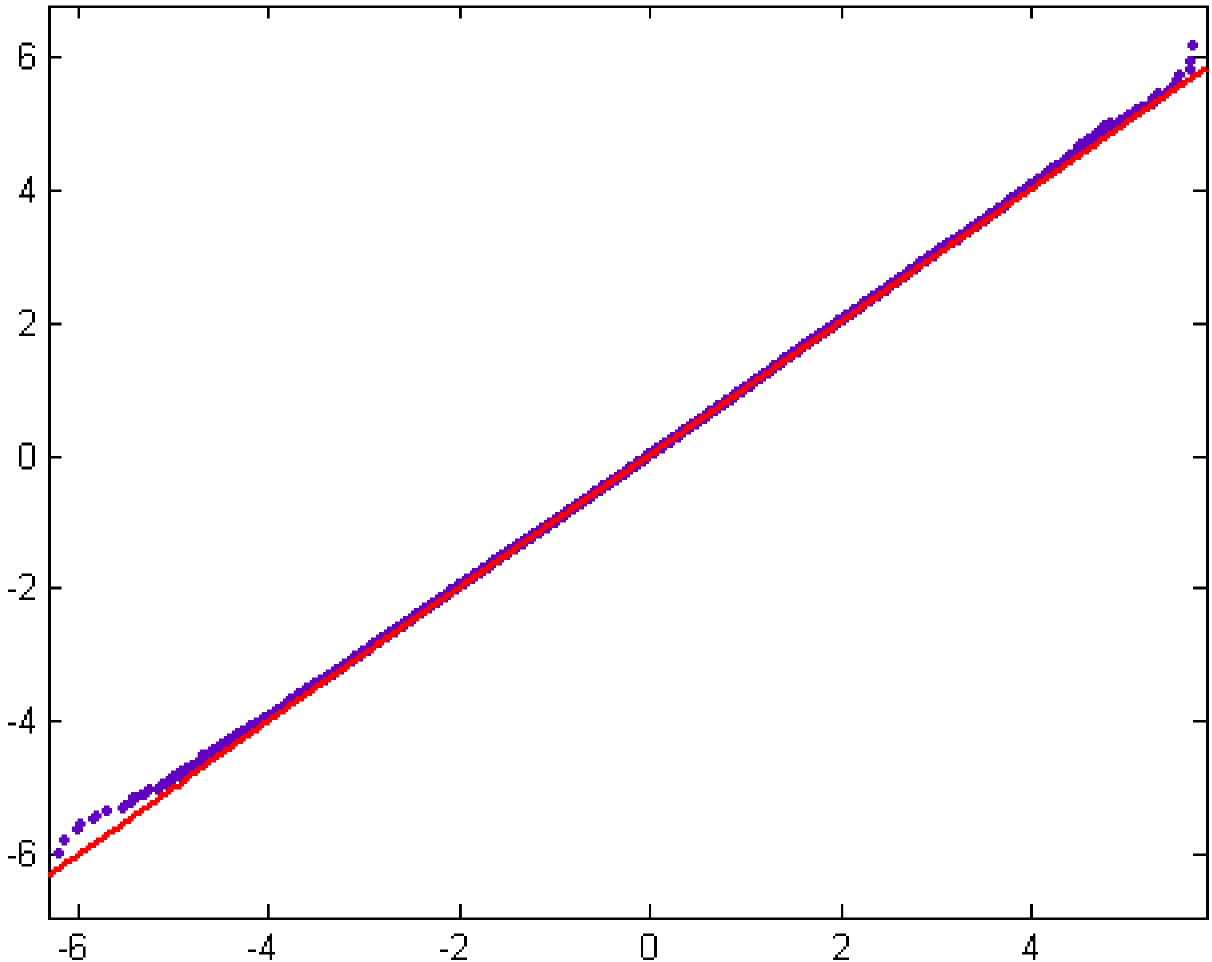
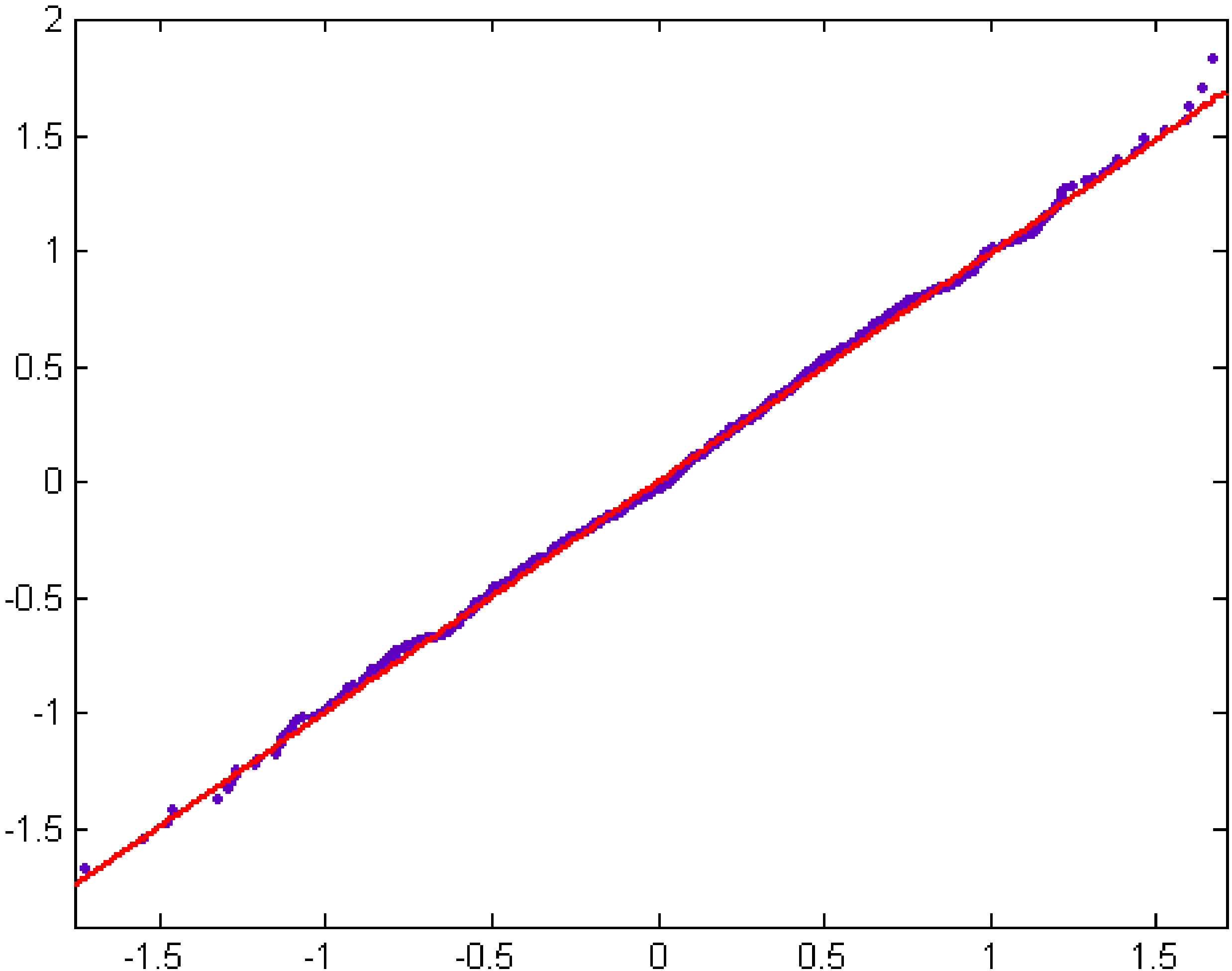


Figure 58 ($n=p^7q$, mean=0.0745, standard deviation=0.5806, size=795)



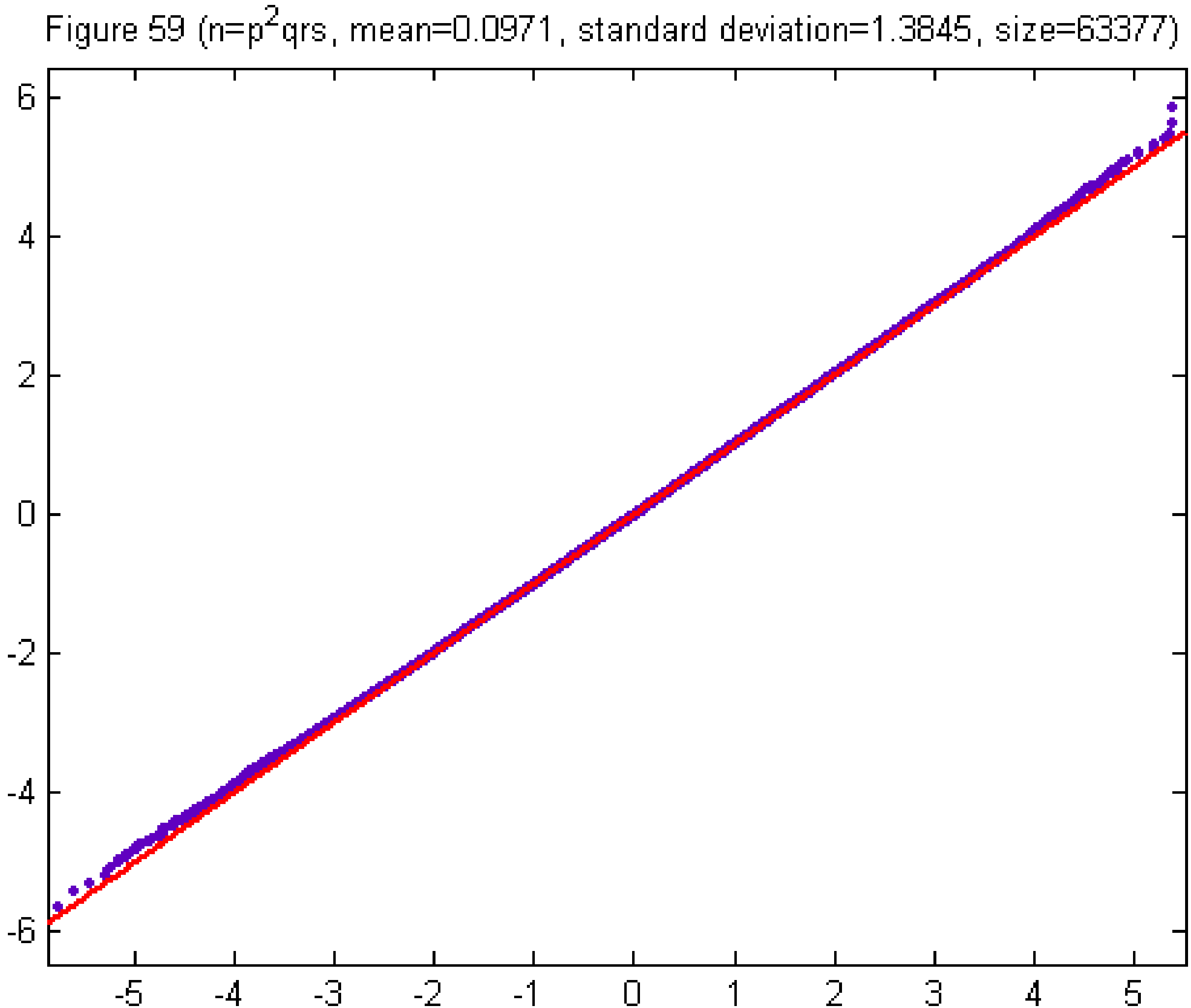


Figure 59 (n=p2qrs, mean=0.0971, standard deviation=1.3845, size=63377)
6
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

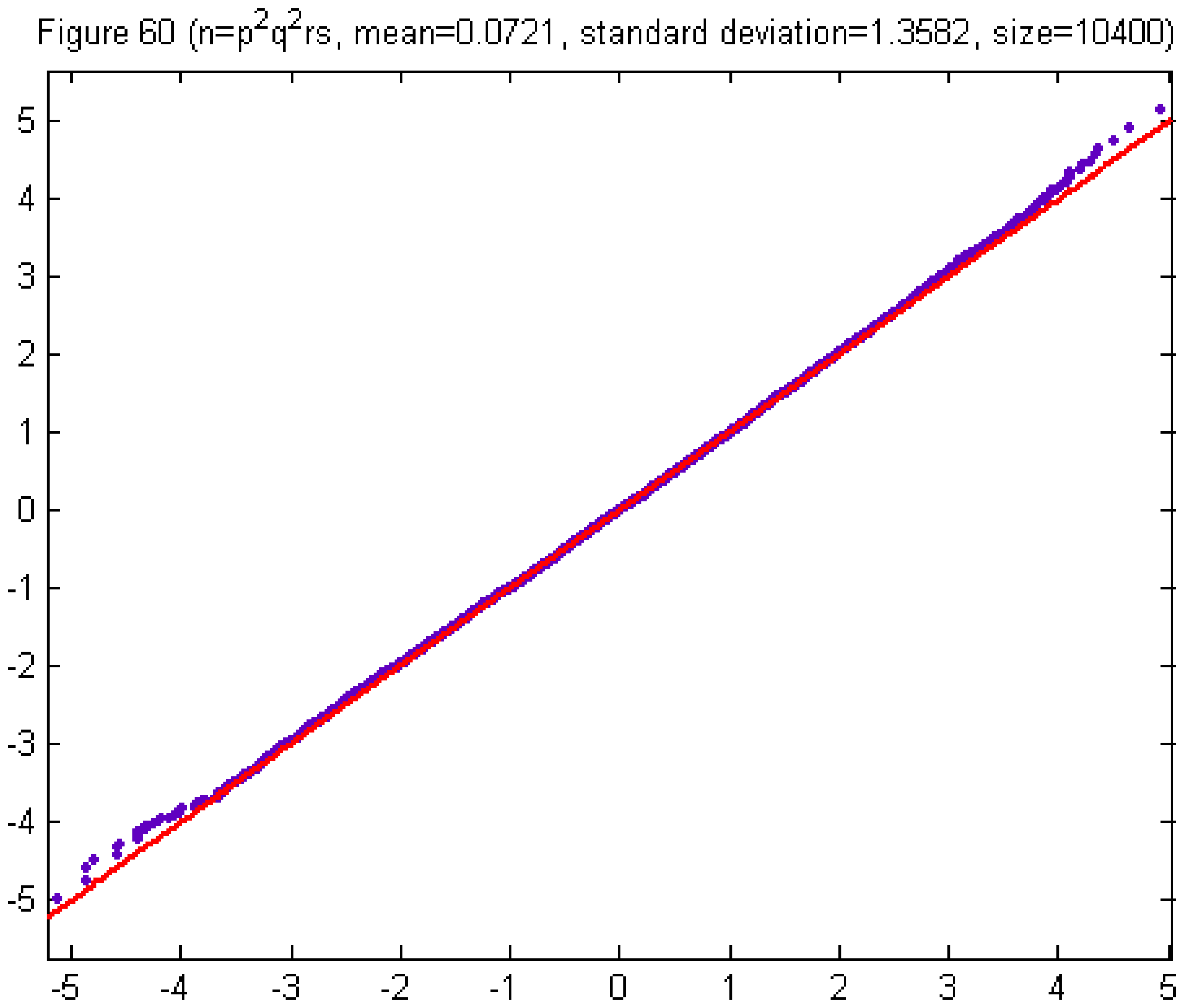
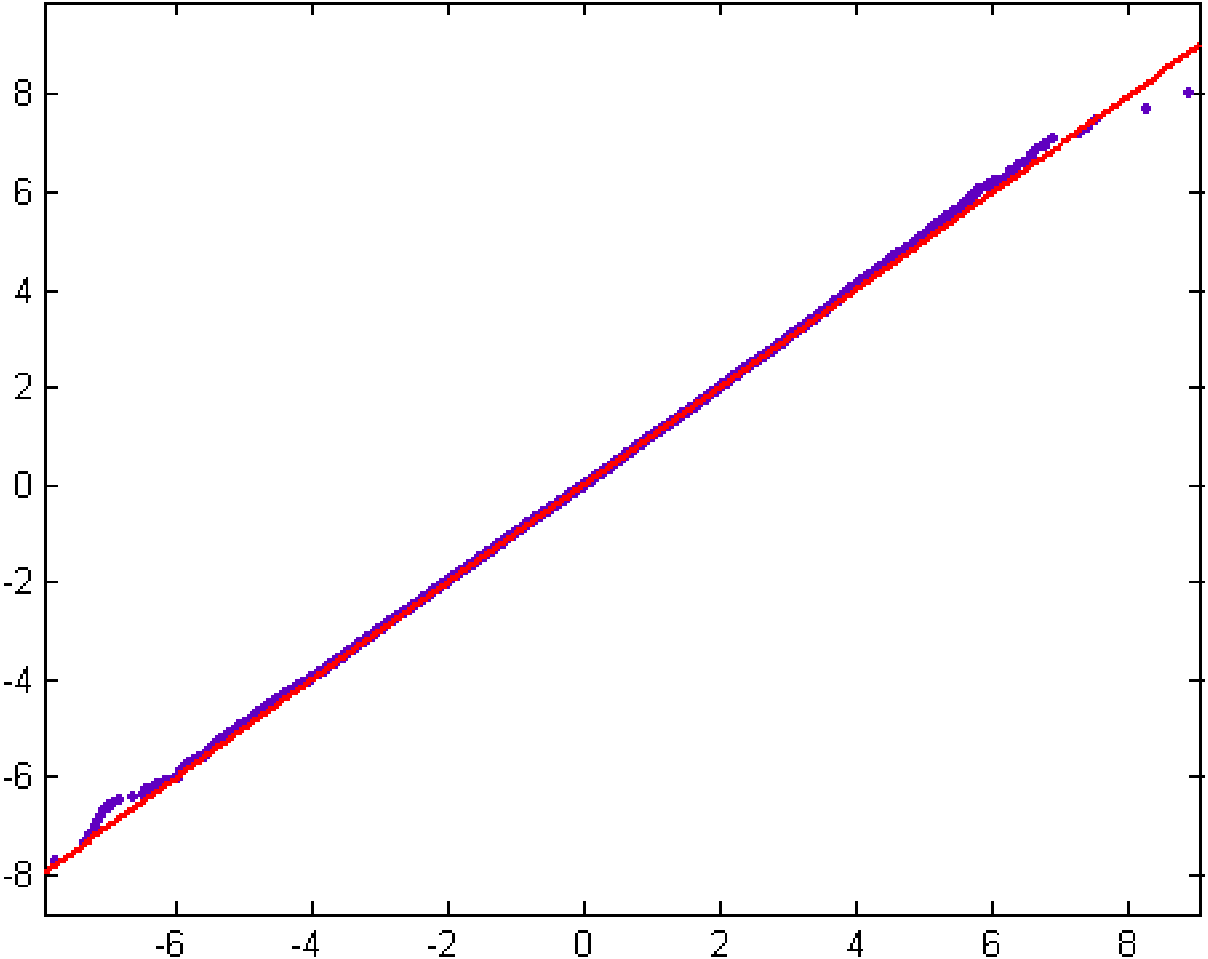


Figure 60 (n=p²q²rs, mean=0.0721, standard deviation=1.3582, size=10400)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 61 ($n=p^2qrst$, mean=0.1505, standard deviation=2.0132, size=21108)



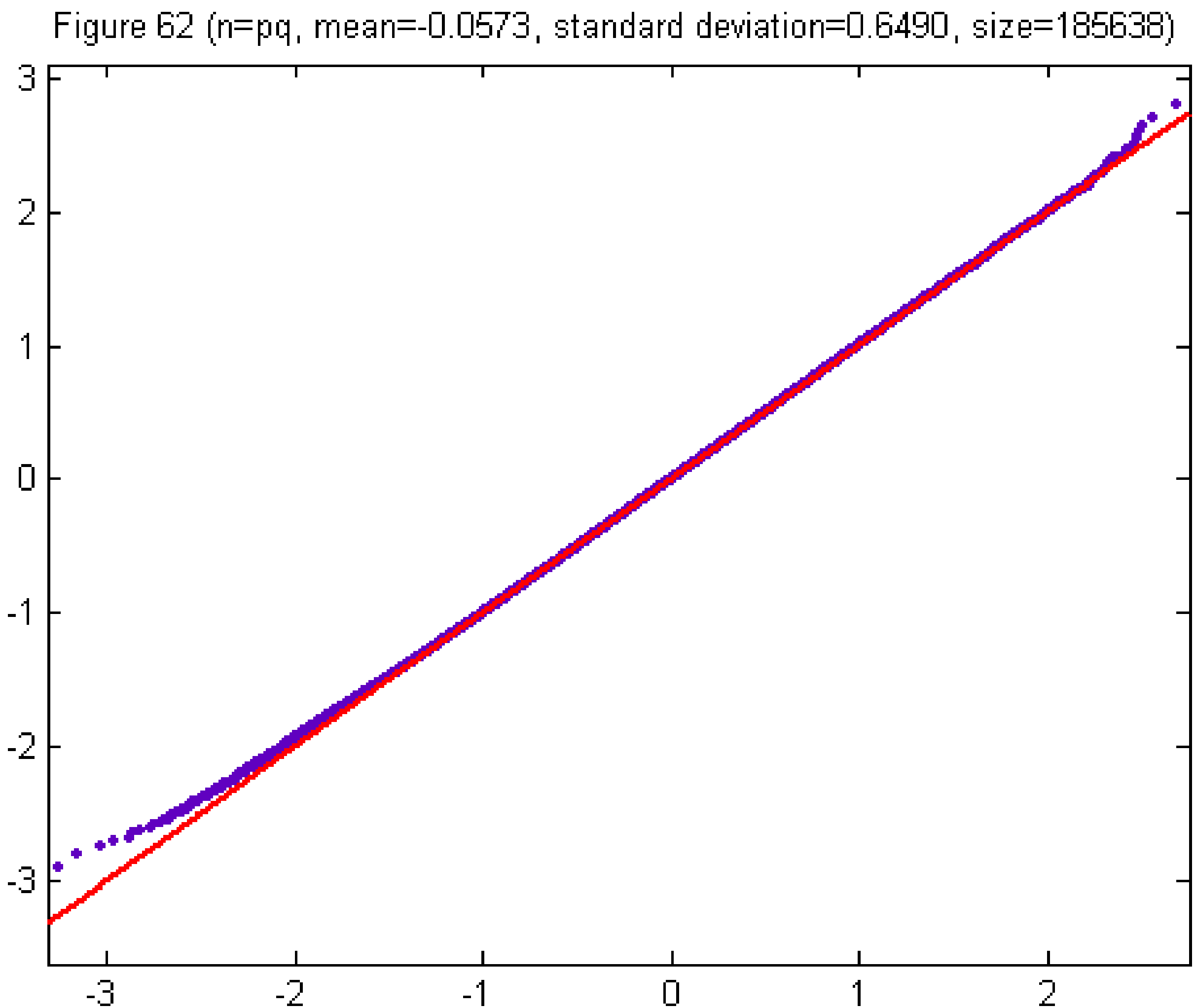


Figure 62 (n=pq, mean=-0.0573, standard deviation=0.6490, size=185638)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2

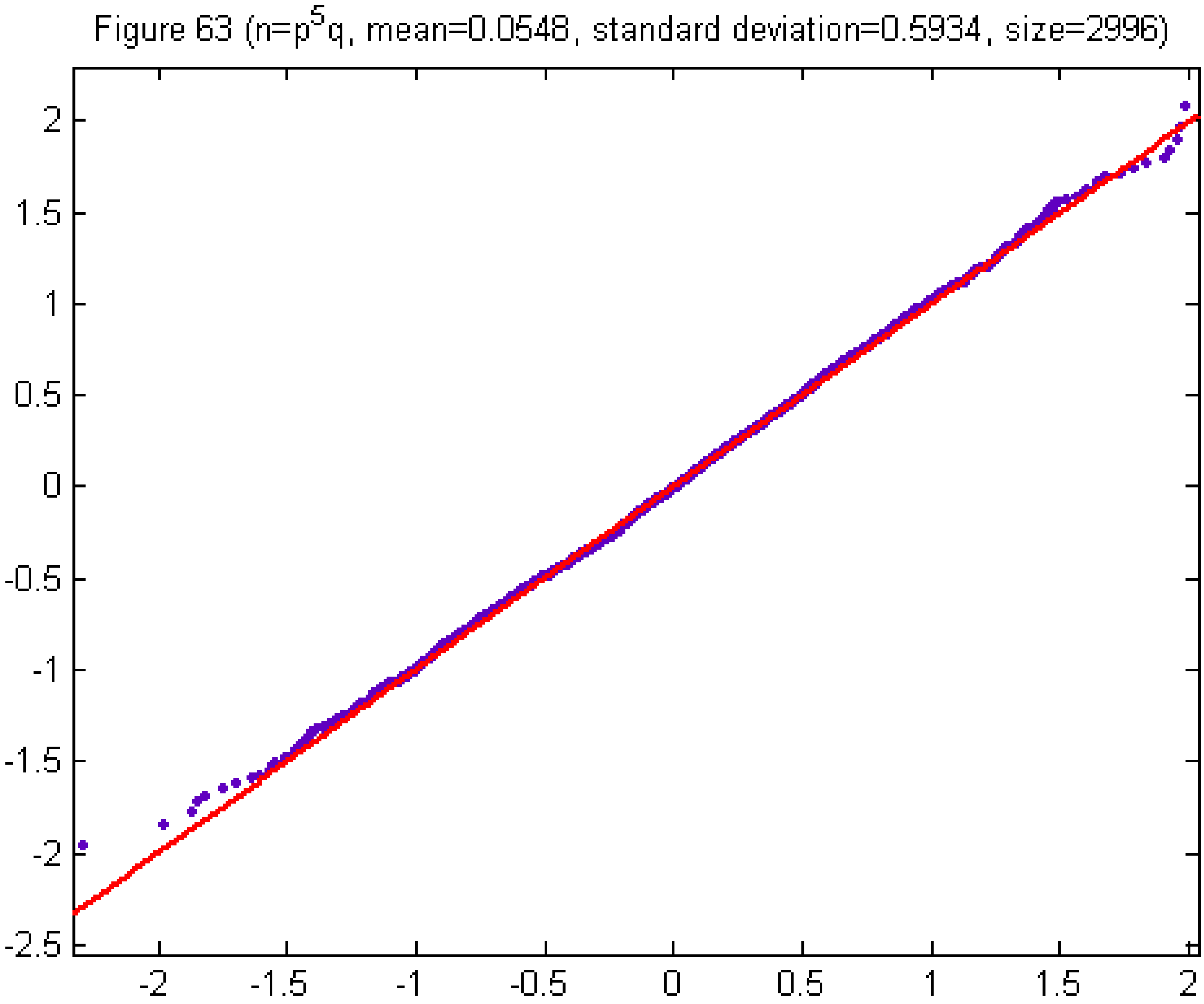


Figure 63 (n=p^5q, mean=0.0548, standard deviation=0.5934, size=2996)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Figure 64 (n=pqrs, mean=0.0729, standard deviation=1.4175, size=114326)

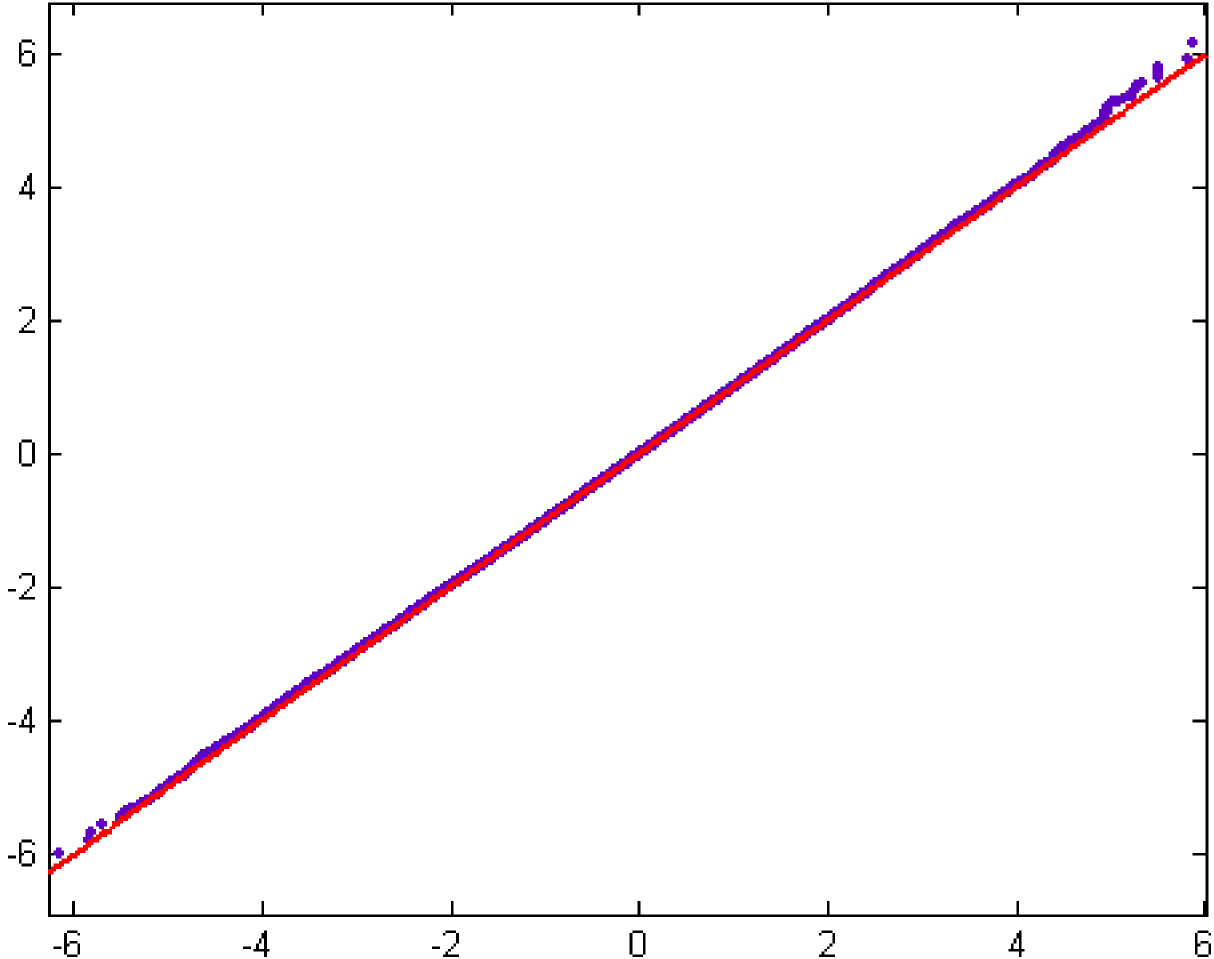


Figure 65 ($n=p^2qrs$, mean=0.0902, standard deviation=1.3952, size=63670)

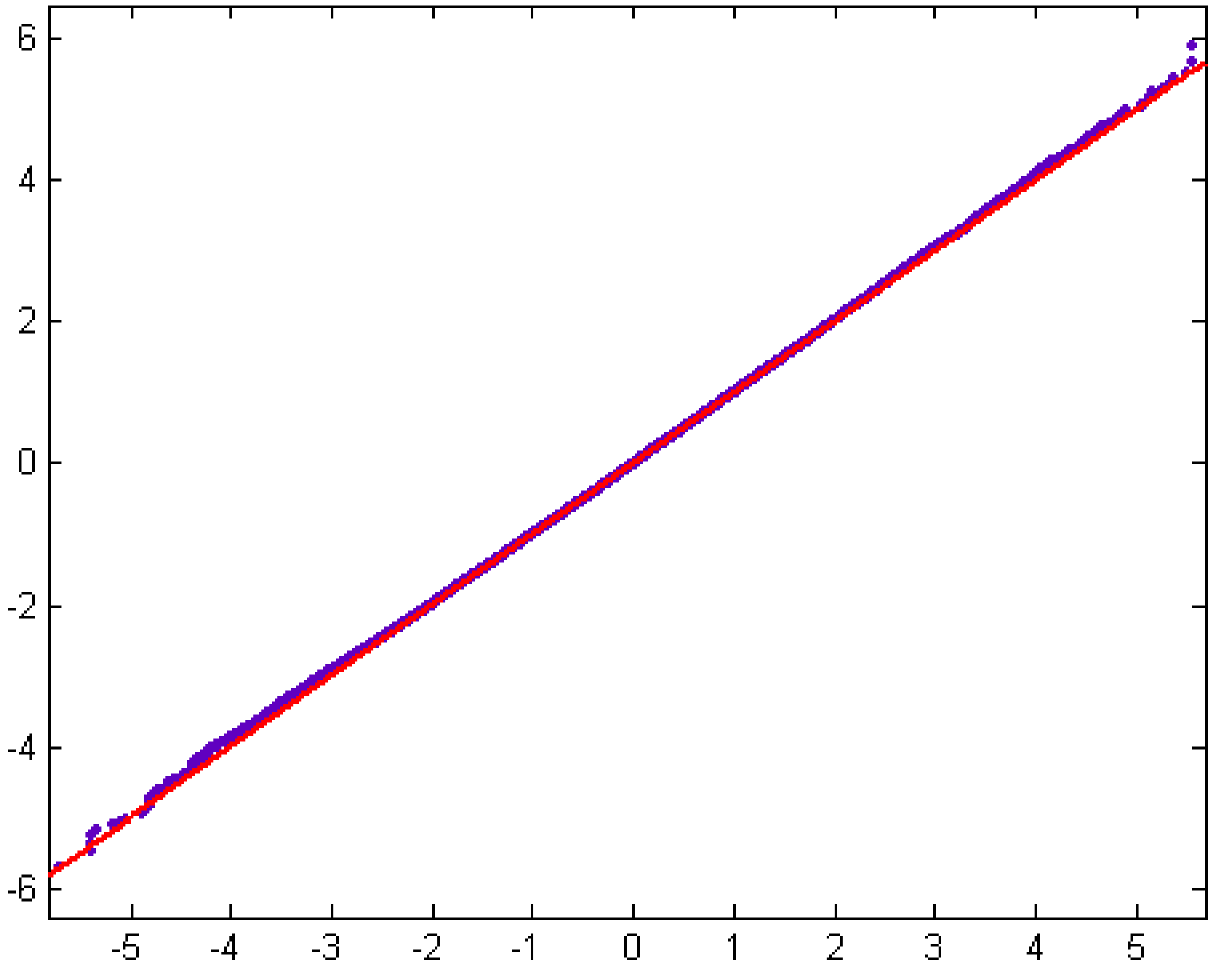


Figure 66 ($n=p^{11}q$, mean=0.0708, standard deviation=0.5771, size=58)

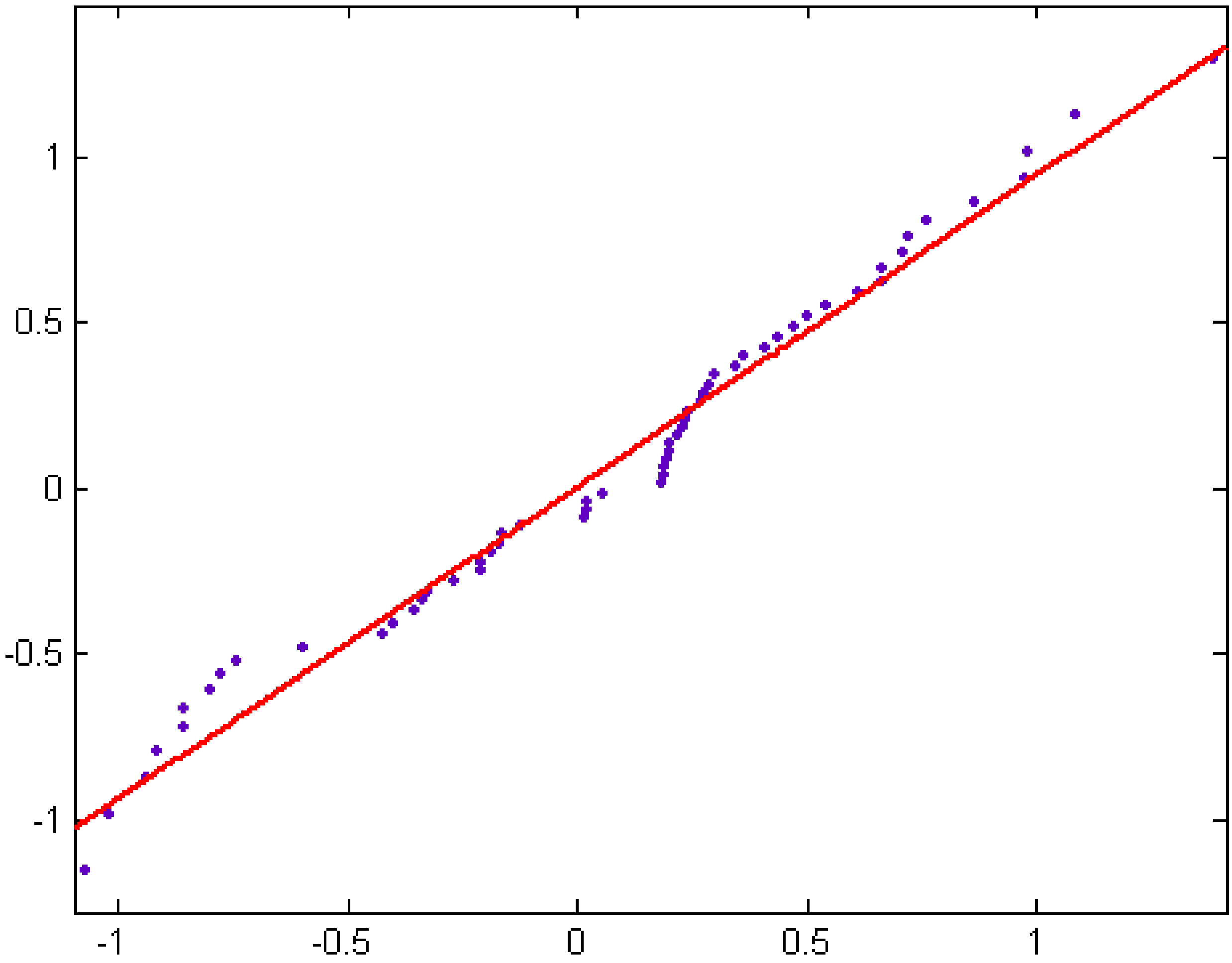
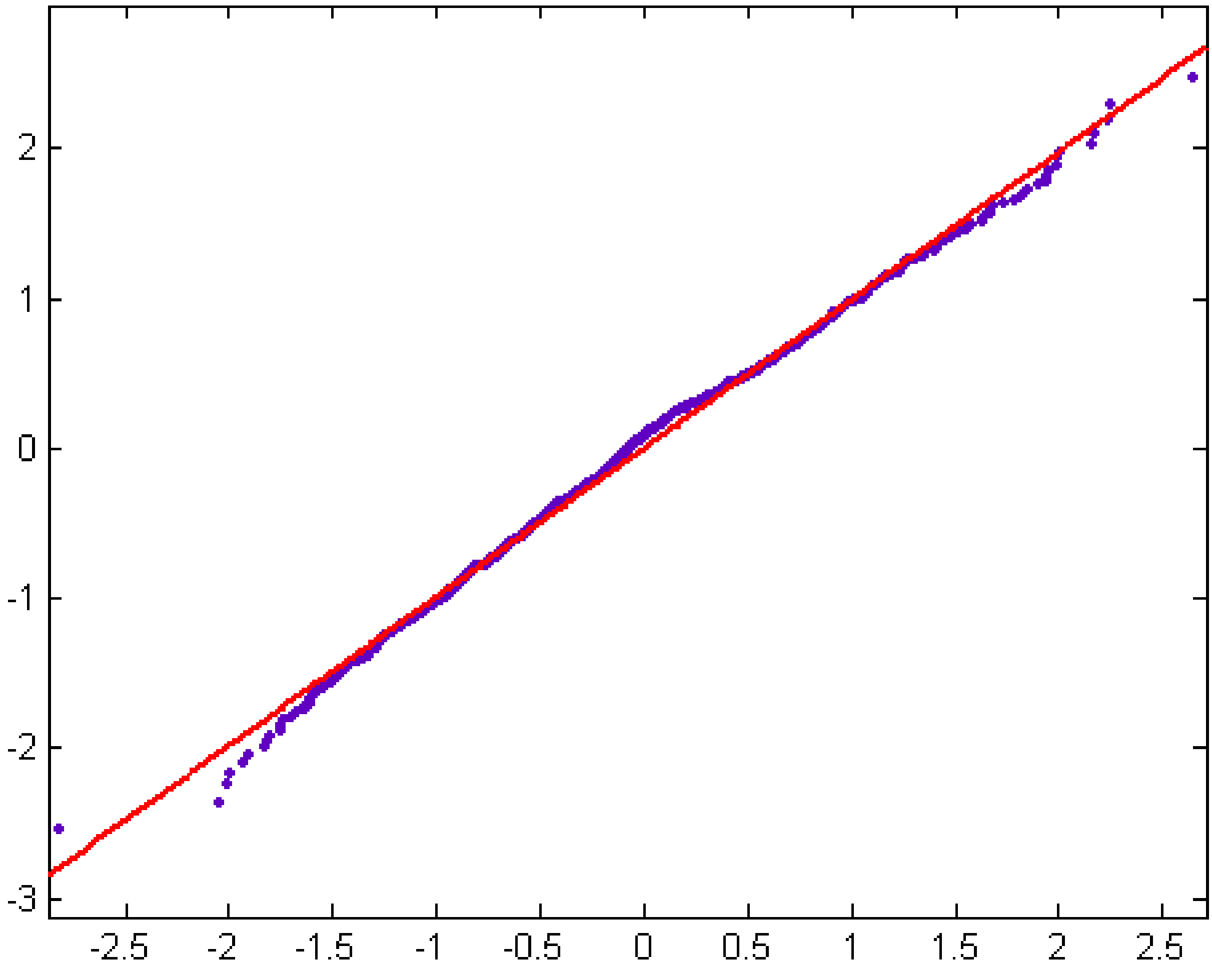


Figure 67 ($n=p^3q^3r$, mean=-0.0329, standard deviation=0.8391, size=725)



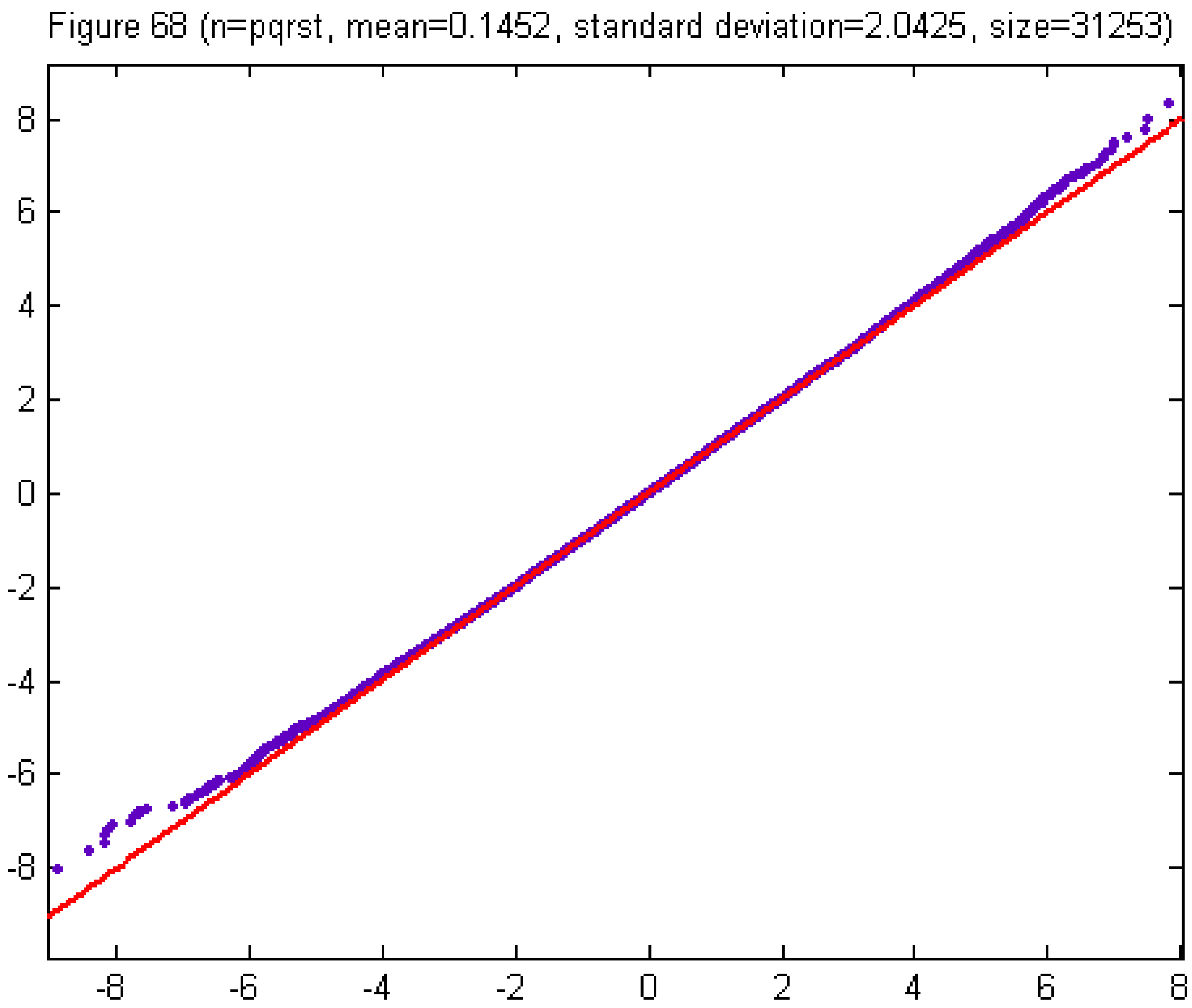


Figure 68 (n=pqrst, mean=0.1452, standard deviation=2.0425, size=31253)
8
6
4
2
0
-2
-4
-6
-8
-8
-6
-4
-2
0
2
4
6
8

Figure 69 ($n=p^3q^2r^2s$, mean=0.0623, standard deviation=1.2929, size=506)

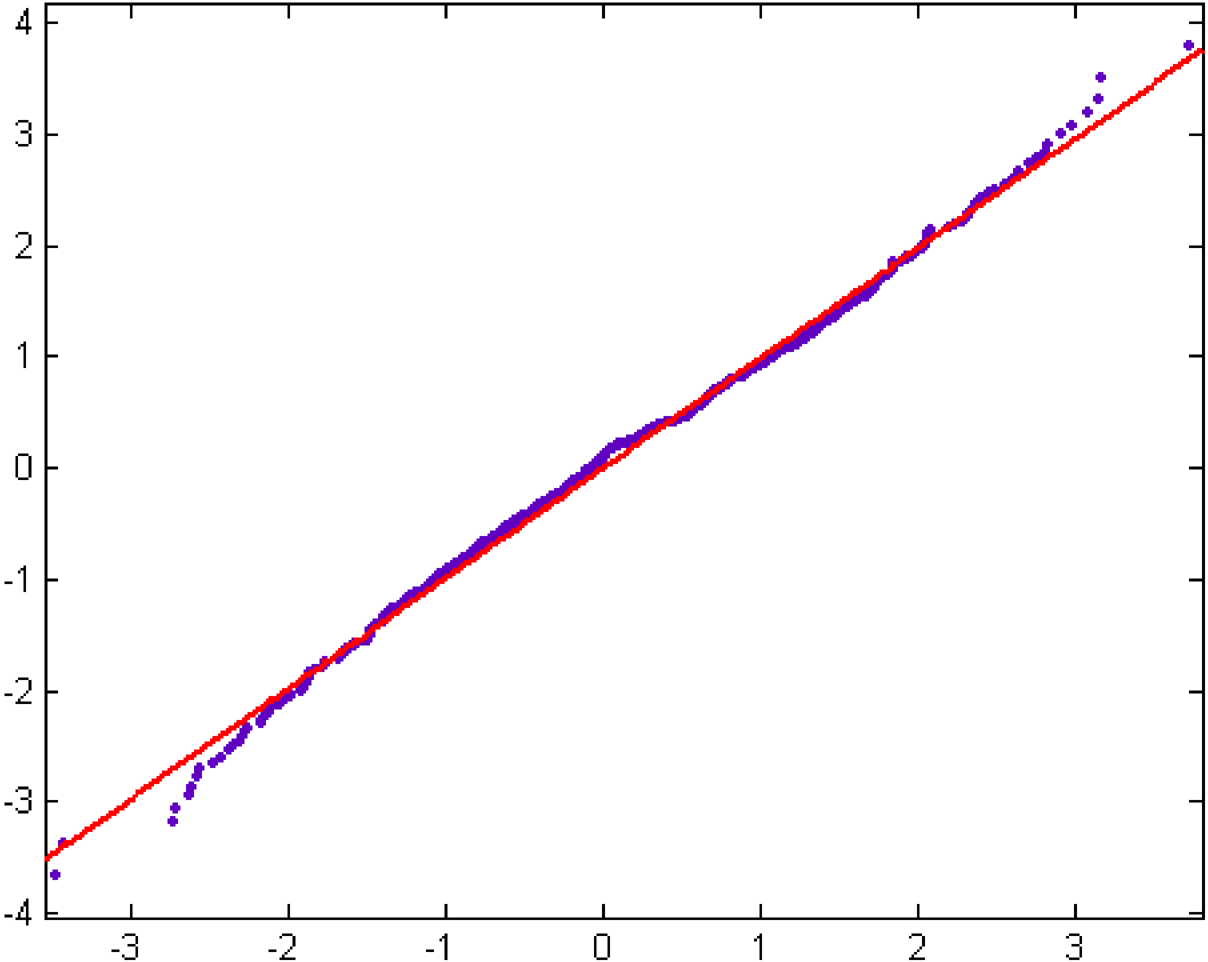


Figure 70 (n=pq, mean=-0.0554, standard deviation=0.6500, size=184760)

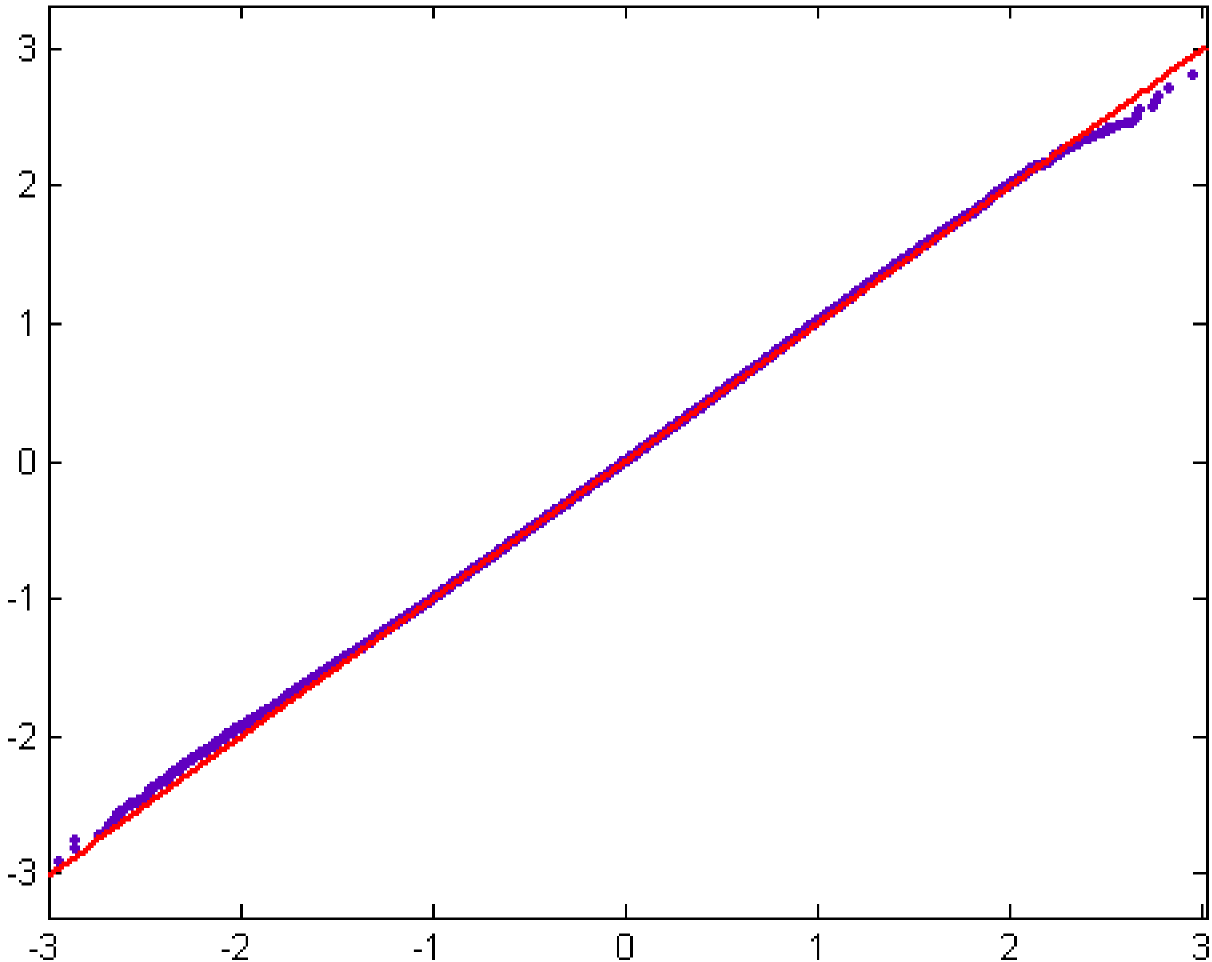
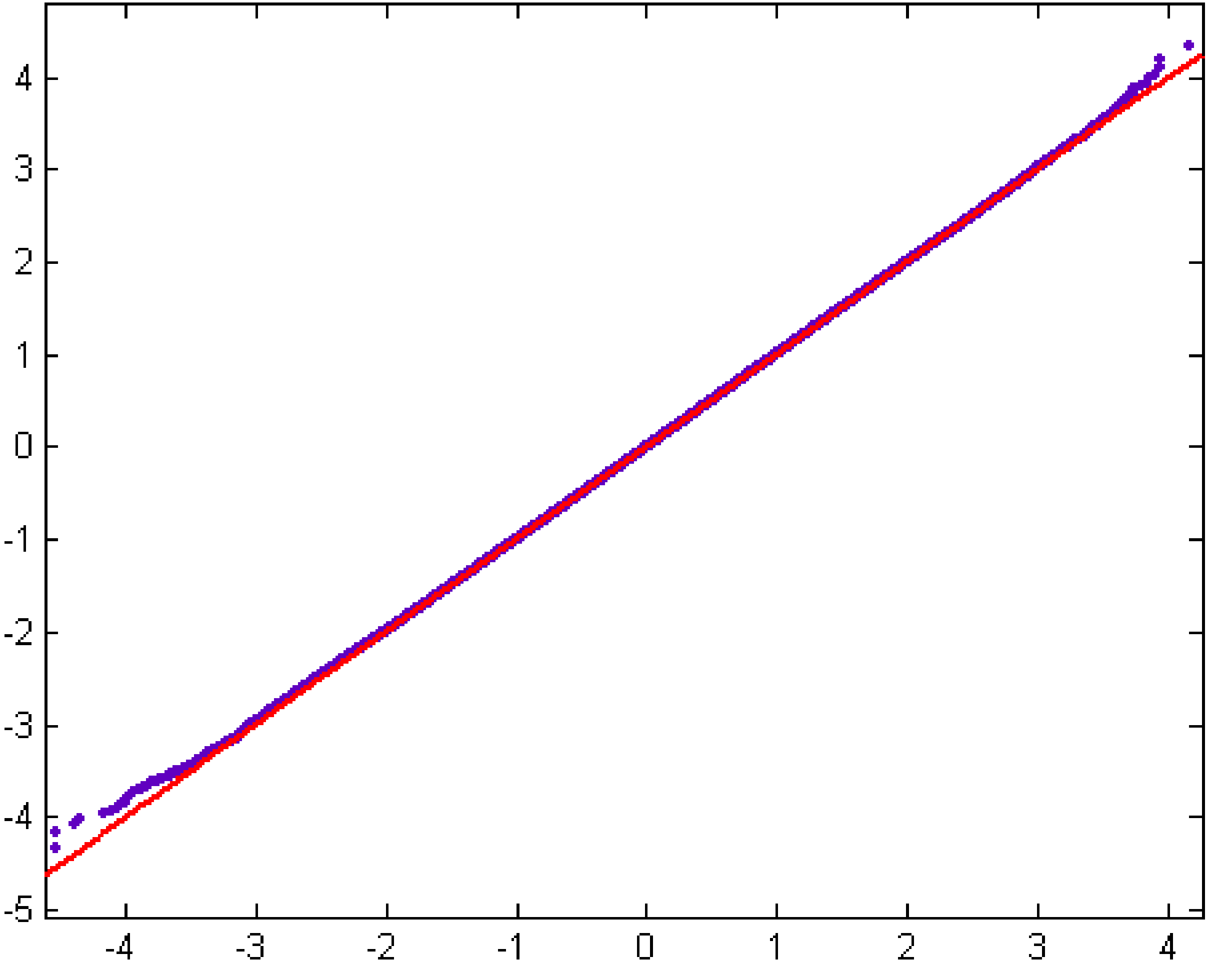


Figure 71 (n=pqr, mean=0.0031, standard deviation=0.9780, size=208784)



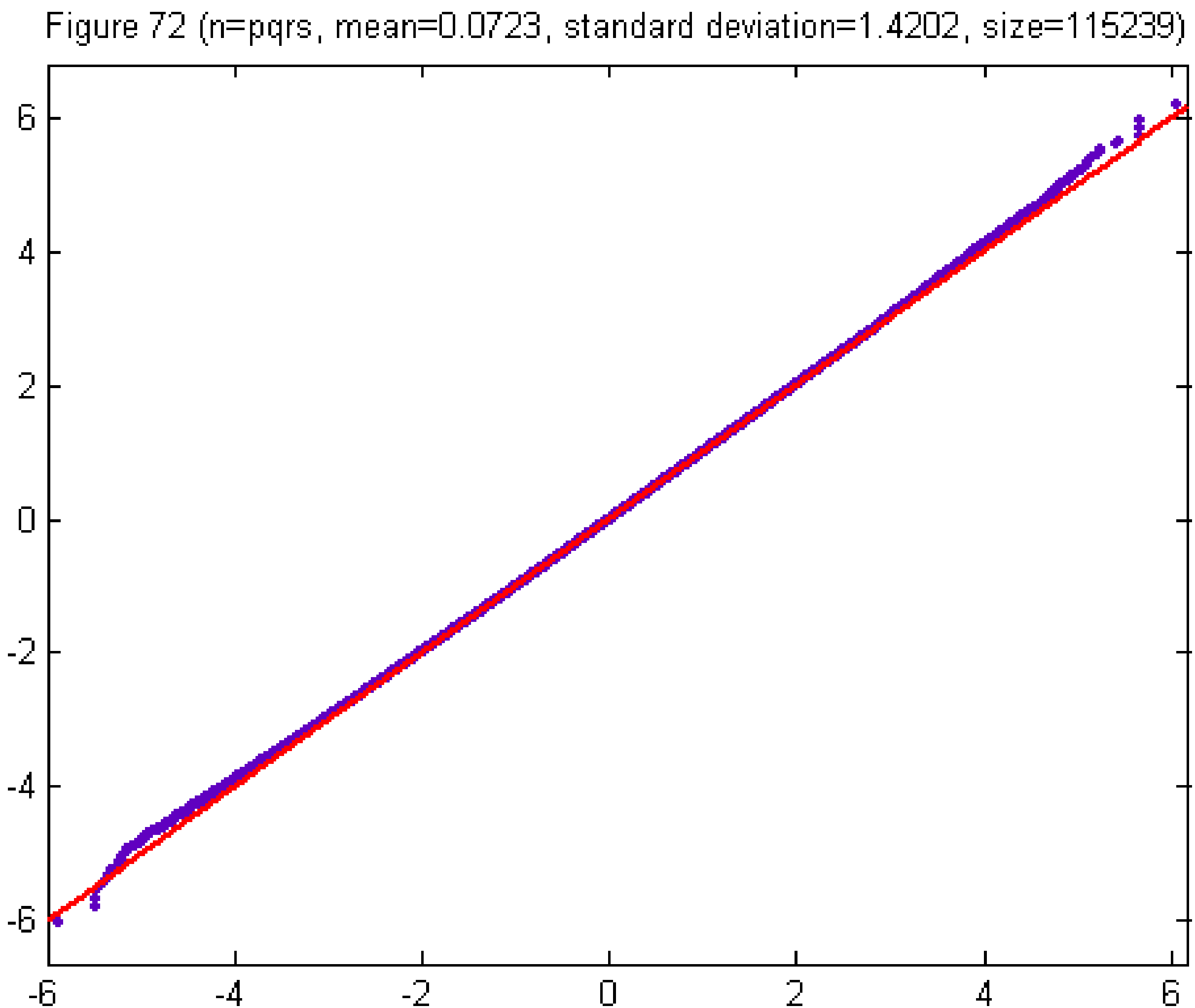


Figure 72 (n=pqrs, mean=0.0723, standard deviation=1.4202, size=115239)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

Figure 73 ($n=p^2qrs$, mean=0.0999, standard deviation=1.3869, size=64000)

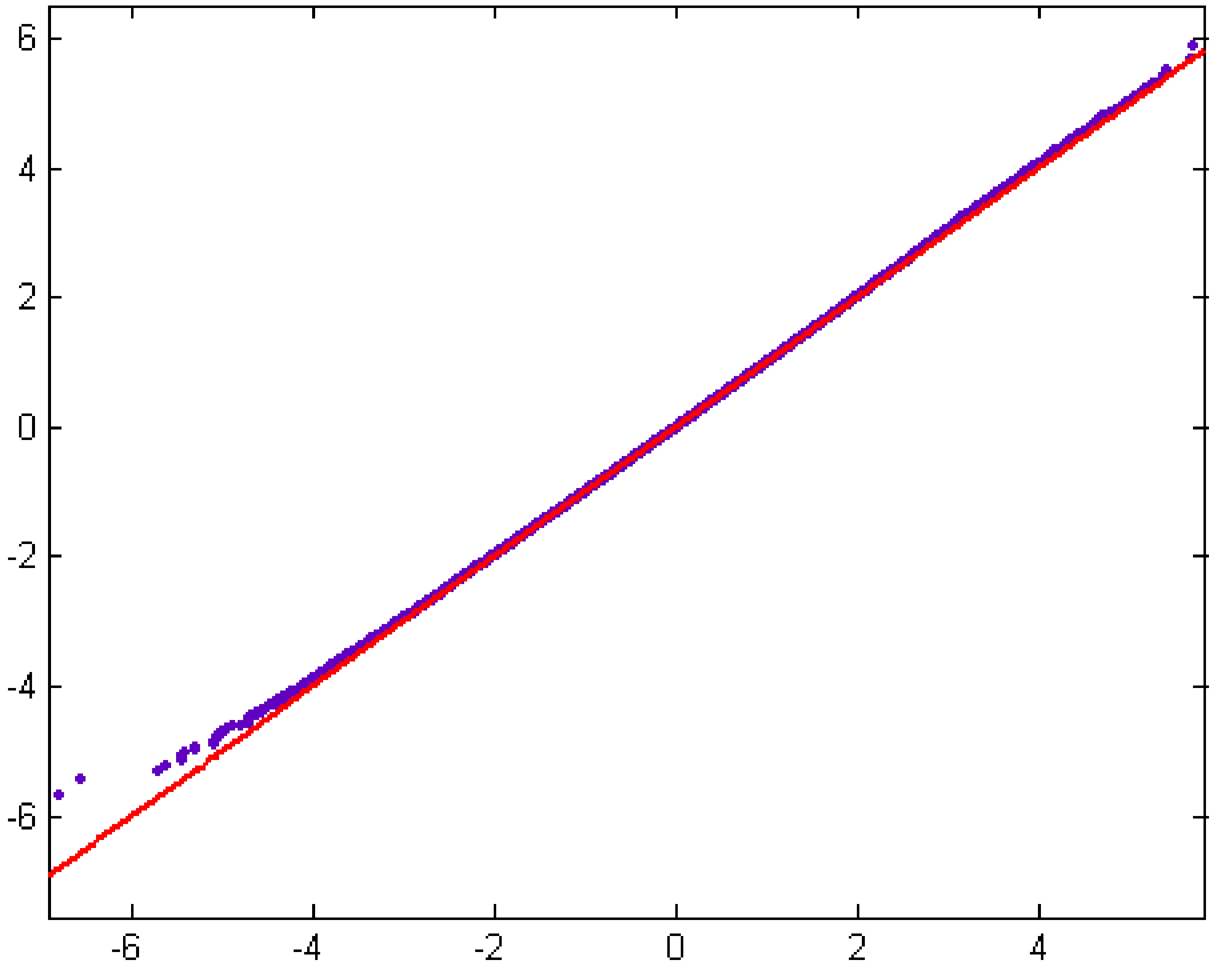
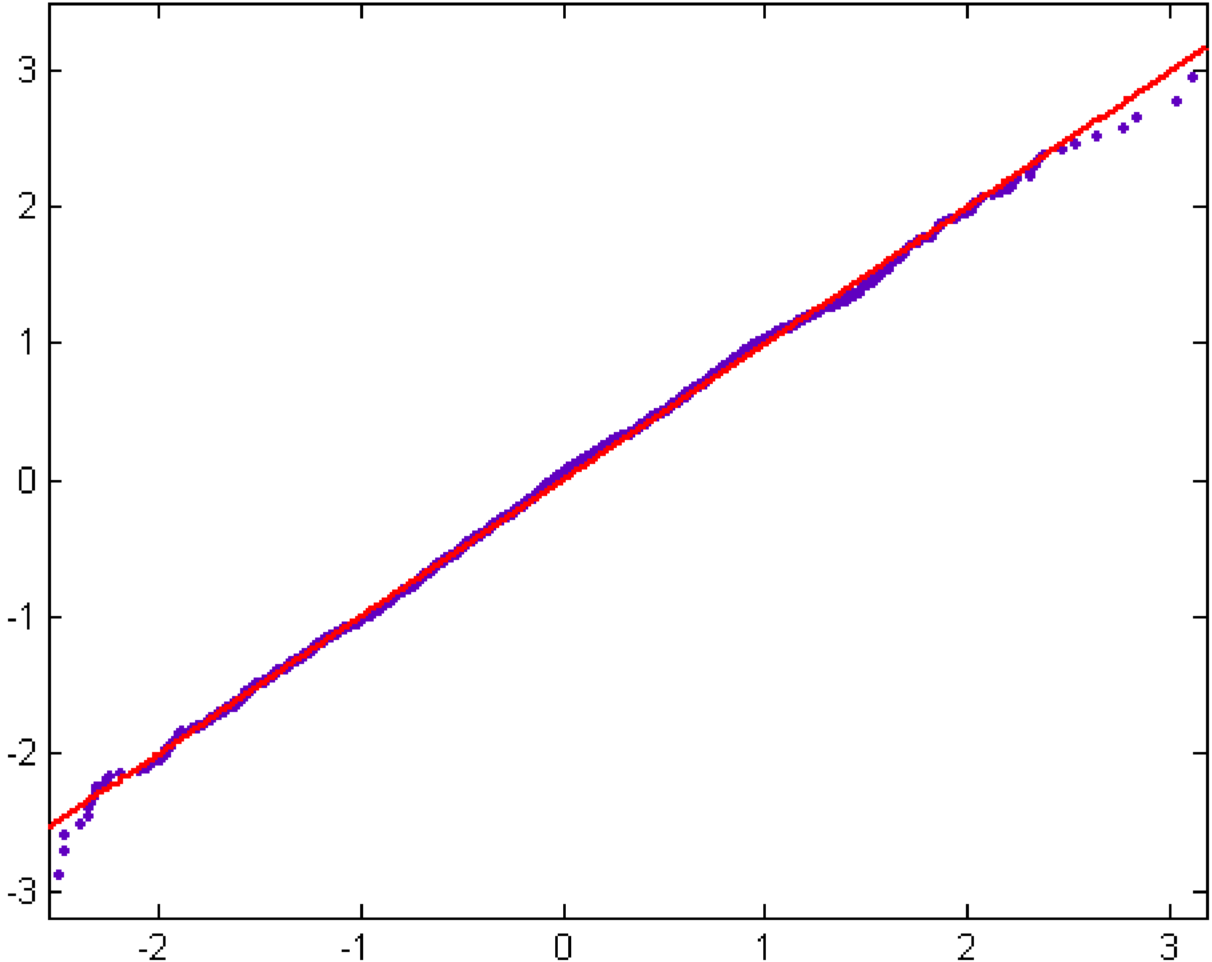


Figure 74 ($n=p^4q^2r$, mean=0.0228, standard deviation=0.8890, size=1893)



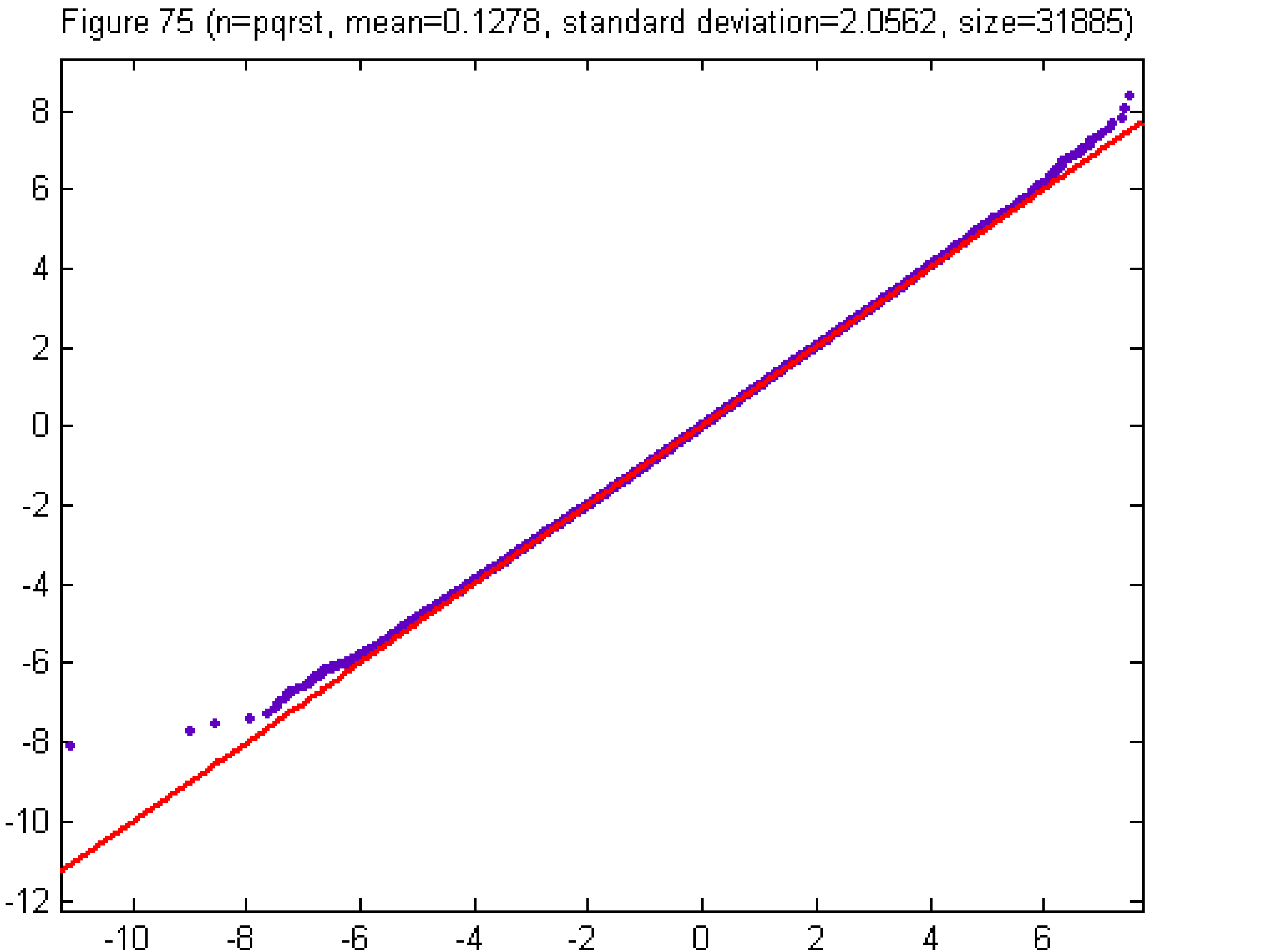


Figure 75 (n=pqrst, mean=0.1278, standard deviation=2.0562, size=31885)
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-10
-8
-6
-4
-2
0
2
4
6

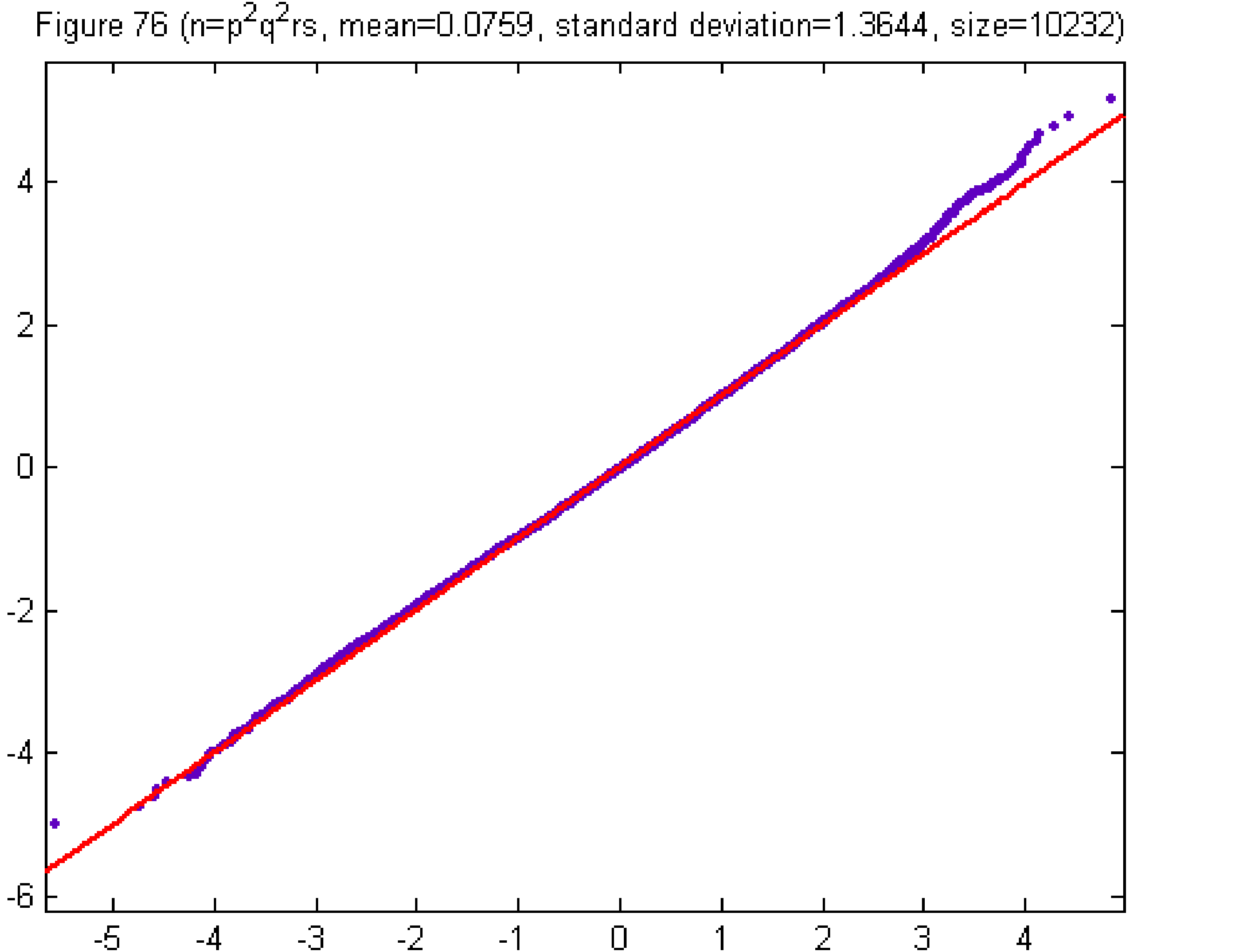
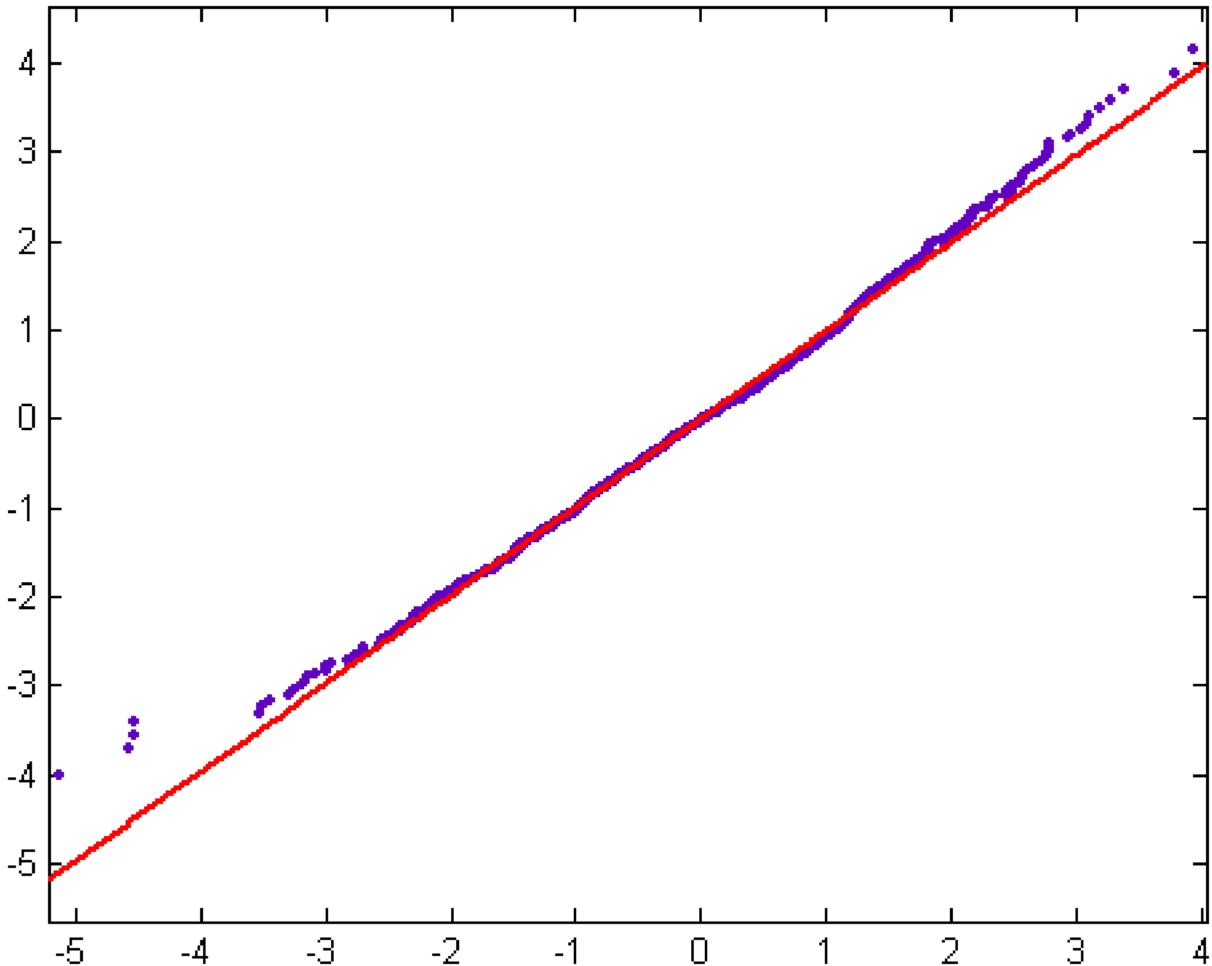


Figure 76 (n=p2q2rs, mean=0.0759, standard deviation=1.3644, size=10232)
4
2
0
-2
-4
-6
-5
-4
-3
-2
-1
0
1
2
3
4

Figure 77 ($n=p^3q^3rs$, mean=0.0784, standard deviation=1.3139, size=1034)



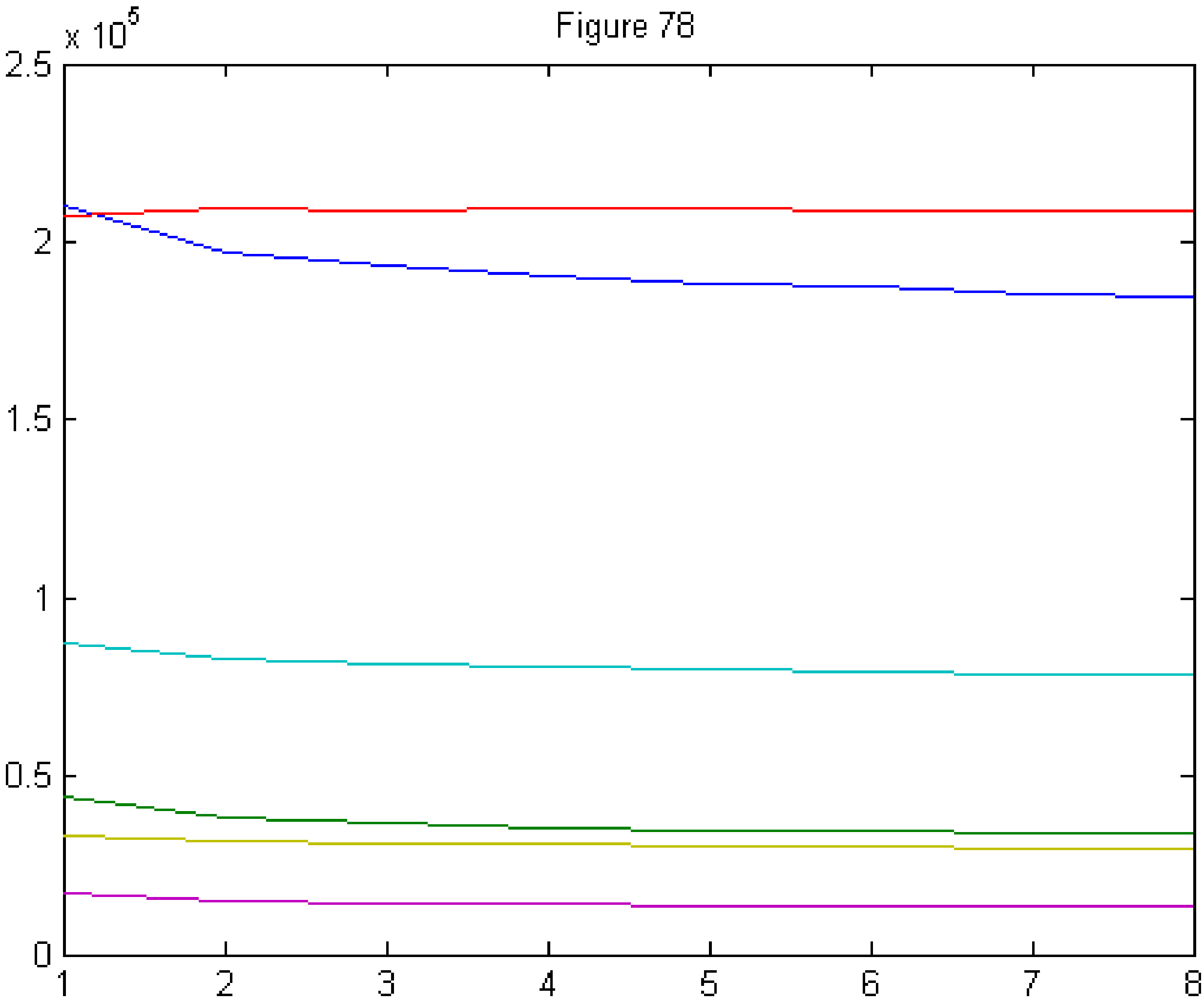


Figure 78
x 10^5
2.5
2
1.5
1
0.5
0
1
2
3
4
5
6
7
8

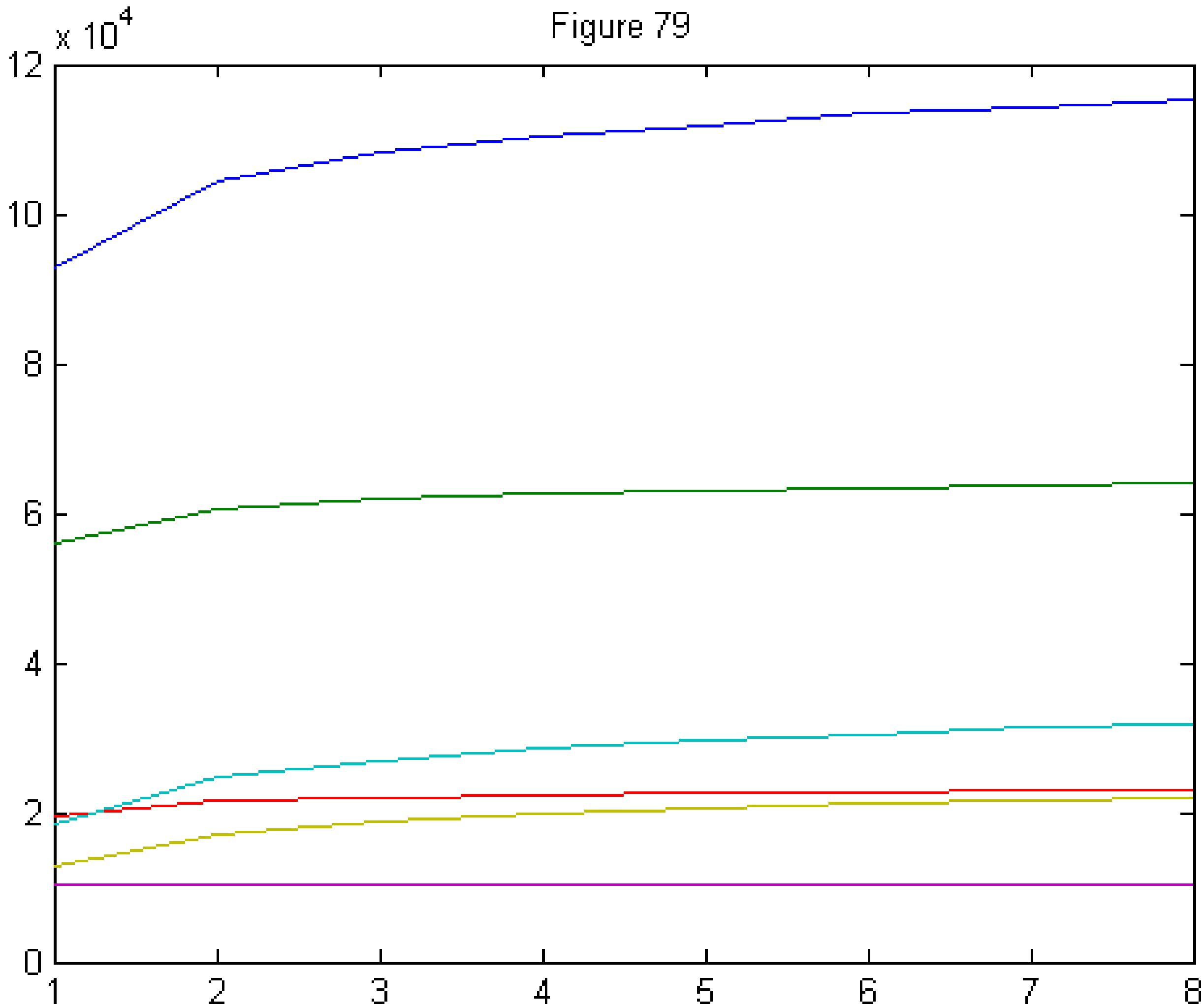


Figure 79
x 10^4
12
10
8
6
4
2
0
1
2
3
4
5
6
7
8

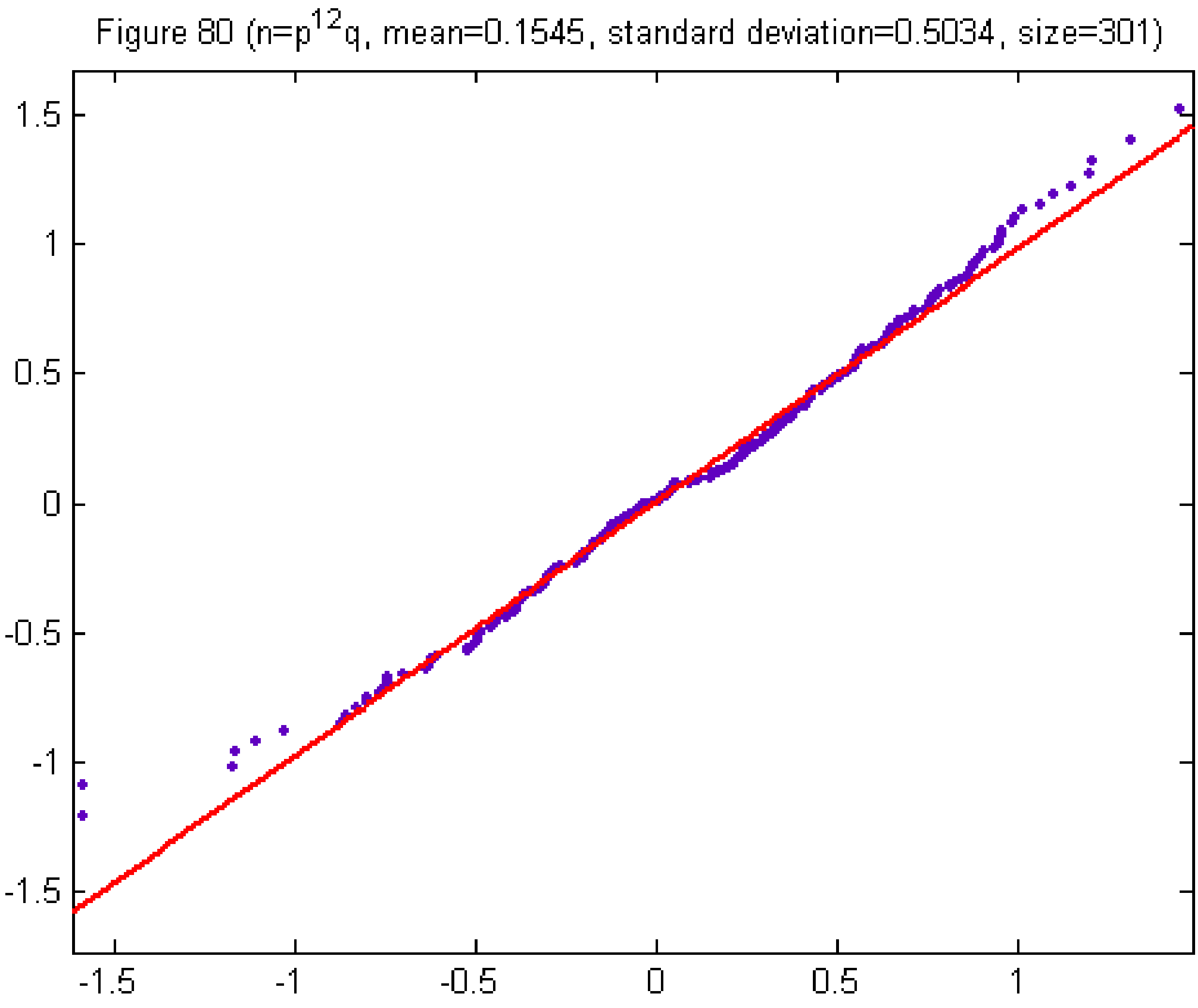


Figure 80 (n=p^12q, mean=0.1545, standard deviation=0.5034, size=301)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1

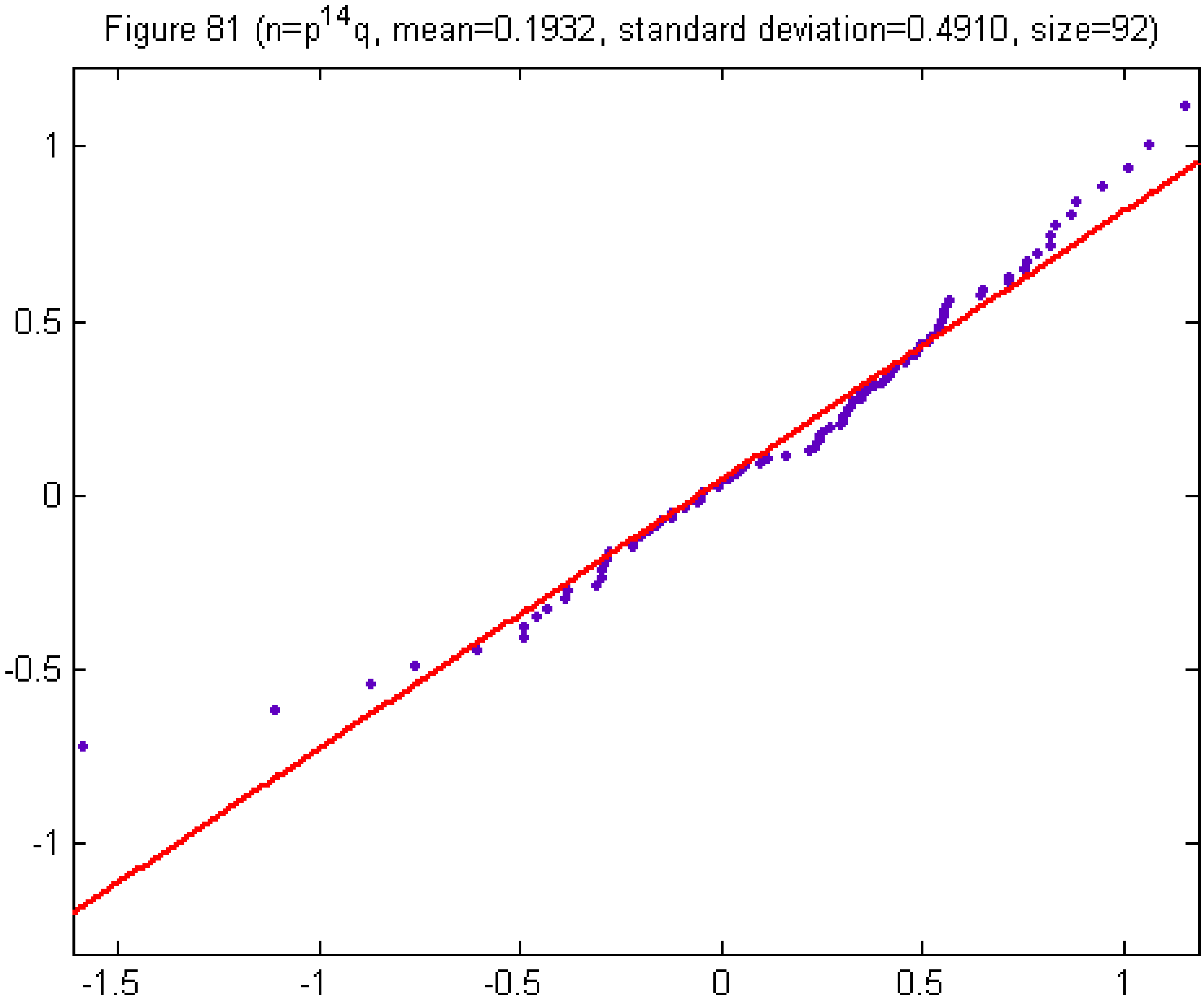
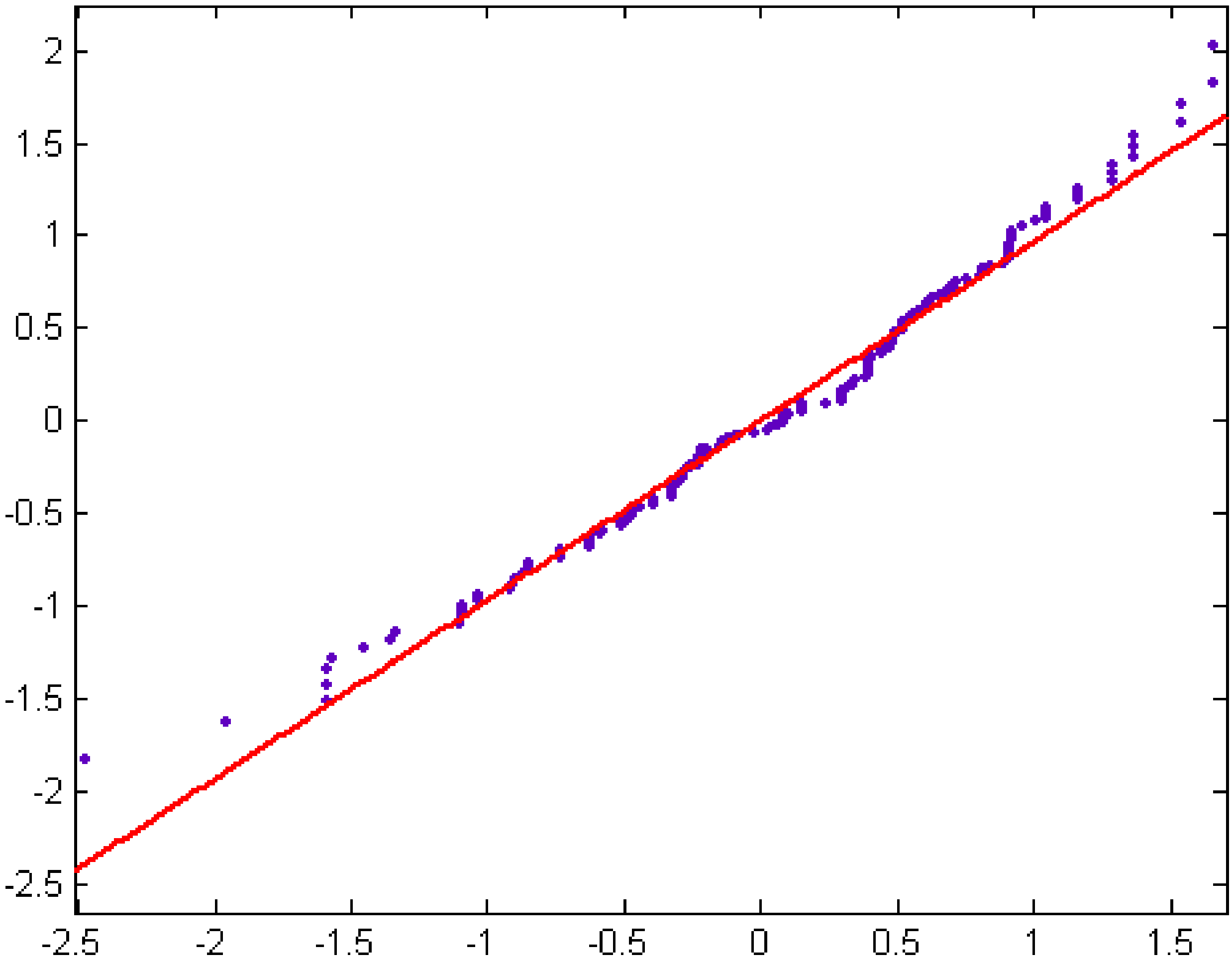


Figure 81 ($n=p^{14}q$, mean=0.1932, standard deviation=0.4910, size=92)
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1

Figure 82 ($n=p^3q^3r^2$, mean=0.0955, standard deviation=0.7664, size=167)



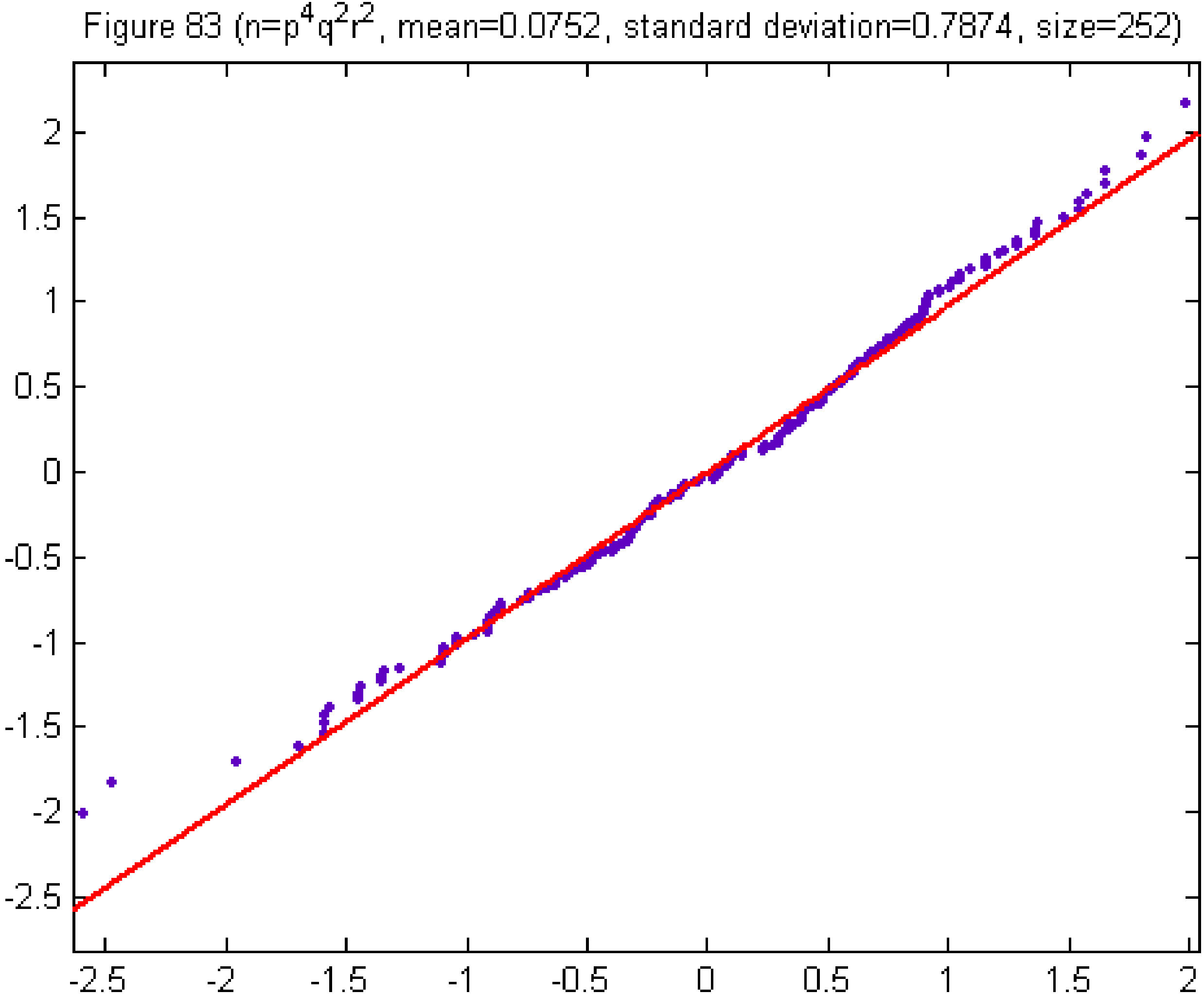


Figure 83 ($n=p^4q^2r^2$, mean=0.0752, standard deviation=0.7874, size=252)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

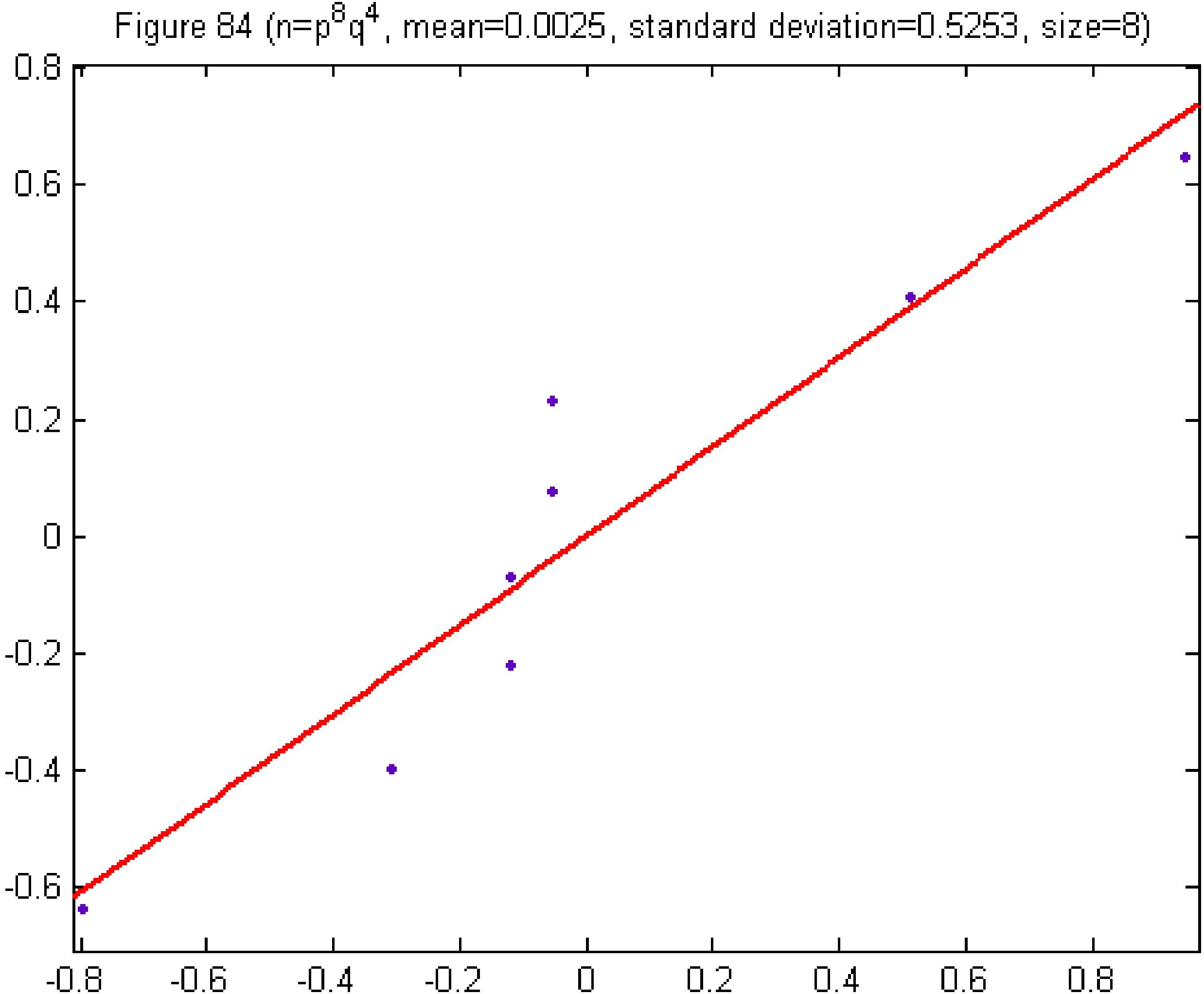
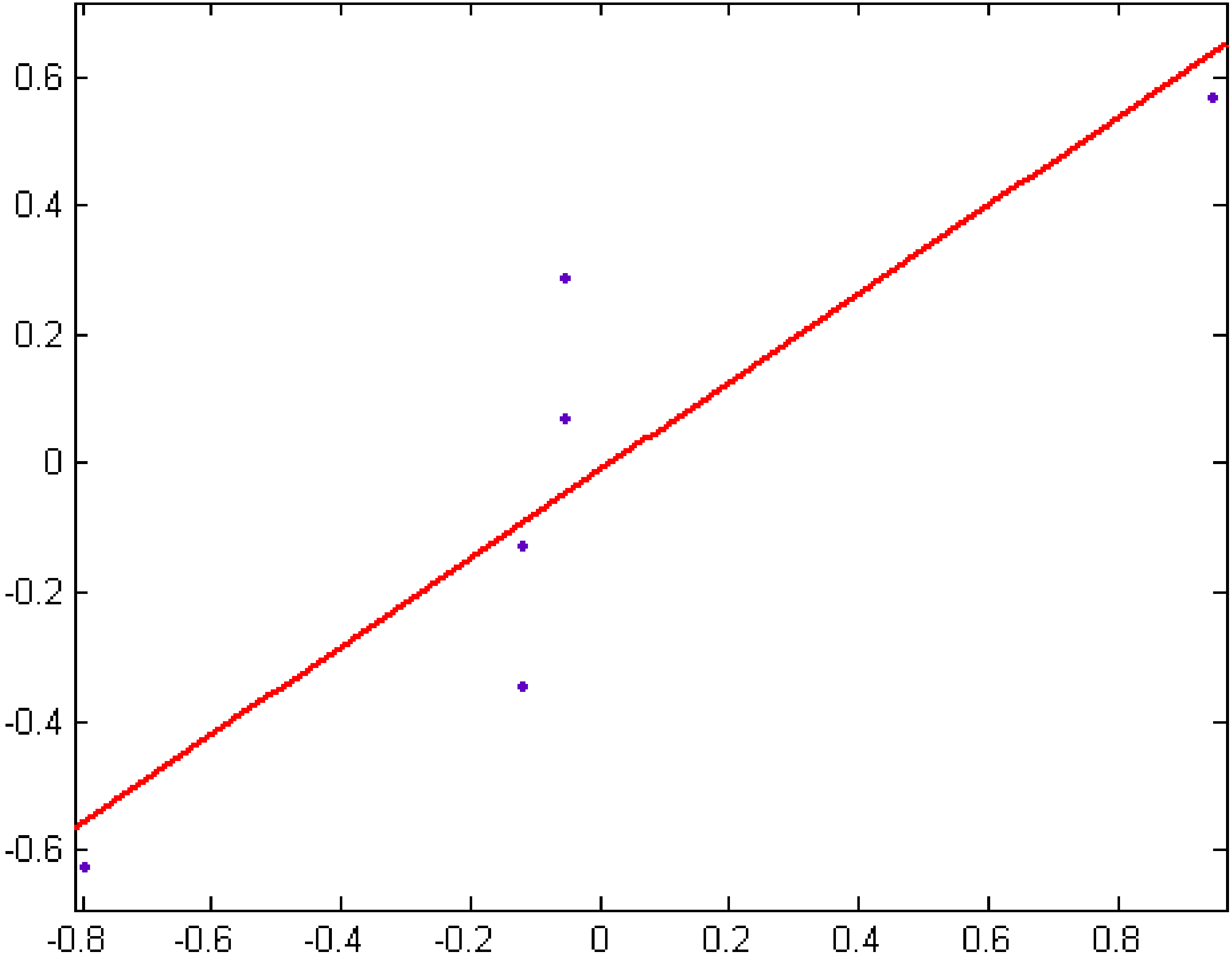


Figure 84 (n=p^8q^4, mean=0.0025, standard deviation=0.5253, size=8)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Figure 85 ($n=p^7q^5$, mean=-0.0312, standard deviation=0.5595, size=6)



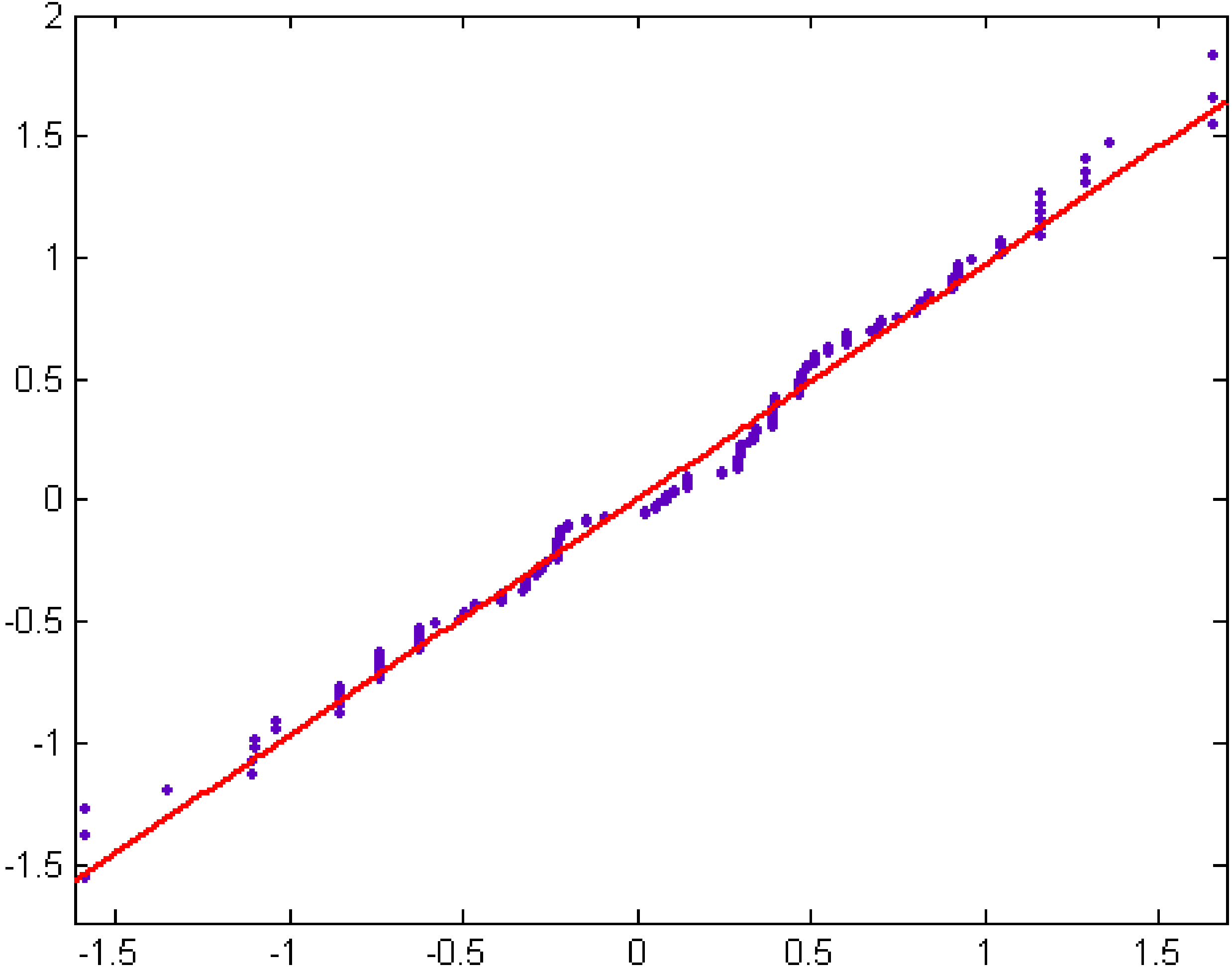


Figure 86 ($n=p^4q^3r^2$, mean=0.1363, standard deviation=0.6746, size=165)

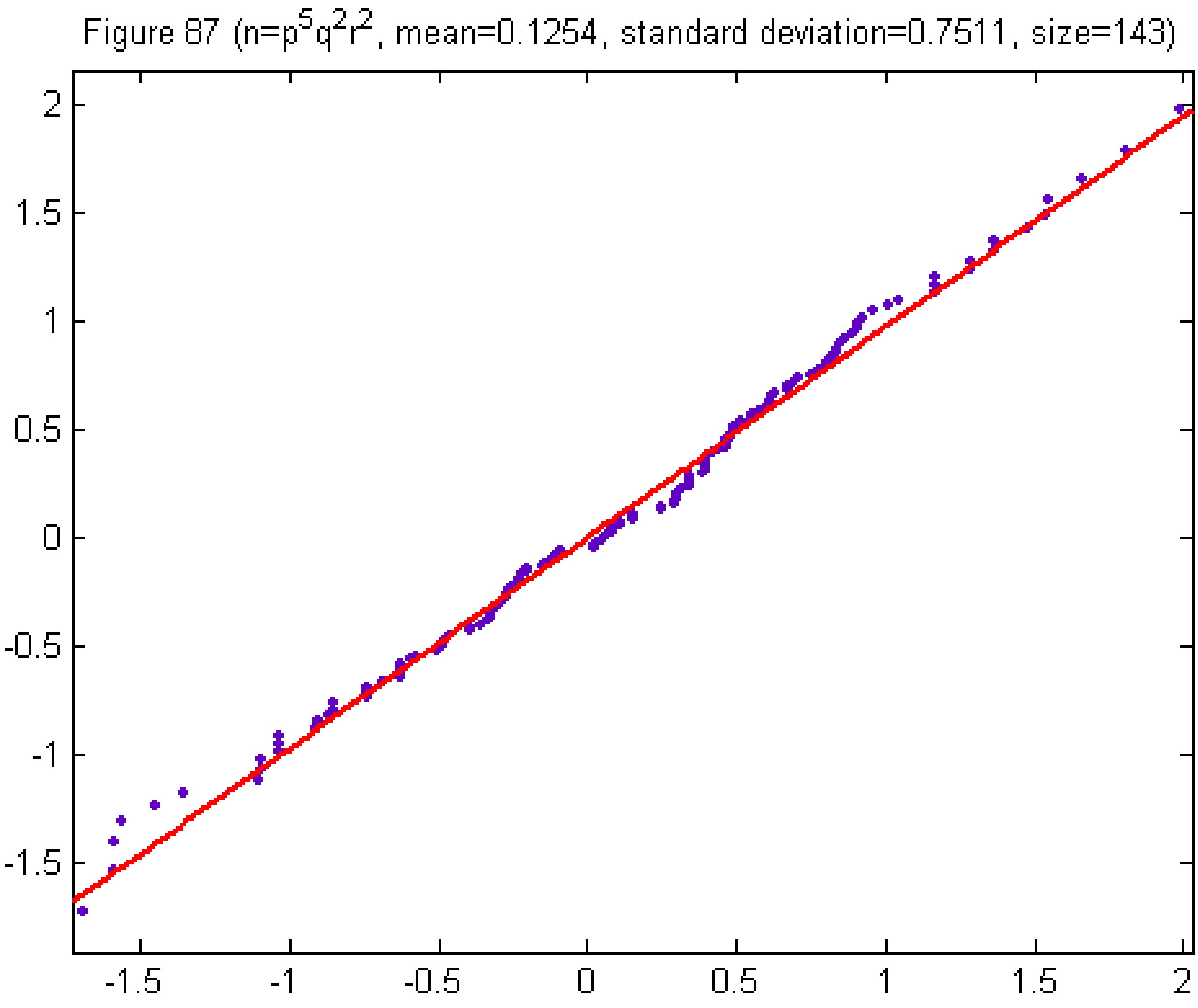


Figure 87 (n=p^5q^2r^2, mean=0.1254, standard deviation=0.7511, size=143)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
2

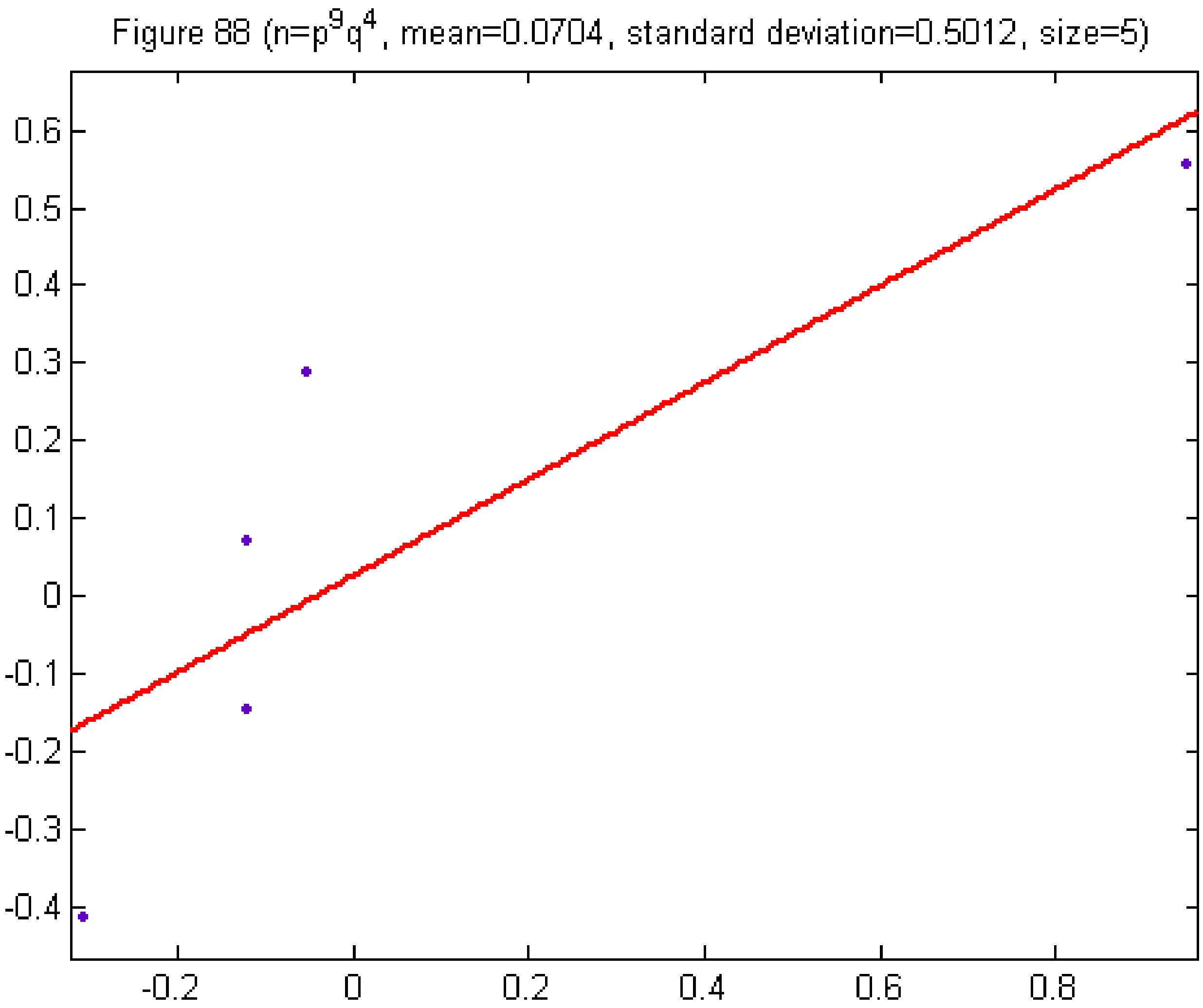


Figure 88 (n=p9q4, mean=0.0704, standard deviation=0.5012, size=5)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.2
0
0.2
0.4
0.6
0.8

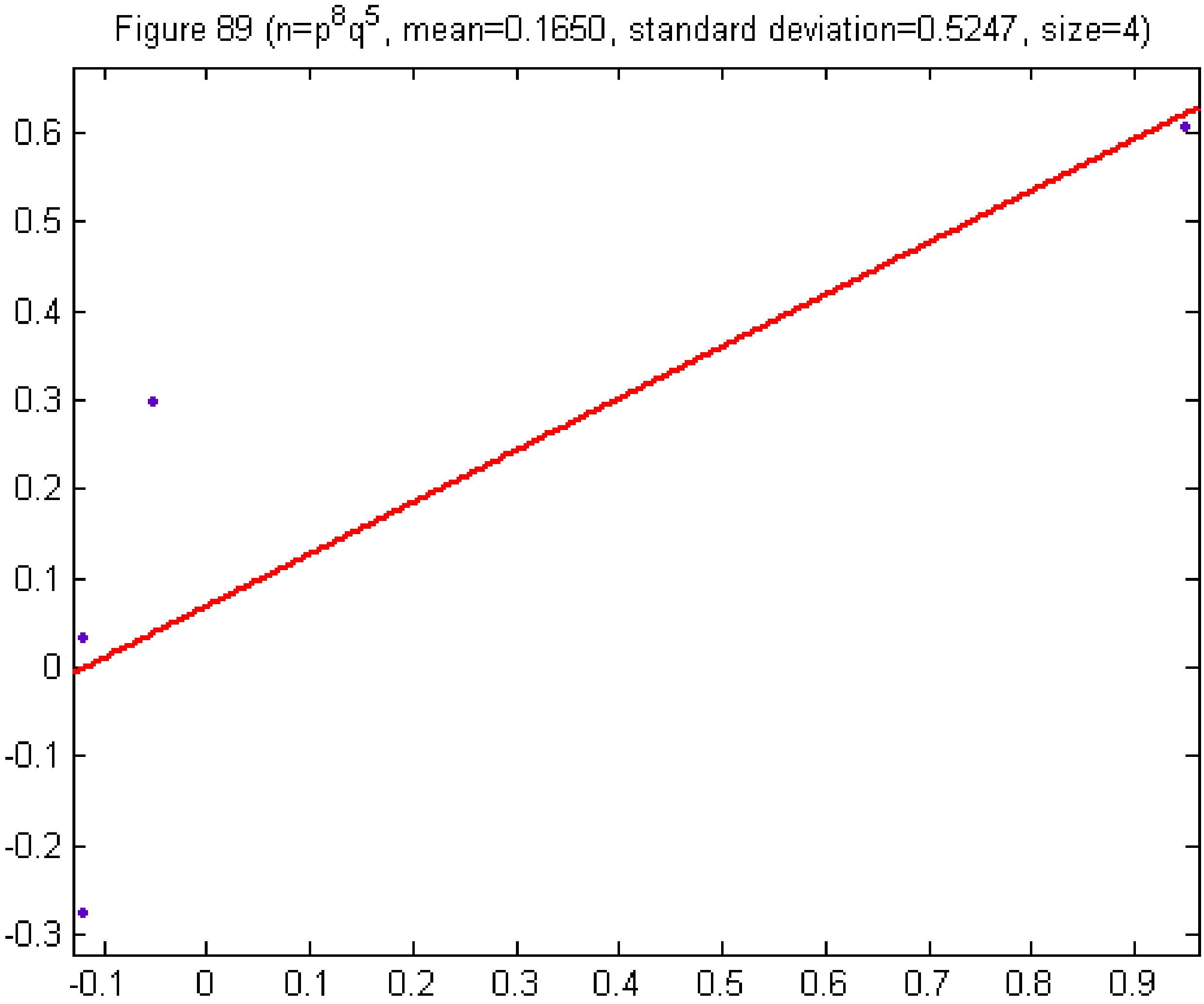


Figure 89 ($n=p^8q^5$, mean=0.1650, standard deviation=0.5247, size=4)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

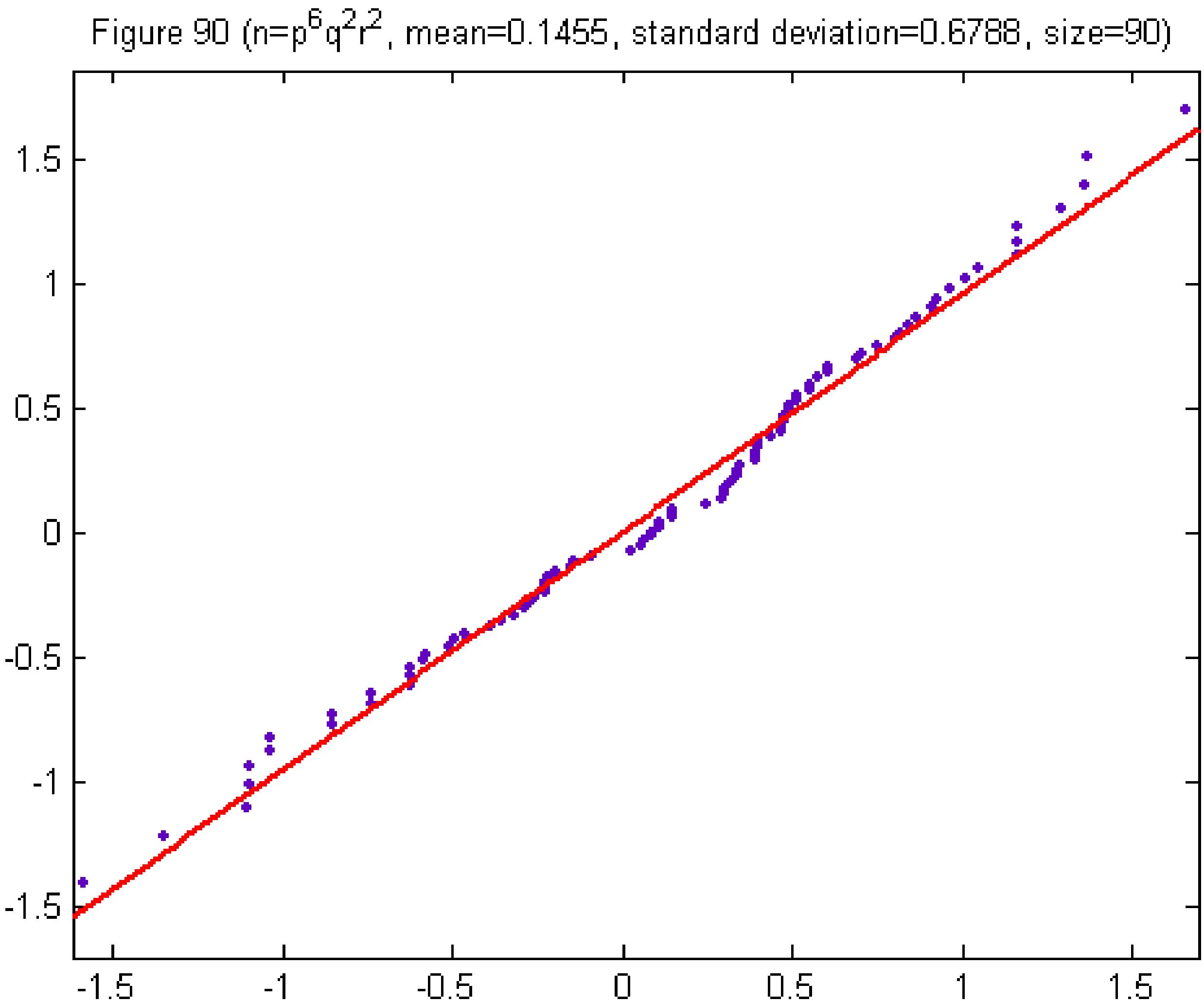


Figure 90 ($n=p^6q^2r^2$, mean=0.1455, standard deviation=0.6788, size=90)
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

Figure 91 ($n=p^{10}q^4$, mean=0.1650, standard deviation=0.5247, size=4)

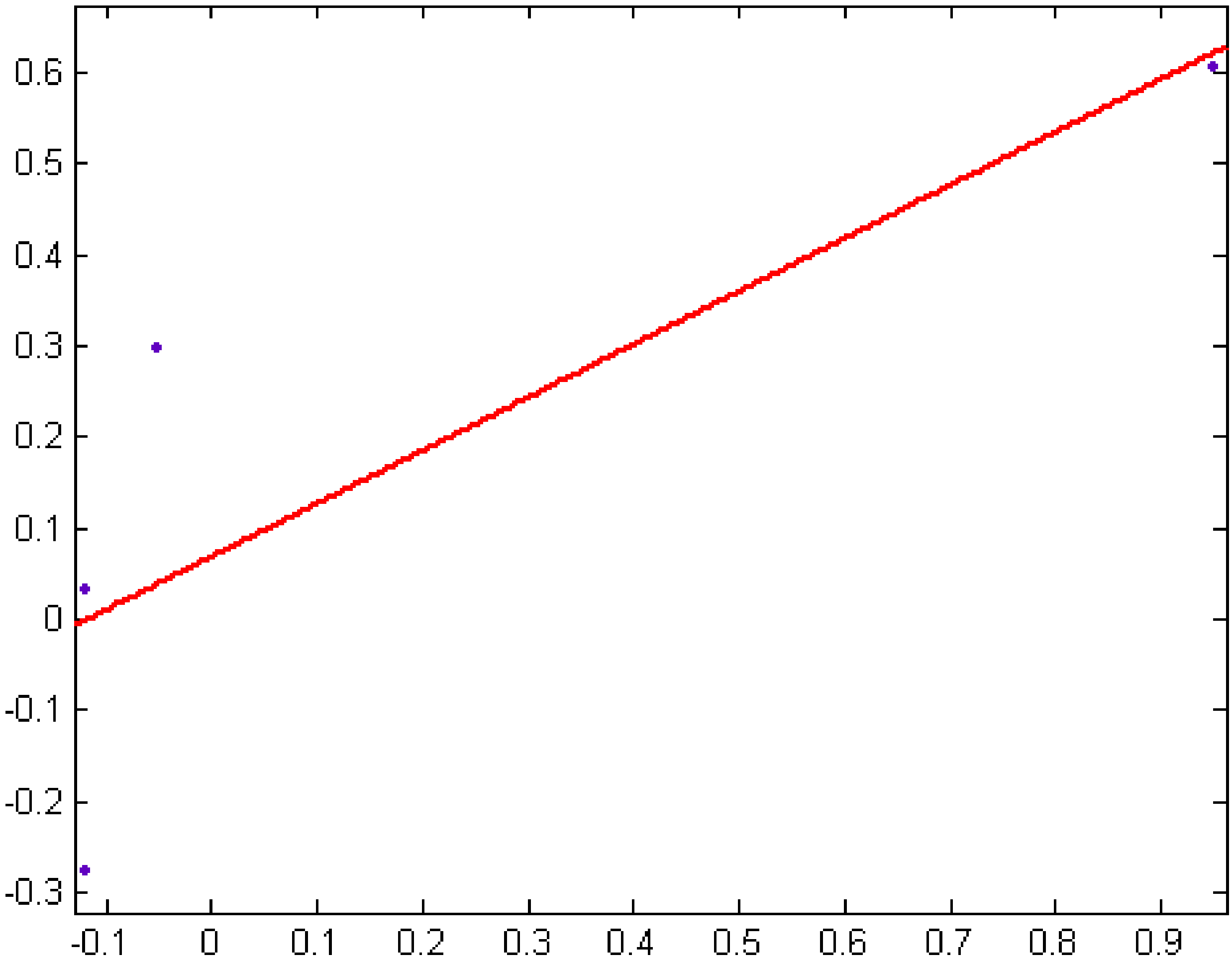


Figure 92 ($n=p^1 1q^3$, mean=0.1446, standard deviation=0.4837, size=6)

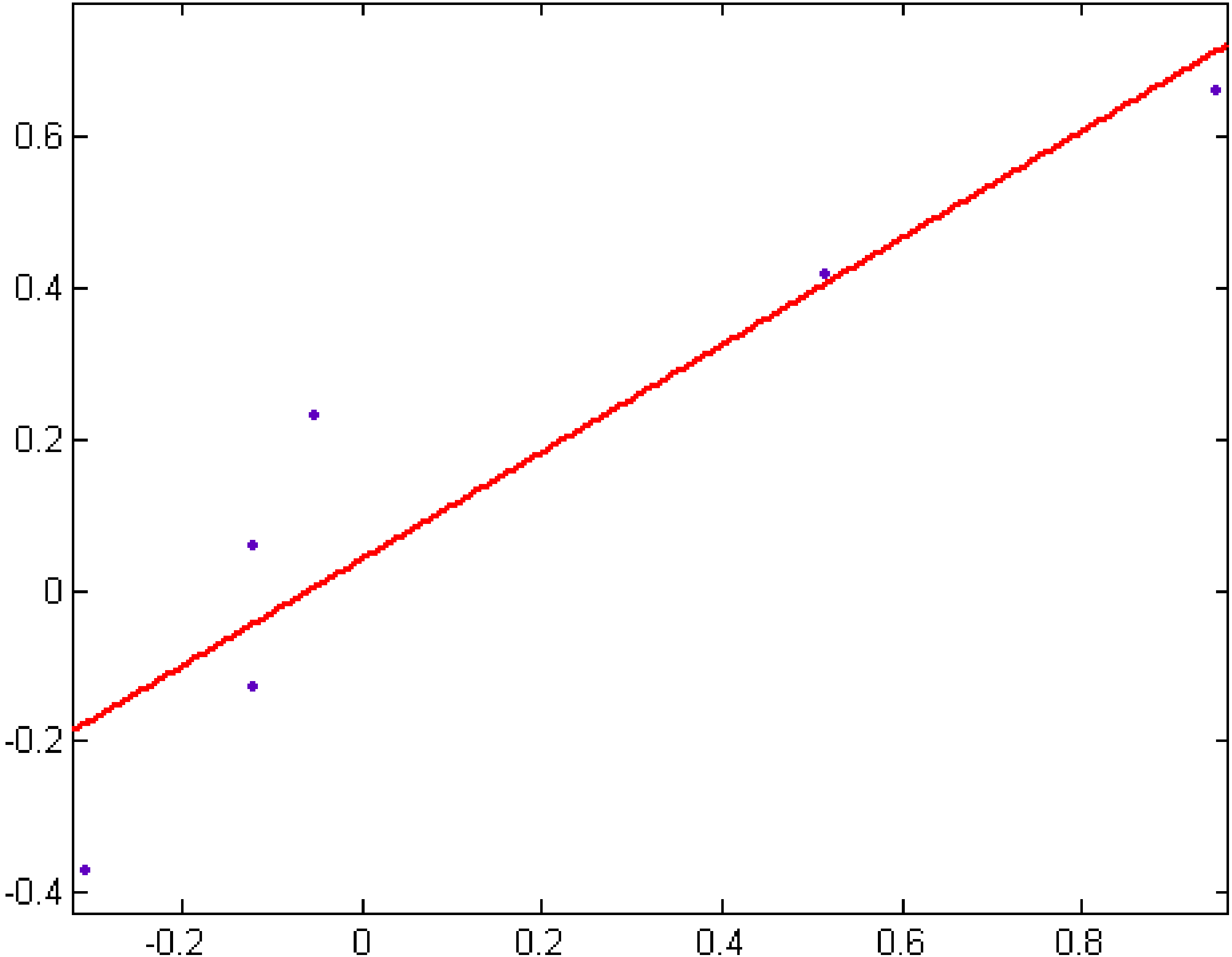


Figure 93 ($n=p^5q^3r^2$, mean=0.1525, standard deviation=0.6824, size=92)

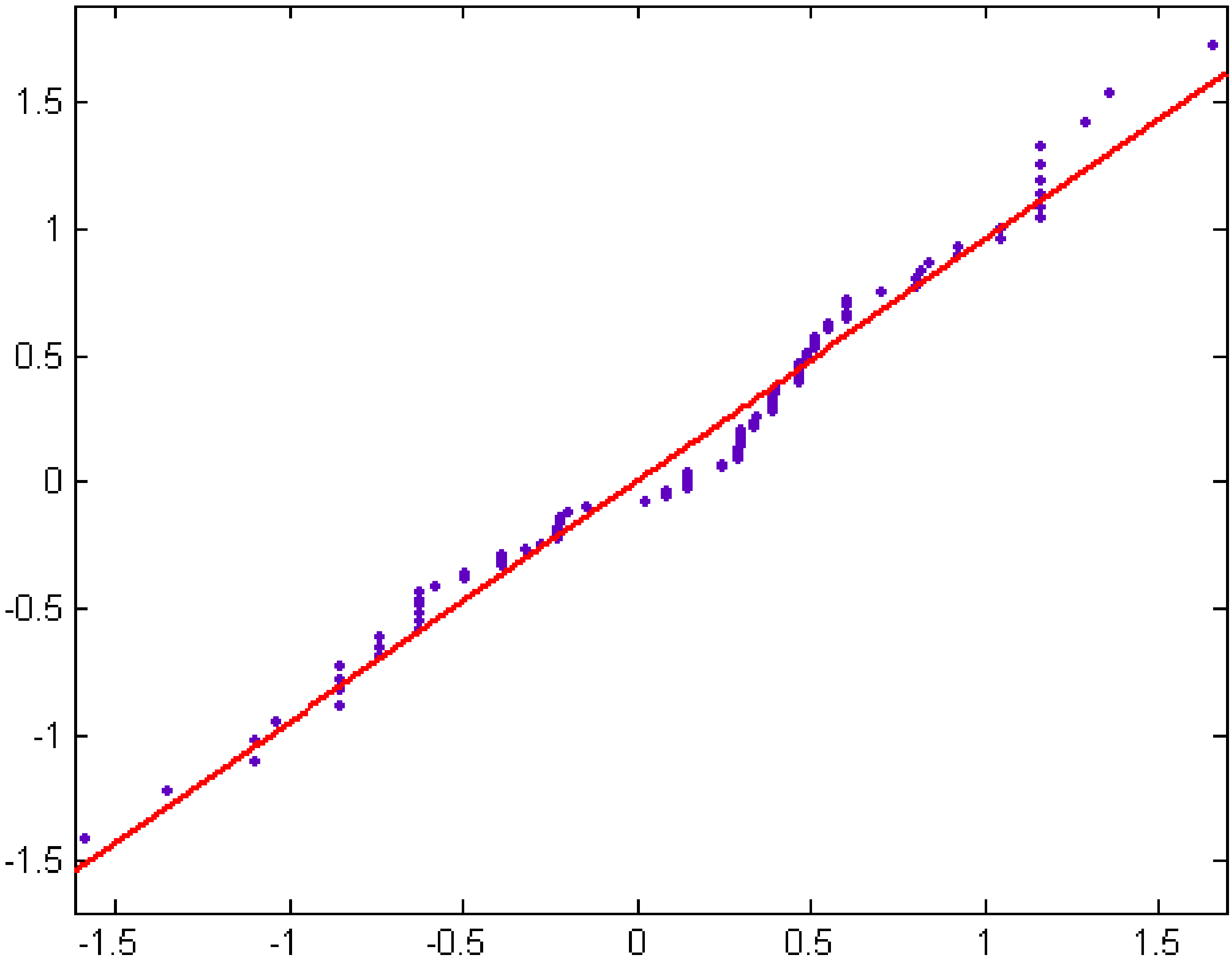
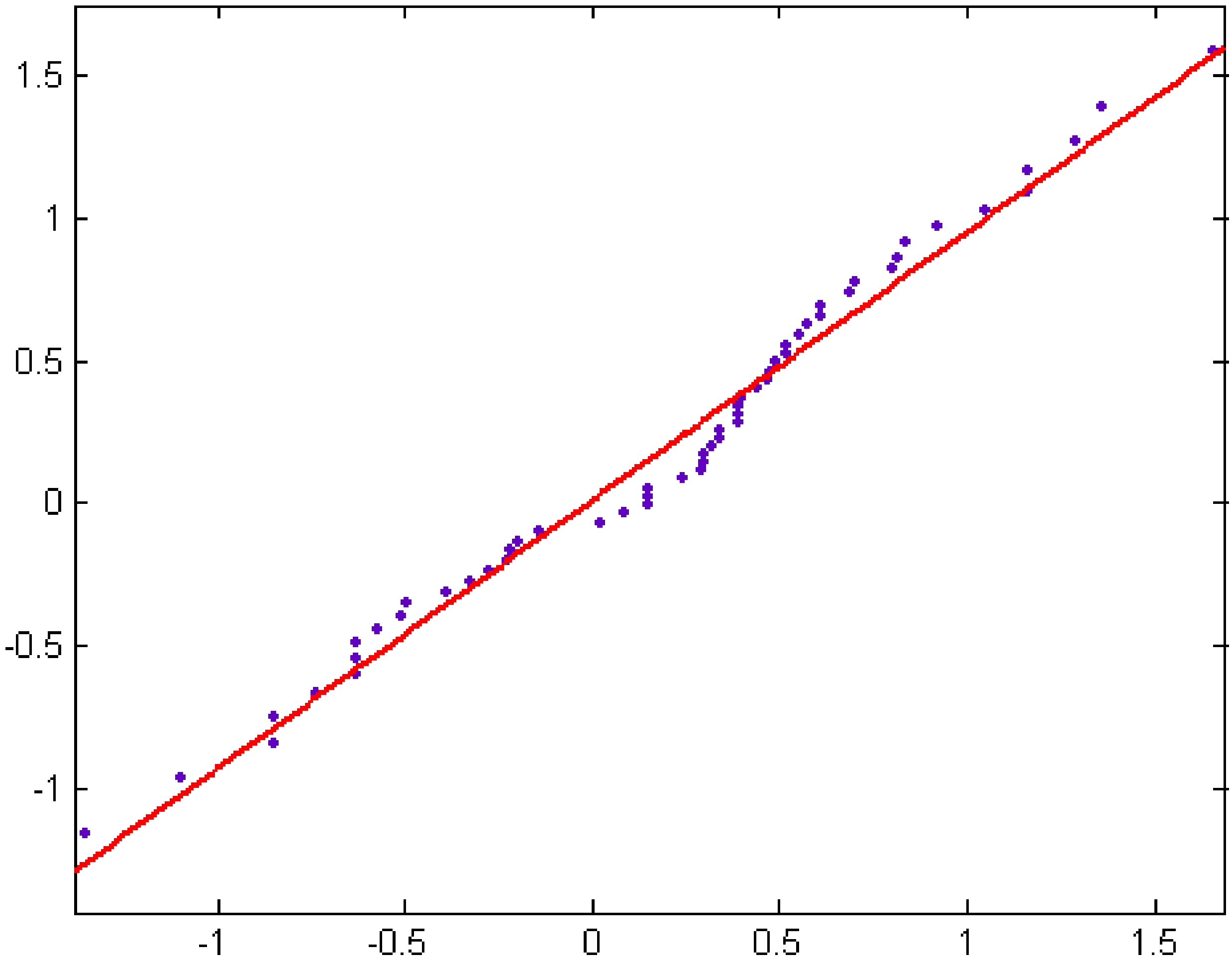


Figure 94 ($n=p^7q^2r^2$, mean=0.2119, standard deviation=0.6477, size=56)



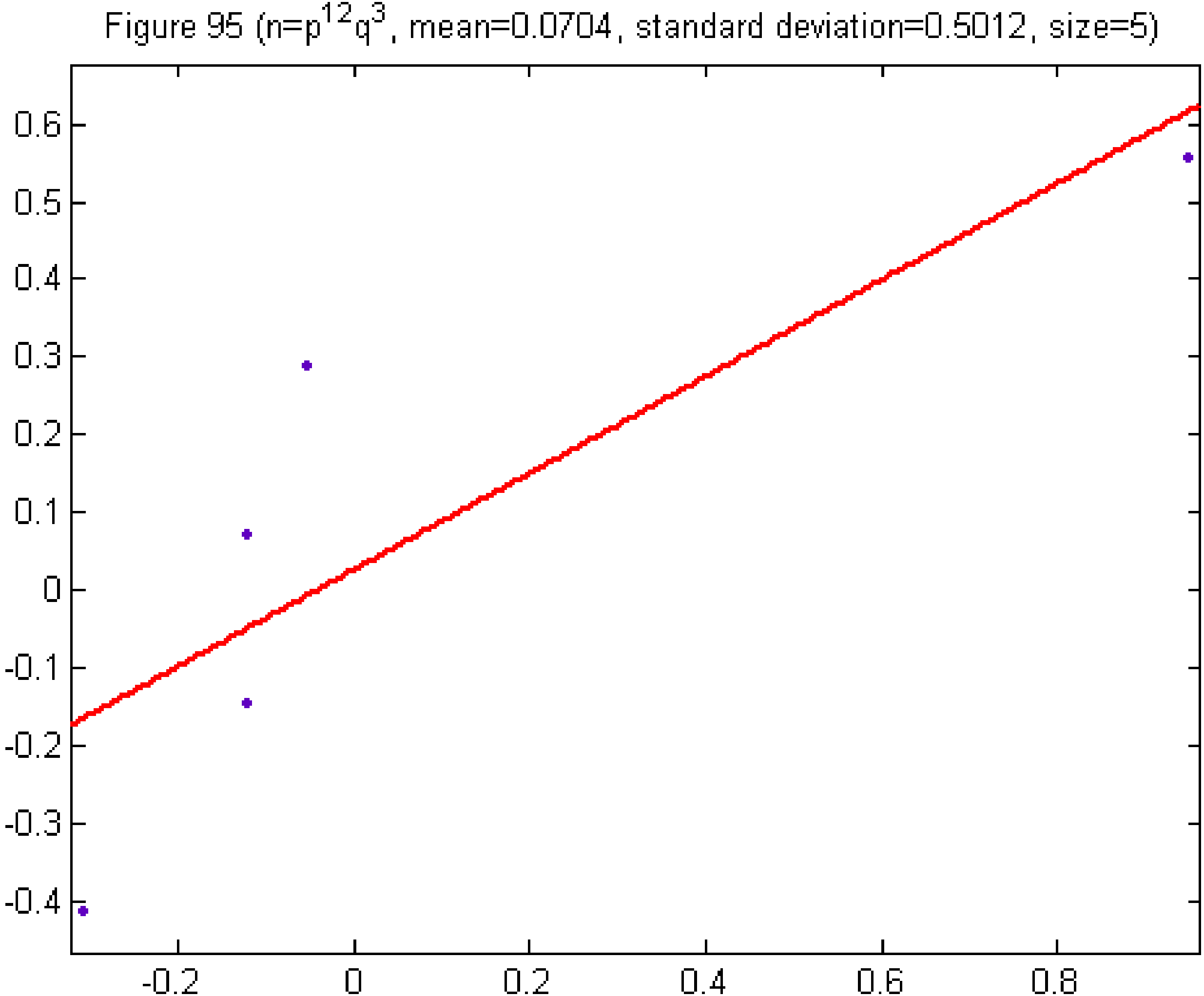


Figure 95 ($n=p^{12}q^3$, mean=0.0704, standard deviation=0.5012, size=5)
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.2
0
0.2
0.4
0.6
0.8