

Zeta Function Zeros, Dirichlet Products, Möbius Inversion, and the Probit Function

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Abstract

Forward and inverse transformations involving the non-trivial Riemann zeta function zeros are investigated. These transformations give a simple method to find and accurately approximate new zeros given a baseline of accurate zeros to extrapolate from. The relevance of the probit function, Ramanujan sums, and Dirichlet inverses to the zeros is also investigated. The generalized probit function is used to model the distribution of the zeros and the differences between them.

1 Introduction

Let $\theta_1, \theta_2, \theta_3, \dots$ denote the imaginary parts of the nontrivial zeros of the Riemann zeta function (Odlyzko's [1] tables will be used). Dirichlet products have the form $\sum_{d|n} f(d)g(n/d)$. Initially, f is set to a constant function having values of 1. The equation $f(n) = \sum_{d|n} (g/d)$ implies $g(n) = \sum_{d|n} f(d)\mu(n/d)$ where μ is the Möbius function. This is known as Möbius inversion. This is the motivation for starting with trivial Dirichlet products. The cumulative distribution function is denoted by $\Phi(p)$ where p are probabilities between 0 and 1. The probit function is the inverse cumulative distribution function for the standard normal distribution. The probit function can be computed using an ordinary differential equation. The more general function $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ where μ and σ are the mean and standard deviation of the distribution will be used here. Generalized Ramanujan sums have the form $\sum_{d|(n,k)} f(d)g(k/d)$ where (n, k) denotes the greatest common divisor function. These sums have a finite Fourier expansion. Fourier analysis of the zeta function zeros can then be done.

2 Dirichlet Products Involving θ

Hutchinson [2] computed the first 138 non-trivial zeros of the Riemann zeta function. See Figure (1) for a plot of $\sum_{i|x} \log(\theta_{x/i})$ for $x = 1, 2, 3, \dots, 135$. The plot consists of 12 different slowly increasing "curves" with gaps in them.

Curve #0 (numbering from the bottom) consists of a single element at $x = 1$ and having a value of $\log(\theta_1)$. Curve #1 consists of elements at $x = 2, 3, 5, \dots$ (the primes) and having values of $\log(\theta_1) + \log(\theta_x)$. Curve #2 consists of elements at $x = 2^2, 3^2, 5^2, \dots$. Curve #3 consists of elements at $x = 2^3, 3^3, 5^3, \dots$ and x values that are the product of two distinct primes. Curve #4 consists of elements at $x = 2^4, 3^4, 5^4, \dots$. Curve #5 consists of elements at $x = 2^5, 3^5, 5^5, \dots$ and x values that are the product of the square of a prime and a different prime. Curve #6 consists of elements at $x = 2^6, 3^6, 5^6, \dots$. Curve #7 consists of elements at $x = 2^7, 3^7, 5^7, \dots$, x values that are the product of the cube of a prime and a different prime, and x values that are the product of three distinct primes. Curve #8 (not shown) consists of elements at $x = 2^8, 3^8, 5^8, \dots$. Curve #9 consists of elements at $x = 2^9, 3^9, 5^9, \dots$, x values that are the product of the fourth power of a prime and a different prime, and x values that are the product of the square of a prime and the square of a different prime (a secondary curve). The values of the secondary curve are somewhat less than those of the primary curve. Curve #10 (not shown) consists of elements at $x = 2^{10}, 3^{10}, 5^{10}, \dots$. Curve #11 consists of elements at $x = 2^{11}, 3^{11}, 5^{11}, \dots$, x values that are the product of the fifth power of a prime and a different prime, x values that are the product of the cube of a prime and the square of a different prime, and x values that are the product of the square of a prime and two other distinct primes. Curve #12 (not shown) consists of elements at $x = 2^{12}, 3^{12}, 5^{12}, \dots$. Curve #13 consists of elements at $x = 2^{13}, 3^{13}, 5^{13}, \dots$, x values that are the product of the sixth power of a prime and a different prime (a secondary curve), x values that are the product of the fourth power of a prime and the square of a different prime (another secondary curve), x values that are the product of the cube of a prime and the cube of a different prime, x values that are the product of the cube of a prime and two other distinct primes, and x values that are the product of four distinct primes. The x values other than those in the secondary curves constitute the primary curve. The first secondary curve has smaller values than those of the second secondary curve. Curve #14 (not shown) consists of elements at $x = 2^{14}, 3^{14}, 5^{14}, \dots$. Curve #15 (not shown) consists of elements at $x = 2^{15}, 3^{15}, 5^{15}, \dots$, x values that are the product of the seventh power of a prime and a different prime (a secondary curve), x values that are the product of the fifth power of a prime and the square of a different prime, x values that are the product of the fourth power of a prime and the cube of a different prime (the primary curve), x values that are the product of the fourth power of a prime and two other distinct primes, and x values that are the product of the square of a prime, the square of a different prime, and another distinct prime. The x values other than those in the first secondary curve and the primary curve constitute the second secondary curve. The first secondary curve has smaller values than those of the second secondary curve. The first secondary curve coincides with the primary curve in Curve #13. Curve #16 (not shown) consists of elements at $x = 2^{16}, 3^{16}, 5^{16}, \dots$. Curve #17 (not shown) consists of elements at $x = 2^{17}, 3^{17}, 5^{17}, \dots$, x values that are the product of the eighth power of a prime and another prime (a secondary curve), x values that are the sixth power of a prime and the square of another prime (another

secondary curve), x values that are the product of the fifth power of a prime and the cube of another prime, x values that are the product of the cube of a prime, the square of another prime, and another distinct prime, and x values that are the product of the square of a prime and three other distinct primes. The x values other than those in the secondary curves constitute the primary curve. The first secondary curve has smaller values than those of the second secondary curve. The first secondary curve coincides with the primary curve in Curve #15. In general, Curve # k consists of at least elements at $x = 2^k, 3^k, 5^k, \dots$. The logarithm of the θ values was used to help identify the curves. See Figure (2) for a plot of $\sum_{i|x} \theta_{x/i}$ for $x = 1, 2, 3, \dots, 135$. Möbius inversion can be used to regenerate the θ values from these curves.

3 A Technique for Finding Zeros Given Other Zeros

See Figure (3) for a plot of the 148933 θ_x values at $x = 2, 3, 5, \dots, 1,999,993$ (the primes less than 2,000,000). For a linear least squares fit of the curve, $p_1 = 7.587$ with a 95% confidence interval of (7.587, 7.587), $p_2 = 3578$ with a 95% confidence interval of (3568, 3587), $SSE=1.323 \cdot 10^{11}$, R-square=1, and RMSE=942.6. See Figure (4) for a plot of the 223 $\sum_{i|x} \theta_{x/i} - \theta_1$ values at $x = 2^2, 3^2, 5^2, \dots, 1409^2$ (x values at the squares of primes less than 2,000,000). For a quadratic least squares fit of the curve, $p_1 = 24.55$ with a 95% confidence interval of (24.31, 24.78), $p_2 = -626.4$ with a 95% confidence interval of (-681.9, -571), $p_3 = 9172$ with a 95% confidence interval of (6480, 11860), $SSE=9.989 \cdot 10^9$, R-square=0.9996, and RMSE=6738. See Figure (5) for a plot of the 30 $\sum_{i|x} \theta_{x/i} - \theta_1$ values at $x = 2^3, 3^3, 5^3, \dots, 117^3$ (x values at the cubes of primes less than 2,000,000). For a cubic least-squares fit of the curve, R-square=0.9947. In general, the $\sum_{i|x} \theta_{x/i} - \theta_1$ values can be predicted fairly accurately for x values that are powers of primes.

Let $\kappa_1(1), \kappa_1(2), \kappa_1(3), \dots$, denote $\theta_1, \theta_2, \theta_3, \dots$ and let $\kappa_n(x)$, $n = 2, 3, 4, \dots$, denote these values and $n - 1$ values that have been linearly interpolated between successive values. In 1903, Gram [3] published a list of the first 15 zeros. Let $y_1, y_2, y_3, \dots, y_{43}$ equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes for x less than or equal to 135. Let $z_1, z_2, z_3, \dots, z_{43}$ equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes for x less than or equal to 135. (Note that only the zeros computed by Gram were used.) See Figure (6) for a plot of y and z . To compare the curves, they would have to be first normalized (reduced in value by subtracting the first element of the array from all the elements) and then the normalized z values could be scaled up by multiplying by the ratio of the last elements of the normalized arrays. These are the parameters needed to predict the y values using the z values. See Figure (7) for a plot of the nor-

malized and scaled y and z values. See Figure (8) for a plot of the differences between the normalized and scaled y and z values divided by the normalized y values (where the first element of the resulting array is not shown). The relative error is usually less than 5%. Let $y_1, y_2, y_3, \dots, y_{19}$ equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another prime for x less than or equal to 135. Let $z_1, z_2, z_3, \dots, z_{19}$ equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another prime for x less than or equal to 135. See Figure (9) for a plot of y and z . See Figure (10) for a plot of the normalized and scaled y and z values. See Figure (11) for a plot of the differences between the normalized and scaled y and z values divided by the normalized y values (where the first element of the resulting array is not shown). The relative error is less than 9%.

Let $y_1, y_2, y_3, \dots, y_{2600}$ equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that the product of two distinct primes for x less than 10,000. See Figure (12) for a plot of the y values. In the following, the corresponding 2850 y values for x less than 11,000 will be approximated. Then Möbius inversion will be used to compare the new 250 zero values to the predicted values. Let $y_1, y_2, y_3, \dots, y_{145}$ equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that the product of two distinct primes for x less than 500. Let $z_1, z_2, z_3, \dots, z_{145}$ equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the product of two distinct primes for x less than 500. The minimum y value is 83.619076 and the minimum z value is 66.318290. The maximum y value is 1266.394397 and the maximum z value is 747.781093. The scaling factor is then $(1266.394397 - 83.619076) / (747.781093 - 66.318290)$ of about 1.73564. See Figure (13) for a plot of the corresponding scaling factors for x values less than 500, 1,000, 1,500, ..., 10,000. The corresponding numbers of x values that are the product of two distinct primes are 145, 288, 424, 563, 694, 825, 962, 1087, 1216, 1346, 1477, 1601, 1724, 1844, 1986, 2105, 2241, 2357, 2472, and 2600 respectively. The sample sizes of multiples of 500 are large enough that the curve is relatively smooth. The curve is locally quadratic. The last four scaling factors are 1.806791, 1.808142, 1.809158, and 1.810204. See Figure (14) for a plot of these values. For a quadratic least-squares fit of the values, $p_1 = -0.00007625$ with a 95% confidence interval of $(-0.0005948, 0.0004423)$, $p_2 = 0.001507$ with a 95% confidence interval of $(-0.001127, 0.00414)$, $p_3 = 1.805$ with a 95% confidence interval of $(1.802, 1.808)$, $\text{SSE} = 6.661 \cdot 10^{-9}$, $\text{R-square} = 0.999$, and $\text{RMSE} = 8.162 \cdot 10^{-5}$. Using this formula gives two more extrapolated scaling factors of 1.810629 and 1.811297. Let w_x equal $\sum_{i|x} \kappa_2(x/i) - \theta_1$ for $x = 1, 2, 3, \dots, 11,000$. Let $w'_x = (w_x + 66.318290) * 1.811297$, $x = 1, 2, 3, \dots, 11,000$. Möbius inversion is then done on w' and the 2850 estimated zeros at x values that are the product of two distinct primes are isolated. See Figure (15) for a plot of these estimated zeros and the actual zeros (similarly isolated). See Figure (16) for a plot of the differences (estimated zeros minus actual zeros) of the two curves. See Figure

(17) for a plot of the relative errors of the approximations (differences divided by actual zeros). See Figure (18) for a plot of the sorted relative errors of the new 250 estimated zeros. The relative errors of the new estimated zeros are between 0.08% and 0.21%.

Trivial Dirichlet products of the standard uniform distribution can be used as a reference point for locating zeros. In the following, the standard uniform distribution (of 2000 values) is scaled up by a factor of 5600.0. See Figure (19) for a plot of these values for $x = 1, 2, 3, \dots, 138$ (corresponding to the zeros that Hutchinson computed). See Figure (20) for a plot of these values at x values that are the product of two distinct primes and Dirichlet products of θ at x values that are the product of two distinct primes.

4 The Generalized Probit Function, Zeta Function Zeros, and the Möbius Function

For convenience, let $w(p)$ denote the probit function. The differential equation than can be used to compute w is $\frac{dw}{dp} = \frac{1}{f(w)}$ where $f(w)$ is the probability density function of w . In the case of the Gaussian distribution, $\frac{d^2w}{dp^2} = w(\frac{dw}{dp})^2$ with initial conditions of $w(\frac{1}{2}) = 0$ and $w'(\frac{1}{2}) = \sqrt{2\pi}$. Steinbrecher and Shaw's [4] method is used to solve this ODE.

Let y_x denote $\sum_{i|x} \theta_i \mu(i)$. For a normal probability distribution fit of y_x for x less than or equal to 9,999, the mean is -21.5933 with a 95% confidence interval of $(-86.9019, 43.7153)$ and the standard deviation is 3331.6 with a 95% confidence interval of $(3286.0, 3378.4)$. See Figure (21) for a plot of the sorted y_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = -21.5933$ and $\sigma = 3331.6$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum y_x values are -9840.91 and 9411.65 and the minimum and maximum F^{-1} values are -10653 and 10610 . For larger samples of zeros, the y_x values are similarly bounded by the generalized probit function. The minimum y_x values are about 92% larger than the minimum F^{-1} values and the maximum y_x values are about 90% smaller than the maximum F^{-1} values. See Figure (22) for a plot of the means of y_x for $x = 999, 1,999, 2,999, \dots, 99,999$. There doesn't appear to be any way to predict what the means of the distributions will be, but they are small and relatively insignificant. See Figure (23) for a plot of the standard deviations of y_x for $x = 999, 1,999, 2,999, \dots, 99,999$. For a cubic least-squares fit of these values, $p_1 = 0.00364$ with a 95% confidence interval of $(0.003284, 0.003995)$, $p_2 = -0.8748$ with a 95% confidence interval of $(-0.9294, -0.8201)$, $p_3 = 301.5$ with a 95% confidence interval of $(299.1, 303.9)$, $p_4 = 349.6$ with a 95% confidence interval of $(321.7, 377.5)$, $SSE=1.099 \cdot 10^5$, $R\text{-square}=1$, and $RMSE=33.83$. The curve of standard deviations is almost quadratic. See Figure (24) for a plot of the maximum y_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots$

19,999. See Figure (25) for the minimum y_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots, 19,999$.

Let θ'_i denote $\theta_i/\log(\theta_i)$. Let z_x denote $\sum_{i|x} \theta'_i \mu(i)$. For a normal probability distribution fit of z_x for x less than or equal to 9,999, the mean is -3.4582 with a 95% confidence interval of $(-10.4901, 3.5736)$ and the standard deviation is 358.7124 with a 95% confidence interval of $(353.8090, 363.7546)$. See Figure (26) for a plot of the sorted z_x values and $F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$ (where $\mu = -3.4582$ and $\sigma = 358.7124$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum z_x values are -1066.36 and 992.0359 and the minimum and maximum F^{-1} values are -1148.1 and 1141.1 . For larger samples of zeros, the z_x values are similarly bounded by the generalized probit function. The minimum z_x values are about 92% larger than the minimum F^{-1} values and the maximum z_x values are about 89% smaller than the maximum F^{-1} values. See Figure (27) for a plot of the means of z_x for $x = 999, 1,999, 2,999, \dots, 19,999$. See Figure (28) for a plot of the standard deviations of z_x for $x = 999, 1,999, 2,999, \dots, 19,999$. For a quadratic least-squares fit of the standard deviations, $p_1 = -0.3416$ with a 95% confidence interval of $(-0.393, -0.2902)$, $p_2 = 35.63$ with a 95% confidence interval of $(34.52, 36.74)$, $p_3 = 36.85$ with a 95% confidence interval of $(31.78, 41.92)$, $\text{SSE}=177.3$, $\text{R-square}=0.9997$, and $\text{RMSE}=3.23$. See Figure (29) for a plot of the maximum z_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots, 19,999$. See Figure (30) for the minimum z_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots, 19,999$.

Let y_x denote $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$. For a normal probability fit of y_x for x less than or equal to 9999, the mean is 2.6505 with a 95% confidence interval of $(2.6095, 2.6914)$ and the standard deviation is 2.0868 with a 95% confidence interval of $(2.0583, 2.1161)$. See Figure (31) for a plot of the sorted y_x values and $F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$ (where $\mu = 2.6505$ and $\sigma = 2.0868$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The curve of sorted y_x values has a characteristic "bump" near the end of the curve. Other than this and values at the beginning of the curve, the generalized probit function approximates the sorted y_x values fairly well. The minimum and maximum y_x values are -5.171634 and 7.786355 and the minimum and maximum F^{-1} values are -4.0085 and 9.3095 . See Figure (32) for a plot of the means of y_x for $x = 999, 1,999, 2,999, \dots, 99,999$. The means are approximately equal to 2.65 . See Figure (33) for a plot of the standard deviations of y_x for $x = 999, 1,999, 2,999, \dots, 99,999$. The standard deviations slowly increase from 2.0057 to 2.1224 . See Figure (34) for a plot of the standard deviations and $\log(x) \cdot 0.014 + 2.0057$ for $x = 1, 2, 3, \dots, 100$. The standard deviations approximately increase at a rate that is proportional to the logarithm. See Figure (35) for a plot of the minima of the y_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots, 39,999$. The minima of y_x are not bounded by the generalized probit function. See Figure (36) for a plot of the maxima of the y_x and F^{-1} values for $x = 999, 1,999, 2,999, \dots, 39,999$. y_x increases in value at $x = 364, 182$. For a normal probability fit of y_x for x less

than or equal to 364,182, the mean is 2.6538 with a 95% confidence interval of (2.6468, 2.6607) and the standard deviation is 2.1363 with a 95% confidence interval of (2.1314, 2.1412). See Figure (37) for a plot of the sorted y_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 2.6538$ and $\sigma = 2.1363$) for $p = 0.00000275, 0.00000549, 0.00000824, \dots, 0.99999725$. The minimum and maximum y_x values are -8.711062 and 10.520115 and the minimum and maximum F^{-1} values are -4.2141 and 9.5217. The maxima of y_x are then not bounded by the generalized probit function either.

Let l_x denote the minimum of the y_x distribution. See Figure (38) for a plot of l_x for x less than or equal to 2000. l_x is a slowly decreasing step function with an initial value of $\theta_2 - \theta_1$. See Figure (39) for a plot of $-l_x$ and the moduli of $\log(\log(\log(x))) + 3$ for $x = 1, 000, 2,000, 3,000, \dots, 500,000$. $-l_x$ appears to increase at the same rate as $\log(\log(\log(x)))$ when the latter is real-valued. For x less than 500,000, l_x decreases in value at $x = 2, 3, 6, 15, 195, 435, 615, 1590, 4305, 4,935, 7,995, 17,355, 32,595, 72,615, 228,165,$ and $261,555$. The prime factorizations of these values are $2, 3, 2 \cdot 3, 3 \cdot 5, 3 \cdot 5 \cdot 13, 3 \cdot 5 \cdot 29, 3 \cdot 5 \cdot 41, 2 \cdot 3 \cdot 5 \cdot 53, 3 \cdot 5 \cdot 7 \cdot 41, 3 \cdot 5 \cdot 7 \cdot 47, 3 \cdot 5 \cdot 13 \cdot 41, 3 \cdot 5 \cdot 13 \cdot 89, 3 \cdot 5 \cdot 41 \cdot 53, 3 \cdot 5 \cdot 47 \cdot 103, 3 \cdot 5 \cdot 7 \cdot 41 \cdot 53,$ and $3 \cdot 5 \cdot 7 \cdot 47 \cdot 53$ respectively. The number of prime factors is non-decreasing. The corresponding l_x values are 2.898497, 1.473297, 0.817020, -1.210552, -2.2385884, -2.584503, -3.612455, -3.717184, -3.834182, -4.450978, -5.171634, -5.675754, -6.993299, -7.754984, -8.439405, and -8.711062. The values of x that are the products of $2, 2 \cdot 3, 2 \cdot 3 \cdot 5, 2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \dots$ are $2, 6, 30, 210, 2,310, 30,030, \dots$. The above value of 195 is between 30 and 210, the above values of 435, 615, 1,590 are between 210 and 2,310, the above values of 4,305, 4,935, 7,995, and 17,355 are between 2,310 and 30,030, etc. This is useful in determining how frequently l_x will decrease in value. The values of y_x at $x = 2, 6, 30, 210, 2,310, 30,030,$ and $510,510$ are 2.898497, 0.817020, -1.078027, 0.307918, -2.025837, -1.763924, and -4.512349 respectively.

Let g_x denote the maximum of the y_x distribution. See Figure (40) for a plot of g_x for x less than or equal to 1000. g_x is a slowly increasing step function with an initial value of $\theta_2 - \theta_1$. See Figure (41) for a plot of g_x and the moduli of $\log(\log(\log(x))) + 6$ for $x = 1, 000, 2,000, 3,000, \dots, 500,000$. g_x appears to increase at the same rate as $\log(\log(\log(x)))$ when the latter is real-valued. For x less than 500,000, g_x increases in value at $x = 2, 1,806, 6,118, 17,822, 25,802, 34,314, 70,518, 131,838, 186,018, 204,078, 213,486,$ and $364,182$. The prime factorizations of these values are $2, 2 \cdot 3 \cdot 7 \cdot 43, 2 \cdot 7 \cdot 19 \cdot 23, 2 \cdot 7 \cdot 19 \cdot 67, 2 \cdot 7 \cdot 19 \cdot 97, 2 \cdot 3 \cdot 7 \cdot 19 \cdot 43, 2 \cdot 3 \cdot 7 \cdot 23 \cdot 73, 2 \cdot 3 \cdot 7 \cdot 43 \cdot 73, 2 \cdot 3 \cdot 7 \cdot 43 \cdot 107, 2 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 23,$ and $2 \cdot 3 \cdot 7 \cdot 13 \cdot 23 \cdot 29$. The number of prime factors is non-decreasing. The corresponding g_x values are 6.887315, 6.938951, 7.786355, 7.926703, 8.147173, 8.684880, 8.930729, 8.999220, 9.061730, 9.071198, 10.498812, and 10.520155.

5 Curves in the Möbius Function Convolution

See Figure (42) for a plot of $\sum_{i|x} \theta_i \mu(i)$ for x less than or equal to 1000. The curves and sub-curves as defined for the Dirichlet products of θ are applicable to $\sum_{i|x} \theta_i \mu(i)$, $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$, $\sum_{i|x} \theta'_i \mu(i)$, and $\sum_{i|x} (\theta'_{i+1} - \theta'_i) \mu(i)$. See Figure (43) for a plot of $\sum_{i|x} \theta_i \mu(i)$ at $x = 2^2, 3^2, 5^2, \dots, 997^2$ (corresponding to Curve #2). For a linear least-squares fit of these 168 values, $p_1 = -8.335$ with a 95% confidence interval $(-8.355, -8.315)$, $p_2 = -4.652$ with a 95% confidence interval of $(-6.59, -2.713)$, $SSE=6661$, $R\text{-square}=0.9998$, and $RMSE=6.335$. See Figure (44) for a plot of the sorted $\sum_{i|x} \theta_i \mu(i)$ values at x values that are the product of two distinct primes and x values that are the cube of a prime (corresponding to Curve #3) for x less than 1,000,000. (The corresponding F^{-1} values are also plotted.) For a normal probability fit of these 209,892 values, the mean is 244,970 with a 95% confidence interval of $(244,310, 245,630)$ and the standard deviation is 154,890 with a 95% confidence interval of $(154,430, 155,360)$. The minimum and maximum $\sum_{i|x} \theta_i \mu(i)$ values are -217.1155 and $597,400$ respectively and the minimum and maximum F^{-1} values are $-252,900$ and $742,840$ respectively. Other than providing upper and lower bounds, the generalized probit function is of little use in accurately modeling these $\sum_{i|x} \theta_i \mu(i)$ values. For Curves #1, #7, #11, #13, and #17, the generalized probit function models $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ fairly accurately.

See Figure (45) for a plot of the sorted $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values at x values that are prime (corresponding to Curve #1) for x less than 2,000,000. For a normal probability fit of these 148,933 values, the mean is 6.3189 with a 95% confidence interval of $(6.3177, 6.3201)$ and the standard deviation is 0.2436 with a 95% confidence interval of $(0.2427, 0.2444)$. The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values are 1.4733 and 6.8753 respectively and the minimum and maximum F^{-1} values are 5.5360 and 7.1018 respectively. Although the difference between the minima is large, less than 0.52% of the $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values are less than 5.5360. The difference between the two curves is approximately zero except at the beginnings and ends of the curves.

See Figure (46) for a plot of the sorted $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values at x values that are the square of a prime (corresponding to Curve #2) for x less than 2,000,000. For a normal probability fit of these 223 values, the mean is 5.5185 with a 95% confidence interval of $(5.4276, 5.6094)$ and the standard deviation is 0.6888 with a 95% confidence interval of $(0.6303, 0.7595)$. The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values are 1.4733 and 6.5528 respectively and the minimum and maximum F^{-1} values are 3.7186 and 7.3184 respectively. Only 6 of the $\sum_{i|x} (\theta_{i+1} - \theta_i) \mu(i)$ values are less than 3.7186. Only 9 of the differences between the two curves are greater than 0.5.

See Figure (47) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the cube of a prime and x values that are the product of two distinct primes (corresponding to Curve #3) for x less than 2,000,000. For a normal probability fit of these 407,091 values, the mean is 3.7078 with a 95% confidence interval of (3.7034, 3.7123) and the standard deviation is 1.4435 with a 95% confidence interval of (1.4404, 1.4466). The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 7.3027 respectively and the minimum and maximum F^{-1} values are -0.9330 and 8.3486 . The two curves cross three times and the differences between the curves are less than 1.5. 94.78% of the differences are between -0.3958 and 0.3904 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (48) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the cube of a prime and x values that are the product of two distinct primes for x less than 100,000. For a normal probability fit of these 23,327 values, the mean is 3.3803 with a 95% confidence interval of (3.3625, 3.3980) and the standard deviation is 1.3825 with a 95% confidence interval of (1.3700, 1.3951). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 7.0130 respectively and the minimum and maximum F^{-1} values are -1.0505 and 7.8111 respectively. The two curves cross three times and the differences between the curves are less than 1.0.

See Figure (49) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another distinct prime (corresponding to Curve #5) for x less than 2,000,000. For a normal probability fit of these 82,152 values, the mean is 2.6517 with a 95% confidence interval of (2.6447, 2.6587) and the standard deviation is 1.0212 with a 95% confidence interval of (1.0163, 1.0262). The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 6.9035 respectively and the minimum and maximum F^{-1} values are -0.6289 and 5.9323 . Only 2 of the $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are less than -0.6289 . The two curves cross four times and the difference between the curves is less than 1.0. 99.31% of the differences are between -0.2735 and 0.5044 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (50) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another distinct prime for x less than 100,000. For a normal probability fit of these 5,627 values, the mean is 2.5699 with a 95% confidence interval of (2.5421, 2.5977) and the standard deviation is 1.0628 with a 95% confidence interval of (1.0435, 1.0828). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 6.6197 respectively and the minimum and maximum F^{-1} values are -0.8019 and 5.9417 . Only 2 of the $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are less than -0.8019 . The two curves cross four times and the differences between the curves are less than 0.7.

See Figure (51) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the seventh power of a prime, x values that are the product of the cube of a prime and a different prime, and x values that are the product of three distinct primes (corresponding to Curve #7) for x less than 2,000,000. For a normal probability fit of these 448,777 values, the mean is 2.3417 with a 95% confidence interval of (2.3372, 2.3463) and the standard deviation is 1.5602 with a 95% confidence interval of (1.5570, 1.5634). The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 8.0631 respectively and the minimum and maximum F^{-1} values are -2.6744 and 7.3578 respectively. 99.46% of the differences between the two curves are between -0.2807 and 0.2123 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (52) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values and x less than 100,000. For a normal probability fit of these 22,181 values, the mean is 2.1075 with a 95% confidence interval of (2.0862, 2.1288) and the standard deviation is 1.6205 with a 95% confidence interval of (1.6055, 1.6357). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 7.3612 respectively and the minimum and maximum F^{-1} values are -3.0852 and 7.3002 respectively.

The 1,000th element of Curve #7 occurs at $x = 4,922$ and the mean and standard deviation of $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ for corresponding x values less than or equal to 4,922 are 1.8622 and 1.7704 respectively (within 95% confidence intervals). The 2,000th element occurs at $x = 9,494$, the 3,000th element occurs at $x = 14,125, \dots$, and the 100,000th element occurs at $x = 446,118$. See Figure (53) for a plot of these x values. For a linear least-squares fit of the curve, $p_1 = 4451$ with a 95% confidence interval of (4449, 4452), $p_2 = 1223$ with a 95% confidence interval of (1141, 1304), SSE=4.107·10⁶, R-square=1, and RMSE=204.7. These sample sizes are then suitable for comparing means and standard deviations of the distributions. Denote these means and standard deviations by $\mu(i)$ and $\sigma(i)$, $i = 1, 2, 3, \dots, 100$. See Figure (54) for a plot of $(\mu(i) - \mu(1)) \cdot 12.630828$ and $\log(i)$ for $i = 1, 2, 3, \dots, 100$. The means approximately increase at a rate that is proportional to the logarithm. See Figure (55) for a plot of $(1/\sigma(i) - 1/\sigma(1)) \cdot 28.054814$ and $\log(\log(i)) - \log(\log(2))$ (excluding the singularity for $i = 1$). The standard deviations approximately decrease at a rate that is proportional to the logarithm of the logarithm.

See Figure (56) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the ninth power of a prime, x values that are the product of the fourth power of a prime and a different prime, and x values that are the product of the square of a prime and the square of a different prime (corresponding to Curve #9) for x less than 2,000,000. For a normal probability fit of these 15,540 values, the mean is 2.5821 with a 95% confidence interval of (2.5701, 2.5942) and the standard deviation is 0.7655 with a 95% confidence interval of (0.7571, 0.7741). The corresponding F^{-1} values are also shown in the above plot. The minimum

and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 6.0884 respectively and the minimum and maximum F^{-1} values are 0.1328 and 5.0314 respectively. The two curves cross twice and the differences between the curves are less than 1.5 . 99.7% of the differences are between -0.3973 and 0.3298 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (57) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values and for x less than $100,000$. For a normal probability fit of these $1,160$ values, the mean is 2.4785 with a 95% confidence interval of $(2.4251, 2.5319)$ and the standard deviation is 0.9271 with a 95% confidence interval of $(0.8909, 0.9665)$. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 5.0827 respectively and the minimum and maximum F^{-1} values are -0.3313 and 5.2883 respectively. The two curves cross twice and the differences between the curves are less than 1.2 . A sub-curve of Curve #9 has a better F^{-1} fit. See Figure (58) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the product of the square of a prime and the square of a different prime for x less than $2,000,000$. For a normal probability fit of these 407 values, the mean is 2.6613 with a 95% confidence interval of $(2.5306, 2.7920)$ and the standard deviation is 1.3411 with a 95% confidence interval of $(1.2548, 1.4407)$. The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -1.2106 and 6.0884 respectively and the minimum and maximum F^{-1} values are -1.0929 and 6.4155 respectively. The differences between the two curves are less than 0.4 .

See Figure (59) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the eleventh power of a prime, x values that are the product of the fifth power of a prime and a different prime, x values that are the product of the cube of a prime and the square of a different prime, and x values that are the square of a prime and two other distinct primes (corresponding to Curve #11) for x less than $2,000,000$. For a normal probability fit of these $178,185$ values, the mean is 1.9910 with a 95% confidence interval of $(1.9838, 1.9982)$ and the standard deviation is 1.5515 with a 95% confidence interval of $(1.5464, 1.5566)$. The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 7.3612 respectively and the minimum and maximum F^{-1} values are -2.9958 and 6.9778 respectively. The differences between the two curves are less than 0.5 . 99.12% of the differences are between -0.2184 and 0.1843 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (60) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values for x less than $100,000$. For a normal probability fit of these $9,954$ values, the mean is 1.8554 with a 95% confidence interval of $(1.8223, 1.8886)$ and the standard deviation is 1.6867 with a 95% confidence interval of $(1.6636, 1.7105)$. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 6.3965 respectively and the minimum and maximum F^{-1} values are -3.5267 and 7.2375 respectively. The differences between the two curves are less than 0.9 .

See Figure (61) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the thirteenth power of a prime, x values that are the product of the sixth power of a prime and a different prime, x values that are the product of the fourth power of a prime and the square of a different prime, x values that are the product of the cube of a prime and the cube of a different prime, x values that are the cube of a prime and two other distinct primes, and x values that are the product of four distinct primes (corresponding to Curve #13) for x less than 2,000,000. For a normal probability fit of these 266,151 values, the mean is 1.7829 with a 95% confidence interval of (1.7753, 1.7905) and the standard deviation is 1.9970 with a 95% confidence interval of (1.9917, 2.0025). The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -7.7550 and 8.8600 respectively and the minimum and maximum F^{-1} values are -4.6367 and 8.2025 respectively. See Figure (62) for a histogram (with 10 bins) of the differences between the two curves. 99.91% of the differences are between -0.4752 and 0.2799 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (63) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values for x less than 100,000. For a normal probability fit of these 10,897 values, the mean is 1.6850 with a 95% confidence interval of (1.6451, 1.7249) and the standard deviation is 2.1254 with a 95% confidence interval of (2.0975, 2.1540). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -7.7550 and 8.3399 respectively and the minimum and maximum F^{-1} values are -5.1014 and 8.4714 respectively. See Figure (64) for a plot of the differences between the two curves. Relatively large differences for a small number of values occur at the beginning of the curve.

The 1,000th element of Curve #13 occurs at $x = 11,584$ and the mean and standard deviation of $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ for corresponding x values less than or equal to 11,584 are 1.6152 and 2.1967 respectively (within 95% confidence intervals). The 2,000th element occurs at $x = 21,448$, the 3,000th element occurs at $x = 30,849$, ..., and the 100,000th element occurs at $x = 789,952$. See Figure (65) for a plot of these x values. For a quadratic least-squares fit of the curve, $p_1 = -5.326$ with a 95% confidence interval of $(-5.675, -4.977)$, $p_2 = 8,318$ with a 95% confidence interval of $(8,282, 8,354)$, $p_3 = 9,158$ with a 95% confidence interval of $(8,361, 9,954)$, $SSE=1.667 \cdot 10^8$, $R\text{-square}=1$, and $RMSE=1,311$. These sample sizes are then suitable for comparing means and standard deviations of the distributions. Denote these means and standard deviations by $\mu(i)$ and $\sigma(i)$, $i = 1, 2, 3, \dots, 100$. See Figure (66) for a plot of $(\mu(i) - \mu(1)) \cdot 33.986715$ and $\log(i)$ for $i = 1, 2, 3, \dots, 100$. The means approximately increase at a rate that is proportional to the logarithm. See Figure (67) for a plot of $(1/\sigma(i) - 1/\sigma(1)) \cdot 100$ and $\log(i)$. The standard deviations approximately decrease at a rate that is proportional to the logarithm.

See Figure (68) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the fifteenth power of a prime, x values that are the product of the

seventh power of a prime and a different prime, x values that are the product of the fifth power of a prime and the square of a different prime, x values that are the product of the fourth power of a prime and the cube of a different prime, x values that are the fourth power of a prime and two other distinct primes, and x values that are the product of the square of a prime, the square of another prime, and another distinct prime (corresponding to Curve #15) for x less than 2,000,000. For a normal probability fit of these 46,802 values, the mean is 1.9034 with a 95% confidence interval of (1.8889, 1.9179) and the standard deviation is 1.6006 with a 95% confidence interval of (1.5904, 1.6109). The corresponding F^{-1} values are also shown in the above plot. The two curves cross twice. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 6.5248 respectively and the minimum and maximum F^{-1} values are -3.2346 and 7.0414 respectively. 99.41883% of the differences are between -0.23932 and 0.2825 . The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (69) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values for x less than 100,000. For a normal probability fit of these 2,866 values, the mean is 1.7631 with a 95% confidence interval of (1.6979, 1.8283) and the standard deviation is 1.7800 with a 95% confidence interval of (1.7351, 1.8273). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -3.6125 and 6.3965 respectively and the minimum and maximum F^{-1} values are -3.8150 and 7.3412 respectively. The two curves cross twice.

See Figure (70) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at x values that are the seventeenth power of a prime, x values that are the product of the eighth power of a prime and a different prime, x values that are the product of the sixth power of a prime and the square of a different prime, x values that are the product of the fifth power of a prime and the cube of a different prime, x values that are the product of the cube of a prime, the square of another prime, and another distinct prime, and x values that are the product of the square of a prime and three other distinct primes (corresponding to Curve #17) for x less than 2,000,000. For a normal probability fit of these 129,128 values, the mean is 1.5749 with a 95% confidence interval of (1.5633, 1.5866) and the standard deviation is 2.1377 with a 95% confidence interval of (2.1295, 2.1460). The corresponding F^{-1} values are also shown in the above plot. The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -7.7550 and 8.6872 respectively and the minimum and maximum F^{-1} values are -5.2949 and 8.4447 respectively. 99.64299% of the differences are between -0.29876 and 0.2426 . See Figure (71) for a histogram (having 10 bins) of the differences. The curve of sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values has almost the same shape for smaller sample sizes. See Figure (72) for a plot of the sorted $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values at the above x values for x less than 100,000. For a normal probability fit of these 5,400 values, the mean is 1.4882 with a 95% confidence interval of (1.4281, 1.5483) and the standard deviation is 2.2540 with a 95% confidence interval of (2.2123, 2.2973). The minimum and maximum $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ values are -6.9938 and 8.3399 respectively and the minimum and maximum F^{-1} values

are -5.6589 and 8.6353 respectively. See Figure (73) for a plot of the differences between the two curves. Relatively large differences for a small number of values occur at the beginning of the curve.

The 1,000th element of Curve #17 occurs at $x = 22,236$ and the mean and standard deviation of $\sum_{i|x}(\theta_{i+1} - \theta_i)\mu(i)$ for corresponding x values less than or equal to 22,236 are 1.4933 and 2.3179 respectively (within 95% confidence intervals). The 2,000th element occurs at $x = 40,612$, the 3,000th element occurs at $x = 58,580$, ..., and the 100,000th element occurs at $x = 1,563,925$. See Figure (74) for a plot of these x values. For a quadratic least-squares fit of the curve, $p_1 = -7.128$ with a 95% confidence interval of $(-7.642, -6.613)$, $p_2 = 16,170$ with a 95% confidence interval of $(16,120, 16,220)$, $p_3 = 14,940$ with a 95% confidence interval of $(13,770, 16,120)$, $SSE=3.619 \cdot 10^8$, $R\text{-square}=1$, and $RMSE=1,932$. These sample sizes are then suitable for comparing means and standard deviations of the distributions. Denote these means and standard deviations by $\mu(i)$ and $\sigma(i)$, $i = 1, 2, 3, \dots, 100$. See Figure (75) for a plot of $\mu(i)$ for $i = 1, 2, 3, \dots, 100$. After $i = 3$, the means approximately increase at a rate that is proportional to the logarithm. See Figure (76) for a plot of $(1/\sigma(i) - 1/\sigma(1)) \cdot 107.18$ and $\log(i)$. The standard deviations approximately decrease at a rate that is proportional to the logarithm.

6 Differences Between Zeros and the Probit Function

Let s_x denote $\theta_{x+1} - \theta_x$. For a normal probability fit of s_x where x is less than or equal to 10,000, the mean is 0.9865 with a 95% confidence interval of $(0.9779, 0.9950)$ and the standard deviation is 0.4377 with a 95% confidence interval of $(0.4317, 0.4439)$. See Figure (77) for a plot of the sorted s_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.9865$ and $\sigma = 0.4377$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum s_x values are 0.037699 and 6.887315 and the minimum and maximum F^{-1} values are -0.4102 and 2.3832. See Figure (78) for a plot of the means of the normal probability fits of the s_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The means approximately decrease at a rate that is proportional to the reciprocal of the logarithm. See Figure (79) for a plot of the reciprocals of the means and $\log(x) \cdot 0.1415 + 1.0/1.4067$ for $x = 1, 2, 3, \dots, 100$. See Figure (80) for a plot of the standard deviations of the normal probability fits of the s_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The standard deviations approximately decrease at a rate that is proportional to the reciprocal of the logarithm. See Figure (81) for a plot of reciprocals of the standard deviations and $\log(x) \cdot 0.361 + 1.0/0.6569$ for $x = 1, 2, 3, \dots, 100$. See Figure (82) for a plot of the minimum F^{-1} values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The minimum F^{-1} values approximately increase at a rate that is proportional to the logarithm. See Figure (83) for a plot of the

maximum F^{-1} values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The maximum F^{-1} values approximately decrease at a rate that is proportional to the reciprocal of the logarithm. See Figure (84) for a plot of the minimum s_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The maximum s_x value is 6.887315. For a normal probability fit of s_x where x is less than or equal to 2,000,000, the mean is 0.5660 with a 95% confidence interval of (0.5656, 0.5663) and the standard deviation is 0.2389 with a 95% confidence interval of (0.2387, 0.2391). See Figure (85) for a plot of the sorted s_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.5660$ and $\sigma = 0.2389$) for $p = 0.0000005, 0.0000010, 0.0000015, \dots, 0.9999995$. The minimum and maximum s_x values are 0.002959 and 6.887315 and the minimum and maximum F^{-1} values are -0.2022 and 1.3342 . For a normal probability fit of s_x where x is less than or equal to 2,000,000 and s_x values greater than 2.0 are excluded, the mean is 0.5657 with a 95% confidence interval of (0.5654, 0.5661) and the standard deviation is 0.2380 with a 95% confidence interval of (0.2377, 0.2382). See Figure (86) for a plot of the sorted s_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.5657$ and $\sigma = 0.2380$) for $p = 0.00000050006, 0.00000100012, 0.00000150018, \dots, 0.99999990$. The minimum and maximum s_x values are 0.002959 and 1.997197 and the minimum and maximum F^{-1} values are -0.1996 and 1.3310 . There appear to be only 242 s_x values greater than 2.0.

Let t_x denote $\theta'_{x+1} - \theta'_x$. For a normal probability fit of t_x where x is less than or equal to 10,000, the mean is 0.1069 with a 95% confidence interval of (0.1057, 0.1080) and the standard deviation is 0.0603 with a 95% confidence interval of (0.0595, 0.0611). See Figure (87) for a plot of the sorted t_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.1069$ and $\sigma = 0.0603$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum t_x values are 0.003777 and 1.565886 and the minimum and maximum F^{-1} values are -0.0855 and 0.2993 . See Figure (88) for a plot of the means of the normal probability fits of the t_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The means approximately decrease at a rate that is proportional to the reciprocal of the logarithm squared. See Figure (89) for a plot of the reciprocals of the means and $\log(x) \cdot \log(x) \cdot 0.31 + 1.0/0.1904$ for $x = 1, 2, 3, \dots, 100$. See Figure (90) for a plot of the standard deviations of the normal probability fits of the t_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The standard deviations approximately decrease at a rate that is proportional to the reciprocal of the logarithm squared. See Figure (91) for a plot of reciprocals of the standard deviations and $\log(x) \cdot \log(x) \cdot 0.74 + 1.0/0.1164$ for $x = 1, 2, 3, \dots, 100$. See Figure (92) for a plot of the minimum F^{-1} values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. See Figure (93) for a plot of the maximum F^{-1} values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. See Figure (94) for a plot of the minimum t_x values for $x = 1,000, 2,000, 3,000, \dots, 100,000$. The maximum t_x value is 1.565886. For a normal probability fit of t_x where x is less than or equal to 2,000,000, the mean is 0.0406 with a 95% confidence interval of (0.0406, 0.0406) and the standard deviation is 0.0189 with a 95% confidence interval of (0.0189, 0.0189). See Figure (95) for a plot of the sorted t_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.0406$ and $\sigma = 0.0189$) for $p = 0.0000005$,

0.0000010, 0.0000015, ..., 0.9999995. The minimum and maximum t_x values are 0.000204 and 1.565886 and the minimum and maximum F^{-1} values are -0.0202 and 0.1014 . For a normal probability fit of t_x where x is less than or equal to 2,000,000 and t_x values greater than 0.4 are excluded, the mean is 0.0406 with a 95% confidence interval of (0.0406, 0.0406) and the standard deviation is 0.0187 with a 95% confidence interval of (0.0187, 0.0187). See Figure (96) for a plot of the sorted t_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 0.0406$ and $\sigma = 0.0187$) for $p = 0.00000050001, 0.00000100002, 0.00000150003, \dots, 0.999999990$. The minimum and maximum t_x values are 0.000204 and 0.394989 and the minimum and maximum F^{-1} values are -0.0195 and 0.1007 . There appear to be only 42 t_x values greater than 0.4.

7 Generalized Ramanujan Sums Involving θ

In the following, the values of both f and g (in the expression $\sum_{d|(n,k)} f(d)g(k/d)$) are set to the θ values. Let $r(x, k)$ denote this convolution. The same curves occur as for the Dirichlet products of θ , but the upper bound of the number of distinct (non-overlapping) curves is determined by the number of divisors of k . (See Figure (104) for a plot of the 5,627 $r(x, 12)$ values at x values in Curve #5, $x \leq 100,000$. There are three curves instead of the possible six. See Figure (105) for a plot of the 22,181 $r(x, 30)$ values at x values in Curve #7, $x \leq 100,000$. There are seven curves instead of the possible eight.) See Figure (106) for a plot of $r(x, 11)$ for x less than or equal to 100,000. See Figure (107) for a plot of the sorted $r(x, 11)$ values. The tail at the end of the curve consists of $r(x, 11)$ values at x values that are a multiple of 11. For a quadratic least-squares fit of the curve (including the tail), $p_1 = 0.0007672$ with a 95% confidence interval of (0.0006829, 0.0008515), $p_2 = 680.2$ with a 95% confidence interval of (679.3, 681.1), $p_3 = 7975$ with a 95% confidence interval of (6090, 9860), $SSE=1.027 \cdot 10^{13}$, $R\text{-square}=0.9997$, and $RMSE=3.205 \cdot 10^4$. Similar curves are obtained for prime k values. See Figure (108) for a plot of $r(x, 5)$ for x less than or equal to 100. Discounting the x values that are a multiple of 5, the values increase almost linearly in groups of 4. See Figure (109) for a plot of $r(x, 6)$ for x less than or equal to 10,000. See Figure (110) for a plot of the moduli of the Fourier transform of $r(x, 6)$. See Figure (111) for a plot of the arguments of the Fourier transform of $r(x, 6)$. In general, there are k relatively large spikes in the moduli other than those at the start and end of the curve. See Figure (112) for a plot of the first 50 moduli and Figure (113) for a plot of the last 50 moduli. There is then no significant “information” in this convolution other than the k spikes.

Let $s(x, k)$ denote the above convolution when both f and g are set to the θ' values. See Figure (114) for a plot of $s(x, 13)$ for x less than or equal to 10,000. See Figure (115) for a plot of the sorted $s(x, 13)$ values. For a linear least-squares fit of the curve (including the tail), $p_1 = 72.12$ with a 95% confidence interval of (72.1, 72.14) and $p_2 = -292.33$ with a 95% confidence in-

terval of $(-388.4, -196.1)$, $SSE=6.011 \cdot 10^8$, $R\text{-square}=0.999$, and $RMSE=2452$. Curves with a better linear least-squares fit can be obtained by using larger prime k values. See Figure (116) for a plot of $s(x, 11)$ for x less than or equal to 100. See Figure (117) for a plot of $s(x, 30)$ for x less than or equal to 10,000. See Figure (118) for a plot of the moduli of the Fourier transform of $s(x, 30)$. See Figure (119) for a plot of the arguments of the Fourier transform of $s(x, 30)$.

8 Dirichlet Inverses of θ

The inverse Dirichlet product of an arithmetical function f where $f(1) \neq 0$ is given by the recursion formula $f^{-1}(1) = 1/f(1)$, $f^{-1}(n) = (-1/f(1)) \sum_{d|n, d < n} f(n/d)f^{-1}(d)$ for $n > 1$. Let y_x denote the Dirichlet inverse of θ_x . For a normal probability fit of y_x for x less than or equal to 9,999, the mean is -1.7016 with a 95% confidence interval of $(-1.9293, -1.4740)$ and the standard deviation is 11.6133 with a 95% confidence interval of $(11.4546, 11.7766)$. See Figure (120) for a plot of the sorted y_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = -1.7016$ and $\sigma = 11.6133$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum y_x values are -49.3269 and 31.6628 and the minimum and maximum F^{-1} values are -38.7599 and 35.3567 . In general, the minimum y_x value is less than the minimum F^{-1} value and the maximum y_x value is less than the maximum F^{-1} value. See Figure (121) for a plot of the means of the distributions of y_x for x less than or equal to 1,000, 2,000, 3,000, ..., 100,000. For a cubic least-squares fit of the means, $p_1 = -2.165 \cdot 10^{-6}$ with a 95% confidence interval of $(-2.433 \cdot 10^{-6}, -1.896 \cdot 10^{-6})$, $p_2 = 0.0005025$ with a 95% confidence interval of $(0.0004612, 0.0005437)$, $p_3 = -0.1513$ with a 95% confidence interval of $(-0.1531, -0.1495)$, $p_4 = -0.198$ with a 95% confidence interval of $(-0.2191, -0.1769)$, $SSE=0.06266$, $R\text{-square}=0.9999$, and $RMSE=0.02555$. The curve of means is almost quadratic. See Figure (122) for a plot of the standard deviations of the distributions of y_x for x less than or equal to 1,000, 2,000, 3,000, ..., 100,000. For a cubic least-squares fit of the standard deviations, $p_1 = 1.543 \cdot 10^{-5}$ with a 95% confidence interval of $(1.38 \cdot 10^{-5}, 1.706 \cdot 10^{-5})$, $p_2 = -0.003588$ with a 95% confidence interval of $(-0.003837, -0.003338)$, $p_3 = 0.9721$ with a 95% confidence interval of $(0.9612, 0.9829)$, $p_4 = 1.991$ with a 95% confidence interval of $(1.864, 2.119)$, $SSE=2.299$, $R\text{-square}=1$, and $RMSE=0.1548$. The curve of standard deviations is almost quadratic. The curves of minimum and maximum y_x values are also cubic.

Let y'_x denote the Dirichlet inverse of θ'_x . For a normal probability fit of y'_x for x less than or equal to 9,999, the mean is -1.9315 with a 95% confidence interval of $(-2.1039, -1.7591)$ and the standard deviation is 8.7933 with a 95% confidence interval of $(8.6731, 8.9169)$. See Figure (123) for a plot of the sorted y'_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = -1.9315$ and $\sigma = 8.7933$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum y'_x values are -37.6306 and 19.8090 and the minimum and maximum F^{-1} val-

ues are -29.9911 and 26.1281 . In general, the minimum y'_x value is less than the minimum F^{-1} value and the maximum y'_x value is less than the maximum F^{-1} value. See Figure (124) for a plot of the means of the distributions of y'_x for x less than or equal to 1,000, 2,000, 3,000, ..., 20,000. For a quadratic least-squares fit of the means, $p_1 = 0.001844$ with a 95% confidence interval of $(0.001564, 0.002124)$, $p_2 = -0.1937$ with a 95% confidence interval of $(-0.1997, -0.1876)$, $p_3 = -0.1805$ with a 95% confidence interval of $(-0.2081, -0.1529)$, SSE=0.005255, R-square=0.9997, and RMSE=0.01758. See Figure (125) for a plot of the standard deviations of the distributions of y'_x for x less than or equal to 1,000, 2,000, 3,000, ..., 20,000. For a quadratic least-squares fit of the standard deviations, $p_1 = -0.008785$ with a 95% confidence interval of $(-0.01017, -0.007401)$, $p_2 = 0.8265$ with a 95% confidence interval of $(0.7965, 0.8564)$, $p_3 = 1.412$ with a 95% confidence interval of $(1.276, 1.548)$, SSE=0.1285, R-square=0.9995, and RMSE=0.08693.

Let z_x denote the Dirichlet inverse of $\theta_{x+2} - \theta_x$. For a normal probability fit of z_x for x less than or equal to 9,999, the mean is -0.00053505 with a 95% confidence interval of $(-0.0010, 0)$ and the standard deviation is 0.0249 with a 95% confidence interval of $(0.0245, 0.0252)$. See Figure (126) for a plot of z_x for x less than or equal to 9,999. See Figure (127) for a plot of the sorted z_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = -0.00053505$ and $\sigma = 0.0249$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum z_x values are -0.2699 and 0.3007 and the minimum and maximum F^{-1} values are -0.08 and 0.0789 . In general, the z_x values bound the F^{-1} values. See Figure (128) for a plot of the means of the distributions of z_x for x less than or equal to 1,000, 2,000, 3,000, ..., 60,000. Other than being small, there isn't much pattern to the means. See Figure (129) for a plot of the standard deviations of the distributions of z_x for x less than or equal to 1,000, 2,000, 3,000, ..., 60,000. The standard deviations slowly decrease. See Figure (130) for a plot of the reciprocals of the standard deviations and $\log(x) \cdot 0.3 + 1.0/0.0283$ for $x = 1, 2, 3, \dots, 60$. The two curves appear to increase at about the same rate.

For x less than or equal to 1,000,000, the maximum z_x values increase at $x = 390, 2,340, 6,930, 13,260, 30,030, 46,410, 79,560, 119,340,$ and $180,180$. The prime factorizations of these values are $2 \cdot 3 \cdot 5 \cdot 13, 2^2 \cdot 3^2 \cdot 5 \cdot 13, 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, 2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 17, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17, 2^3 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17, 2^2 \cdot 3^3 \cdot 5 \cdot 13 \cdot 17,$ and $2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. The number of prime factors (counting powers of primes) is non-decreasing and even. The corresponding maximum z_x values are $0.091944, 0.148888, 0.212224, 0.300685, 0.359688, 0.431405, 0.437987, 0.547892, 0.682205,$ and 1.091032 . See Figure (131) for a plot of the maxima of the z_x distributions for x less than or equal to 5,000.

For x less than or equal to 1,000,000, the minimum z_x values decrease at $x = 2, 78, 630, 780, 1,170, 2,310, 2,730, 6,630, 16,380, 25,740, 39,780, 90,090, 360,360, 540,540,$ and $900,900$. The prime factorizations of these values are $2, 2 \cdot 3 \cdot 13, 2 \cdot 3^2 \cdot 5 \cdot 7, 2^2 \cdot 3 \cdot 5 \cdot 13, 2 \cdot 3^2 \cdot 5 \cdot 13, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13, 2 \cdot 3 \cdot 5 \cdot 13 \cdot 17,$

$2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, $2^2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13$, $2^2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17$, $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, $2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, and $2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. The number of prime factors (counting powers of primes) is non-decreasing and odd. The corresponding minimum z_x values are 0.091944, -0.079490 , -0.086033 , -0.095726 , -0.133309 , -0.168437 , -0.193491 , -0.205926 , -0.269875 , -0.317359 , -0.339756 , -0.615993 , -0.715986 , -0.913493 , -1.359183 , and -1.385499 . See Figure (132) for a plot of the minima of the z_x distributions for x less than or equal to 5,000.

Let z'_x denote the Dirichlet inverse of $\theta'_{x+2} - \theta'_x$. For a normal probability fit of z'_x for x less than or equal to 9,999, the mean is -0.0013 with a 95% confidence interval of $(-0.0025, -0.0001)$ and the standard deviation is 0.0614 with a 95% confidence interval of $(0.0605, 0.0622)$. See Figure (133) for a plot of z'_x for x less than or equal to 9,999. See Figure (134) for a plot of the sorted z'_x values and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = -0.0013$ and $\sigma = 0.0614$) for $p = 0.0001, 0.0002, 0.0003, \dots, 0.9999$. The minimum and maximum z'_x values are -0.652787 and 0.721099 and the minimum and maximum F^{-1} values are -0.1972 and 0.1946 . In general, the z'_x values bound the F^{-1} values. See Figure (135) for a plot of the means of the distributions of z'_x for x less than or equal to 1,000, 2,000, 3,000, ..., 60,000. Other than being small, there isn't much pattern to the means. See Figure (136) for a plot of the standard deviations of the distributions of z'_x for x less than or equal to 1,000, 2,000, 3,000, ..., 60,000. The standard deviations slowly decrease. See Figure (137) for a plot of the reciprocals of the standard deviations and $\log(x) \cdot 1.925726 + 1.0/0.0853$ for $x = 1, 2, 3, \dots, 60$. The standard deviations are accurately modeled.

For x less than or equal to 1,000,000, the maximum z'_x values increase at $x = 390, 2,340, 6,930, 13,260, 30,030, 80,010, 119,340$, and $180,180$. The number of prime factors (counting powers of primes) is non-decreasing and even. The corresponding maximum z'_x values are 0.442263, 0.548131, 0.721099, 0.813133, 0.958113, 1.082810, 1.345843, and 2.081319.

For x less than or equal to 1,000,000, the minimum z'_x values decrease at $x = 2, 780, 1,170, 2,310, 2,730, 6,630, 16,380, 25,740, 39,780, 90,090, 360,360, 510,510, 540,540$, and $900,900$. The number of prime factors (counting powers of primes) is non-decreasing and odd. The corresponding minimum z'_x values are $-0.339049, -0.369013, -0.466524, -0.508299, -0.542218, -0.652787, -0.731160, -0.768962, -1.295561, -1.468219, -1.623613, -1.632653, -2.428308$, and -2.471748 .

9 Curves in the Dirichlet Inverses of θ

See Figure (138) for a plot of y_x (the Dirichlet inverse of θ_x) for x less than or equal to 1000. The curves and sub-curves as defined for the Dirichlet products of θ_x are applicable to the Dirichlet inverses of $\theta_x, \theta_{x+1} - \theta_x, \theta'_x$, and $\theta'_{x+1} - \theta'_x$. See Figure (139) for a plot of y_x at x values of 2, 3, 5, ..., 99,991

(the 9,592 primes less than 100,000). For a linear least-squares fit of the values, $p_1 = -0.039$ with a 95% confidence interval of $(-0.039, -0.03899)$, $p_2 = -1.77$ with a 95% confidence interval of $(-1.789, -1.751)$, $SSE=2152$, $R\text{-square}=1$, and $RMSE=0.4737$. See Figure (140) for a plot of y_x at x values of $2^2, 3^2, 5^2, \dots, 313^2$ (the 65 squares of primes less than 100,000). For a quadratic least-squares fit of the values, $p_1 = -0.07464$ with a 95% confidence interval of $(-0.07697, -0.07231)$, $p_2 = 0.9818$ with a 95% confidence interval of $(0.823, 1.141)$, $p_3 = -3.952$ with a 95% confidence interval of $(-6.223, -1.682)$, $SSE=543$, $R\text{-square}=0.9986$, and $RMSE=2.959$. See Figure (141) for a plot of y_x at x values corresponding to Curve #3, $x \leq 100,000$. For a normal probability fit of these 23,327 values, the mean is 15.5662 with a 95% confidence interval of $(14.5423, 16.5902)$ and the standard deviation is 79.7863 with a 95% confidence interval of $(79.0689, 80.5170)$. See Figure (142) for a plot of the sorted y_x values and the corresponding F^{-1} values. The minimum and maximum y_x values are -148.7407 and 226.9803 respectively and the minimum and maximum F^{-1} values are -240.1434 and 271.2758 respectively. See Figure (143) for a plot of y_x at x values corresponding to Curve #5, $x \leq 100,000$. For a normal probability fit of these 5,627 values, the mean is -13.2197 with a 95% confidence interval of $(-13.5901, -12.8494)$ and the standard deviation is 14.1716 with a 95% confidence interval of $(13.9146, 14.4384)$. See Figure (144) for a plot of the sorted y_x values and the corresponding F^{-1} values. The minimum and maximum y_x values are -62.9634 and 158.2191 respectively and the minimum and maximum F^{-1} values are -58.1804 and 31.7410 respectively. See Figure (145) for a plot of y_x at x values corresponding to Curve #7, $x \leq 100,000$. For a normal probability fit of these 22,181 values, the mean is 9.0014 with a 95% confidence interval of $(8.6392, 9.3637)$ and the standard deviation is 27.5240 with a 95% confidence interval of $(27.2703, 27.7826)$. See Figure (146) for a plot of the sorted y_x values and the corresponding F^{-1} values. The minimum and maximum y_x values are -79.1304 and 90.6123 respectively and the minimum and maximum F^{-1} values are -79.1961 and 97.1989 respectively. See Figure (147) for a plot of y_x at x values corresponding to Curve #7, $x \leq 2,000,000$. For a normal probability fit of these 434,036 values, the mean is 321.2527 with a 95% confidence interval of $(319.9216, 322.5838)$ and the standard deviation is 447.4374 with a 95% confidence interval of $(446.4981, 448.3806)$. See Figure (148) for a plot of the sorted y_x values and the corresponding F^{-1} values. The minimum and maximum y_x values are -848.9640 and 1962.3 respectively and the minimum and maximum F^{-1} values are -1117.3 and 1759.8 respectively.

The 1,000th element of Curve #7 occurs at $x = 4,922$. The 2,000th element occurs at $x = 9,494$, the 3,000th element occurs at $x = 14,125, \dots$, and the 100,000th element occurs at $x = 446,118$. See Figure (53) for a plot of these x values. Denote the means and standard deviations of y_x for x less than or equal to these values by $\mu(i)$ and $\sigma(i)$, $i = 1, 2, 3, \dots, 100$. See Figure (149) for a plot of $\mu(i)$ for $i = 1, 2, 3, \dots, 100$. For a quadratic least-squares fit of these values, $p_1 = 0.0005875$ with a 95% confidence interval of $(0.0005, 0.000675)$, $p_2 = 0.592$ with a 95% confidence interval of $(0.5828, 0.6011)$, $p_3 = -4.021$ with

a 95% confidence interval of $(-4.22, -3.821)$, $SSE=10.46$, $R\text{-square}=0.9997$, and $RMSE=0.3284$. See Figure (150) for a plot of $\sigma(i)$. For a quadratic least-squares fit of these values, $p_1 = -0.0007492$ with a 95% confidence interval of $(-0.0007934, -0.000705)$, $p_2 = 1.198$ with a 95% confidence interval of $(1.193, 1.202)$, $p_3 = 1.122$ with a 95% confidence interval of $(1.021, 1.222)$, $SSE=2.67$, $R\text{-square}=1$, and $RMSE=0.1659$. See Figure (151) for a plot of the minima of the distributions. For a cubic least-squares fit of these values, $p_1 = -6.414 \cdot 10^{-5}$ with a 95% confidence interval of $(-7.081 \cdot 10^{-5}, -5.747 \cdot 10^{-5})$, $p_2 = 0.01491$ with a 95% confidence interval of $(0.01389, 0.01594)$, $p_3 = -3.497$ with a 95% confidence interval of $(-3.542, -3.453)$, $p_4 = -7.772$ with a 95% confidence interval of $(-8.296, -7.248)$, $SSE=38.71$, $R\text{-square}=0.9999$, and $RMSE=0.635$. See Figure (152) for a plot of the maxima of the distributions. These values have a saw-tooth pattern.

Let z_x denote the Dirichlet inverse of $\theta_{x+1} - \theta_x$. See Figure (153) for a plot of z_x at x values corresponding to Curve #3, $x \leq 100,000$. For a normal probability fit of these 23,327 values, the mean is 0.0028 with a 95% confidence interval of $(0.0027, 0.0030)$ and the standard deviation is 0.0121 with a 95% confidence interval of $(0.0120, 0.0122)$. See Figure (154) for a plot of the sorted z_x values and the corresponding F^{-1} values. The minimum and maximum z_x values are -0.0655 and 0.1127 respectively and the minimum and maximum F^{-1} values are -0.0360 and 0.0416 respectively. See Figure (155) for a plot of z_x at x values corresponding to Curve #5, $x \leq 100,000$. For a normal probability fit of these 5,627 values, the mean is -0.0036 with a 95% confidence interval of $(-0.0040, -0.0032)$ and the standard deviation is 0.0145 with a 95% confidence interval of $(0.0142, 0.0147)$. See Figure (156) for a plot of the sorted z_x values and the corresponding F^{-1} values. The minimum and maximum z_x values are -0.0927 and 0.0521 respectively and the minimum and maximum F^{-1} values are -0.0496 and 0.0424 respectively. See Figure (157) for a plot of z_x at x values corresponding to Curve #7, $x \leq 100,000$. For a normal probability fit of these 22,181 values, the mean is 0.0038 with a 95% confidence interval of $(0.0036, 0.0041)$ and the standard deviation is 0.0186 with a 95% confidence interval of $(0.0185, 0.0188)$. See Figure (158) for a plot of the sorted z_x values and the corresponding F^{-1} values. The minimum and maximum z_x values are -0.1443 and 0.1214 respectively and the minimum and maximum F^{-1} values are -0.0558 and 0.0634 respectively.

The 1,000th element of Curve #7 occurs at $x = 4,922$. The 2,000th element occurs at $x = 9,494$, the 3,000th element occurs at $x = 14,125, \dots$, and the 100,000th element occurs at $x = 446,118$. See Figure (53) for a plot of these x values. Denote the means and standard deviations of z_x for x less than or equal to these values by $\mu(i)$ and $\sigma(i)$, $i = 1, 2, 3, \dots, 100$. See Figure (159) for a plot of $\mu(i)$. The mean decreases at a rate that is proportional to the logarithm. See Figure (160) for a plot of $0.0379929 \cdot (1/\mu(i) - 1/\mu(1))$ and $\log(i)$. See Figure (161) for a plot of $\sigma(i)$. The standard deviation decreases at a rate that is approximately proportional to the logarithm. See Figure (162) for a plot of

$0.128688 \cdot (1/\sigma(i) - \sigma(1))$ and $\log(i)$. The minima of the z_x distributions are greater than -0.1443 and the maxima are less than 0.1214 . The minimum z_x value occurs at $x = 615$ and the maximum z_x value occurs at $x = 24$.

See Cox [5] for the software used to determine the above results.

References

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- [5] Cox, D., “www.darrellcox.website/riemann.htm”

Figure 1

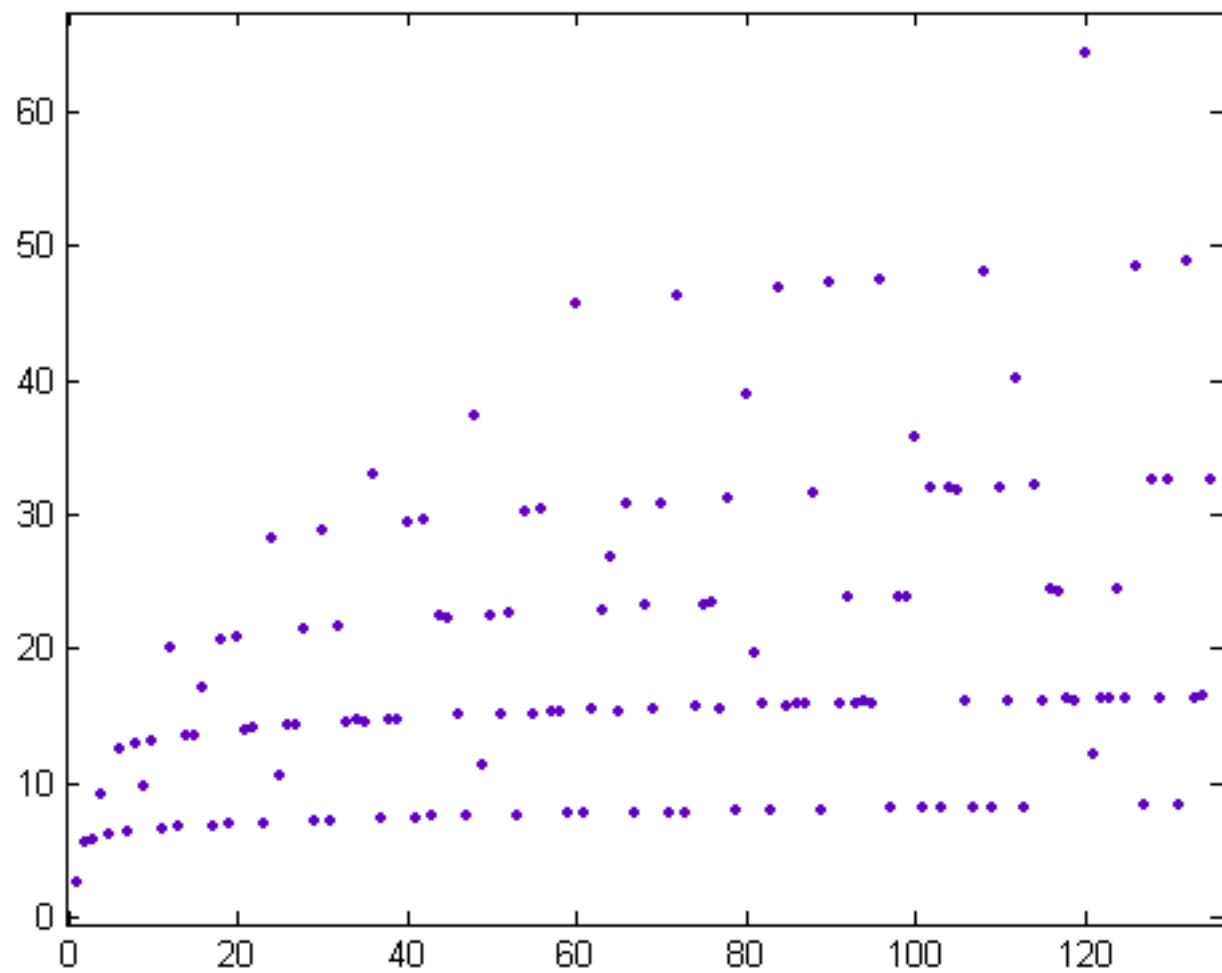


Figure 2

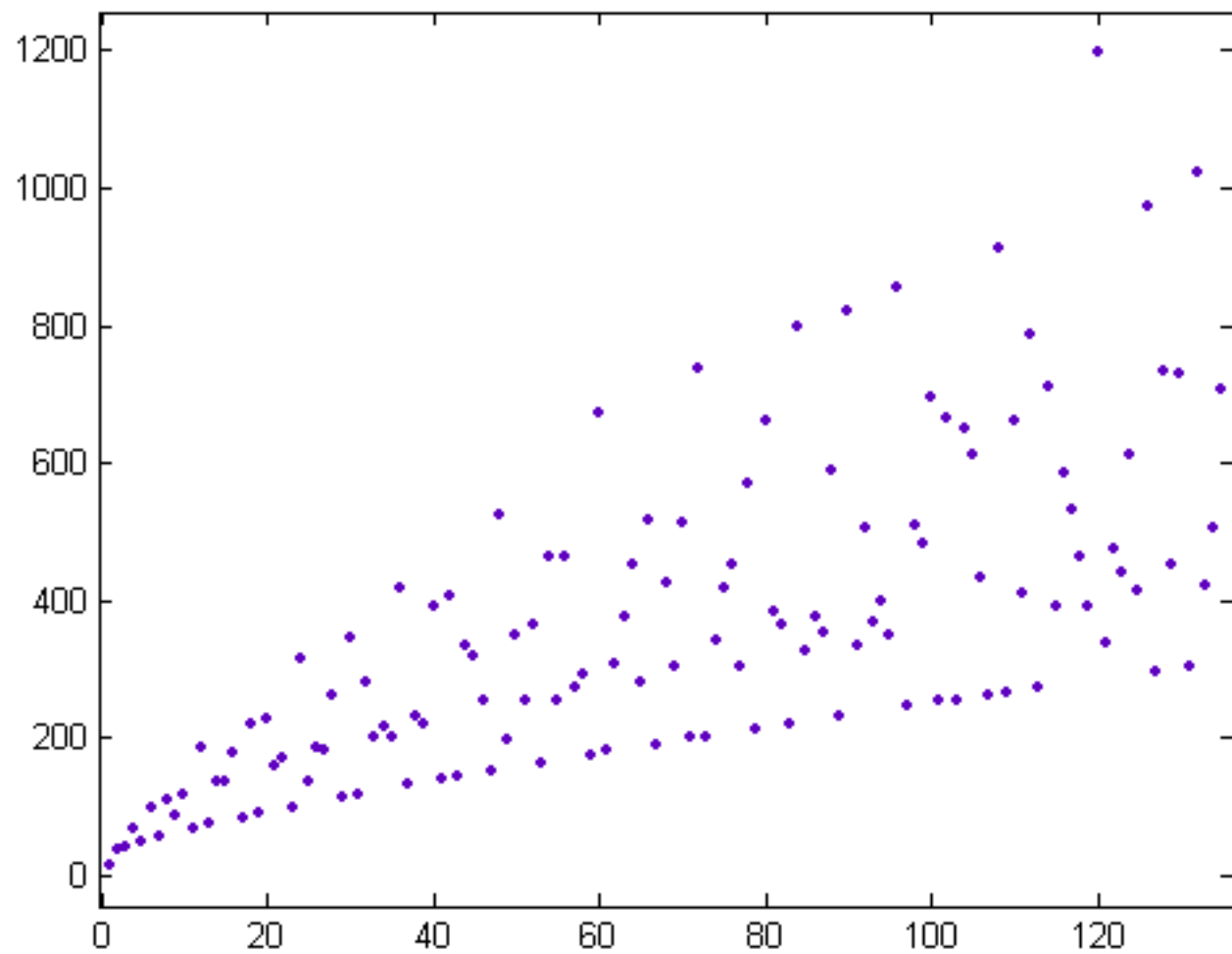


Figure 3

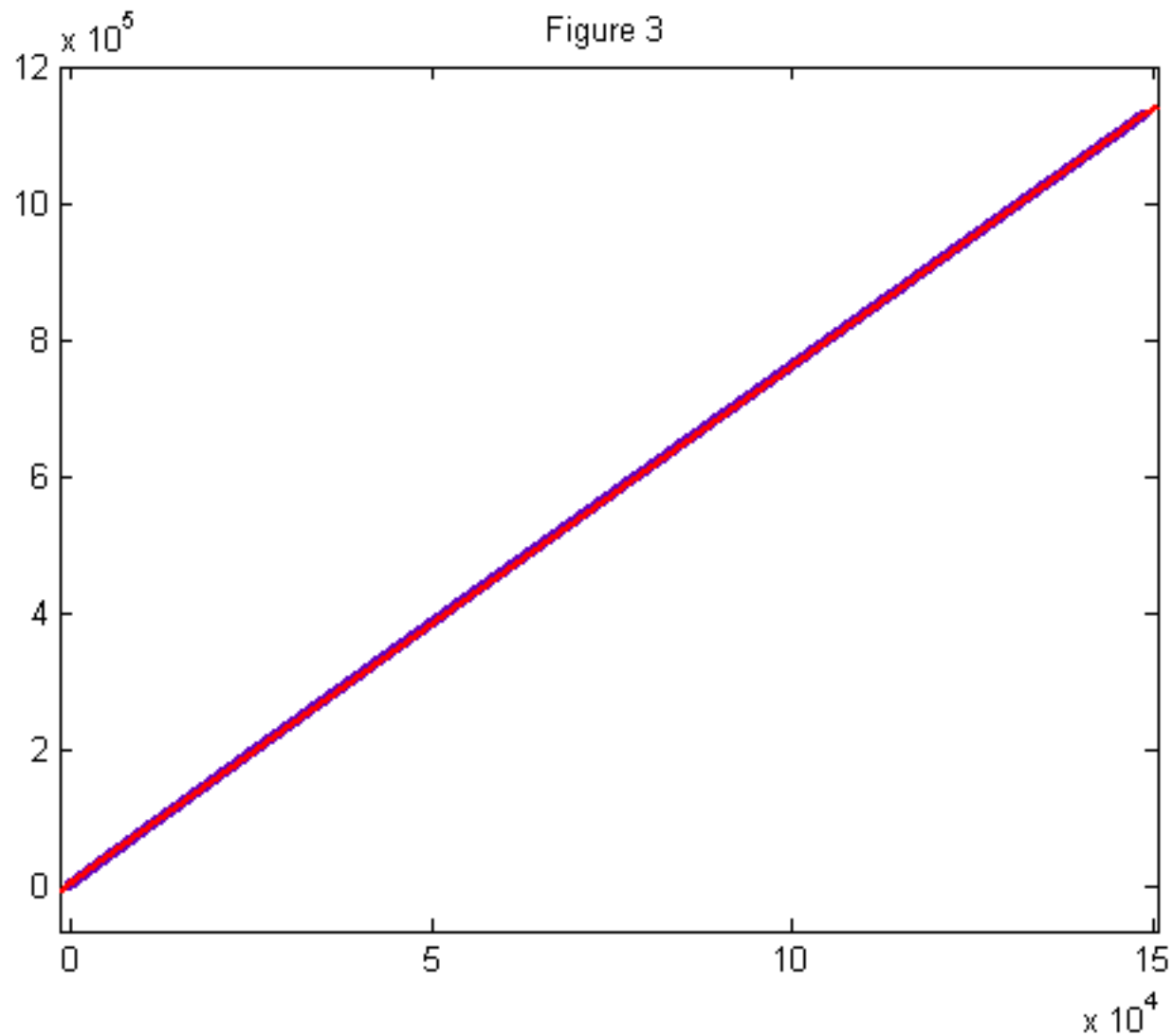


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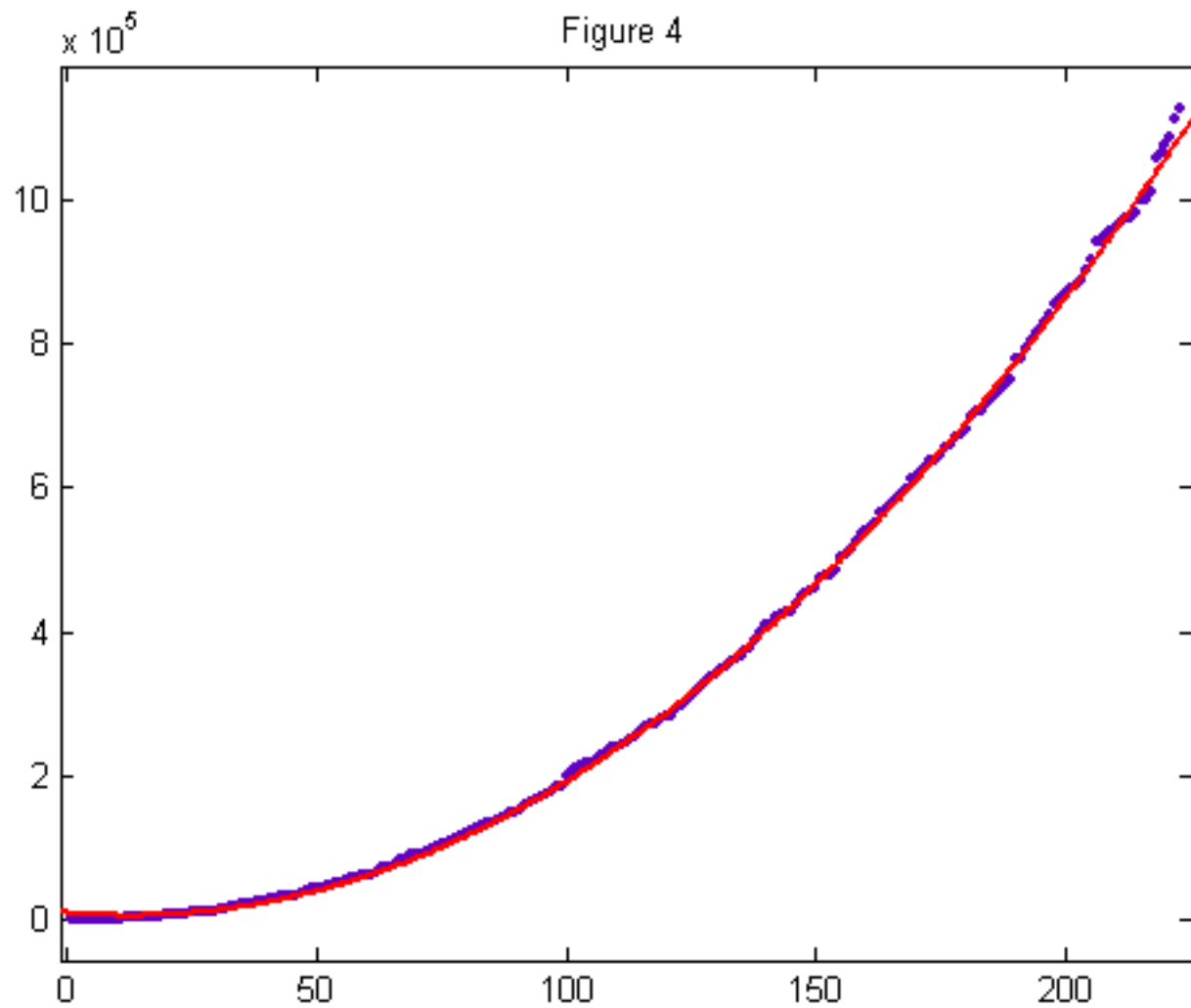


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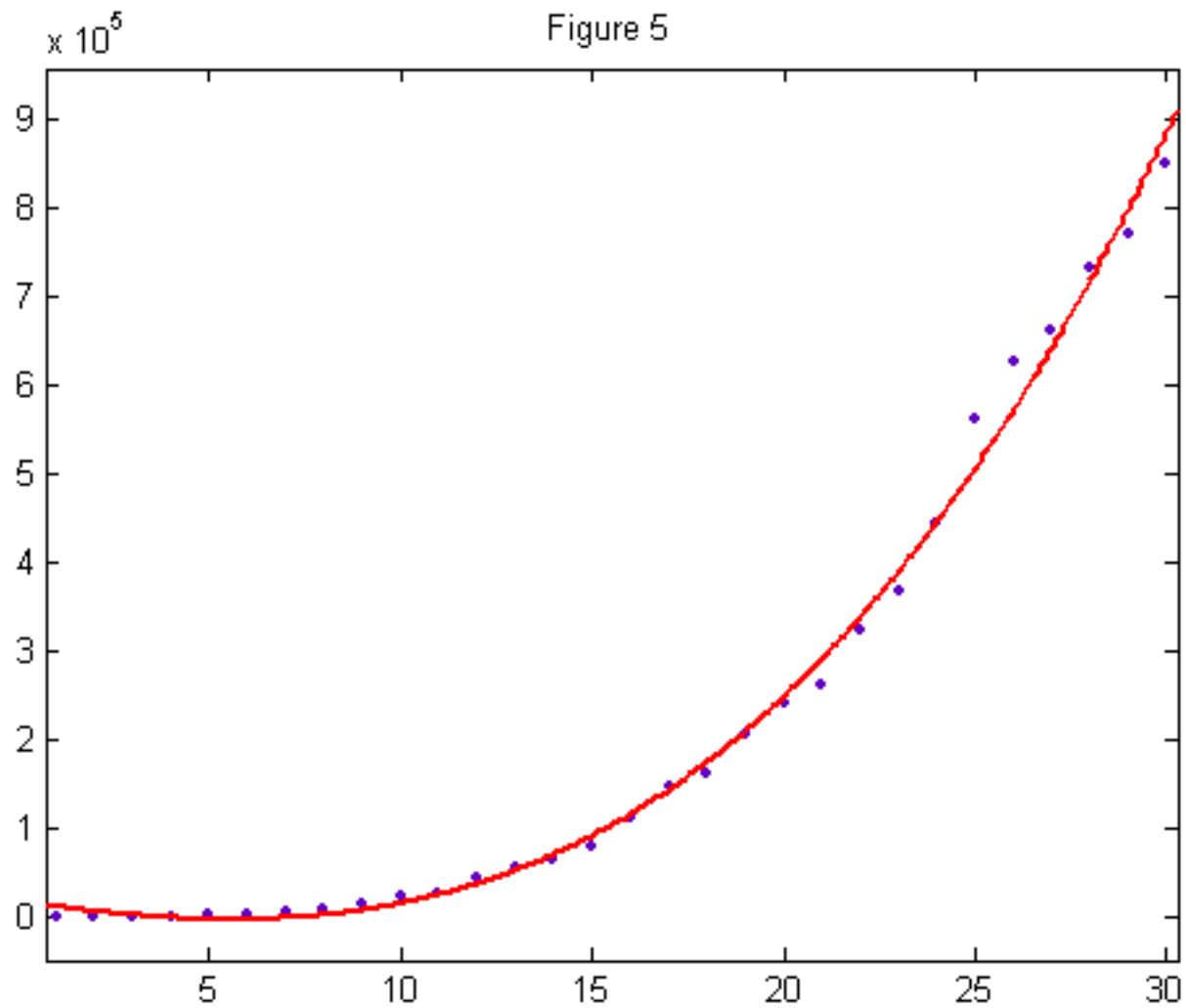


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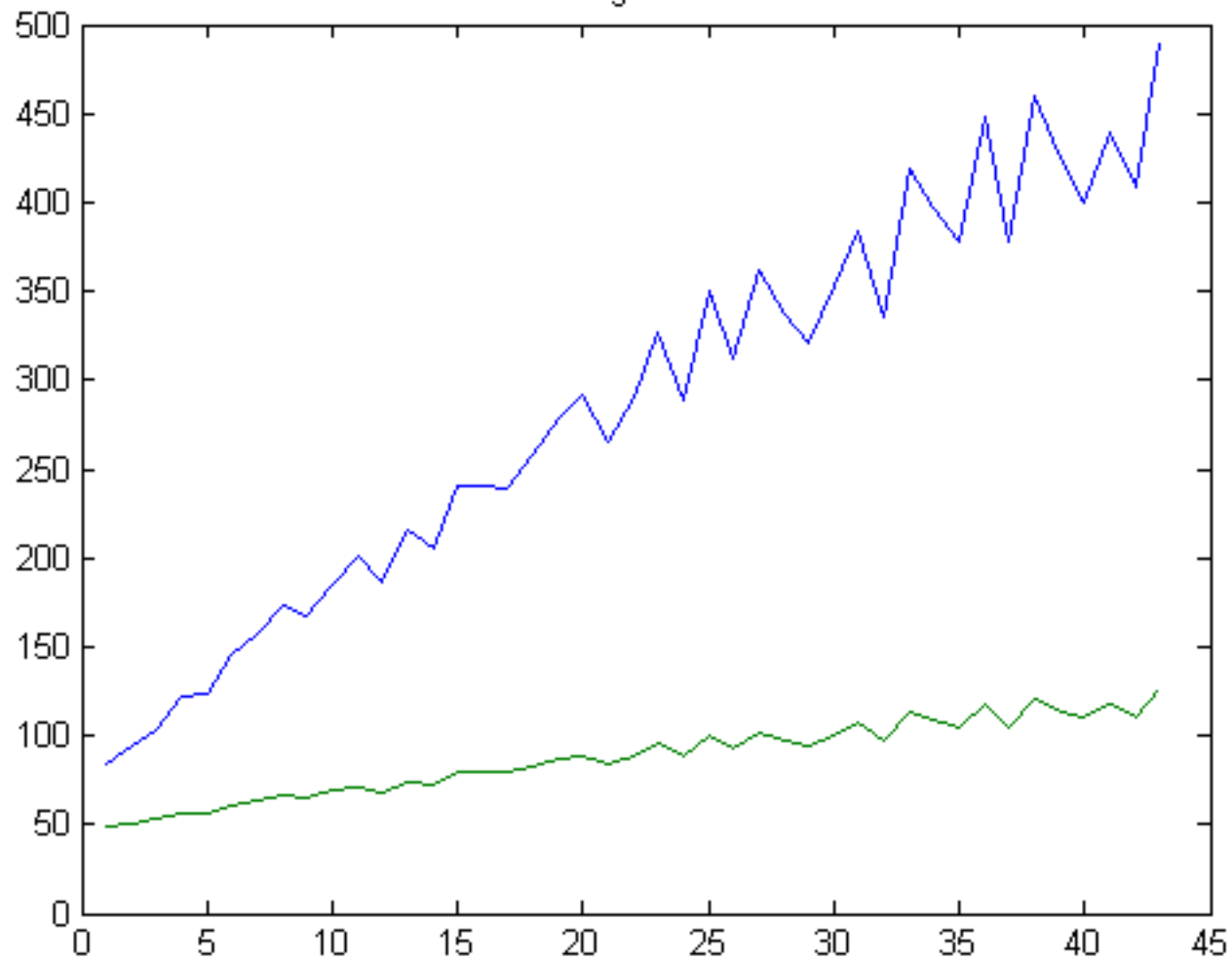


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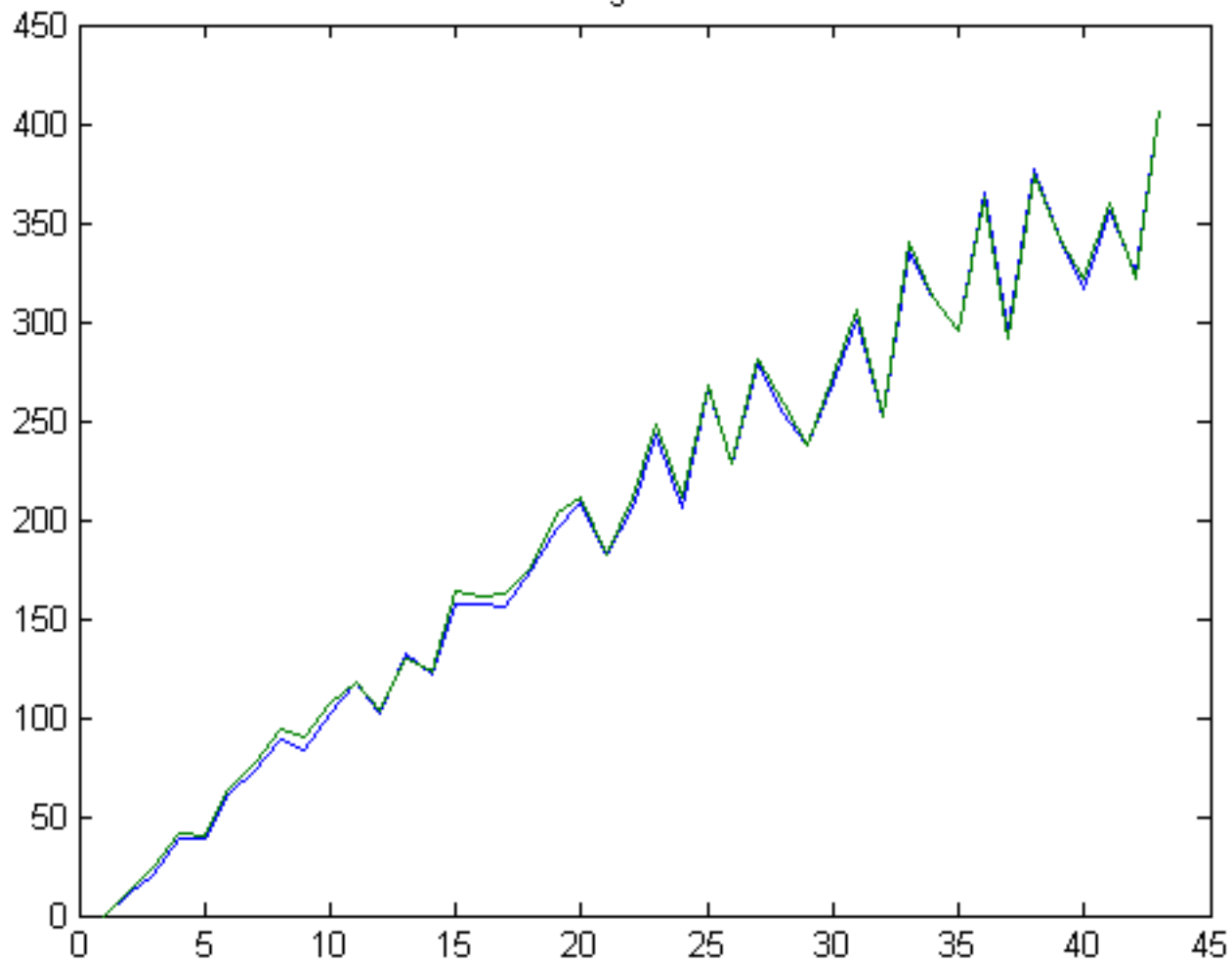


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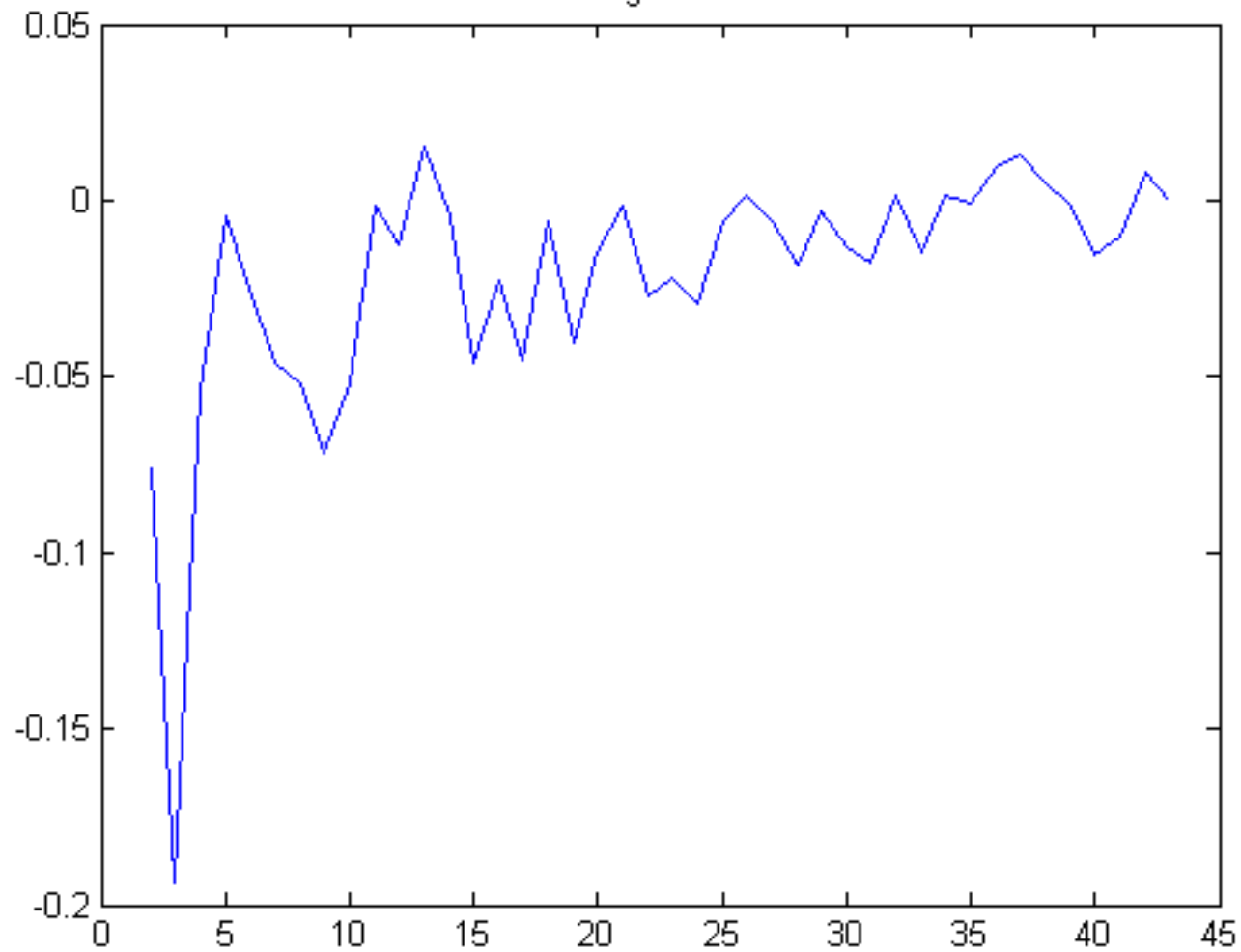


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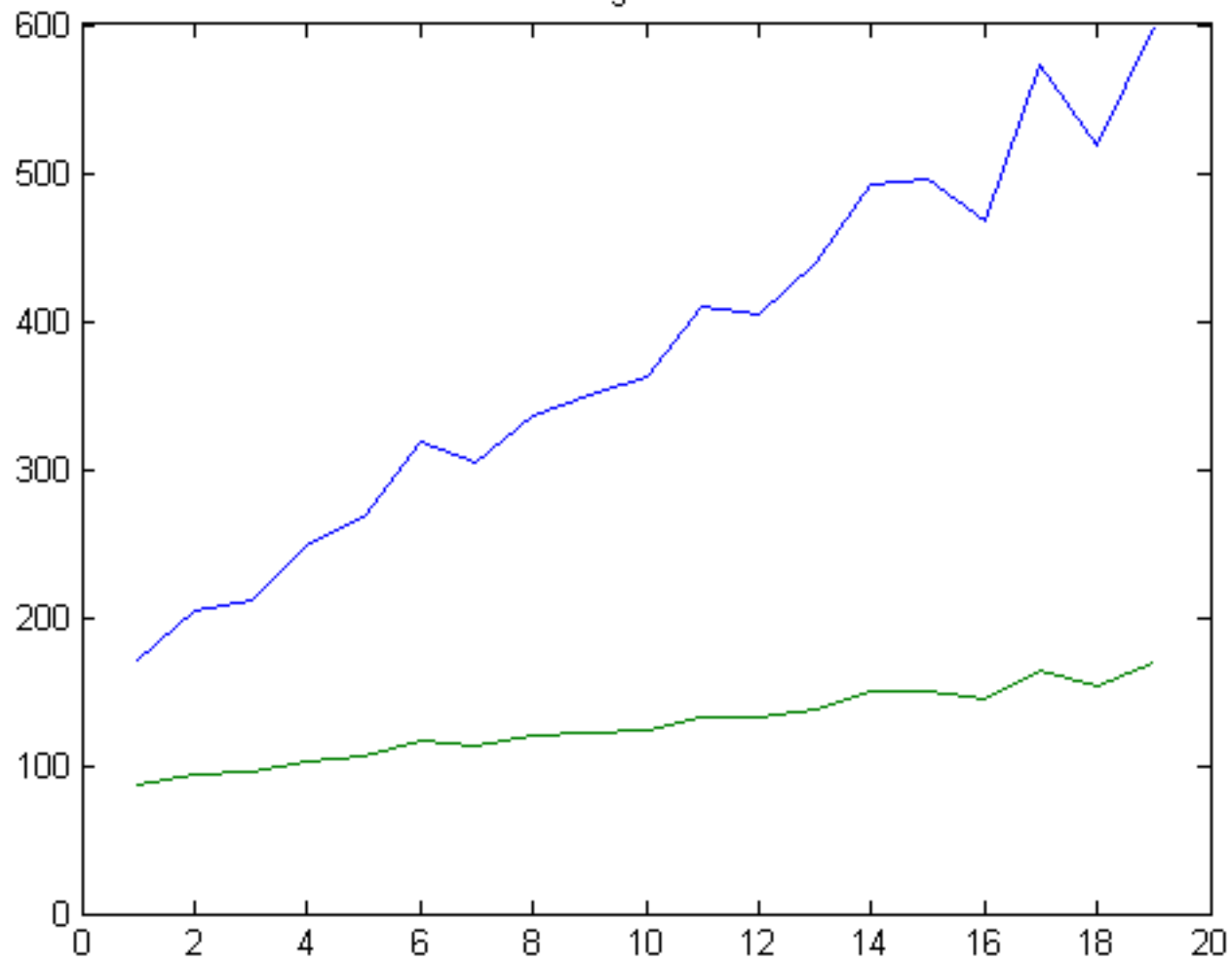


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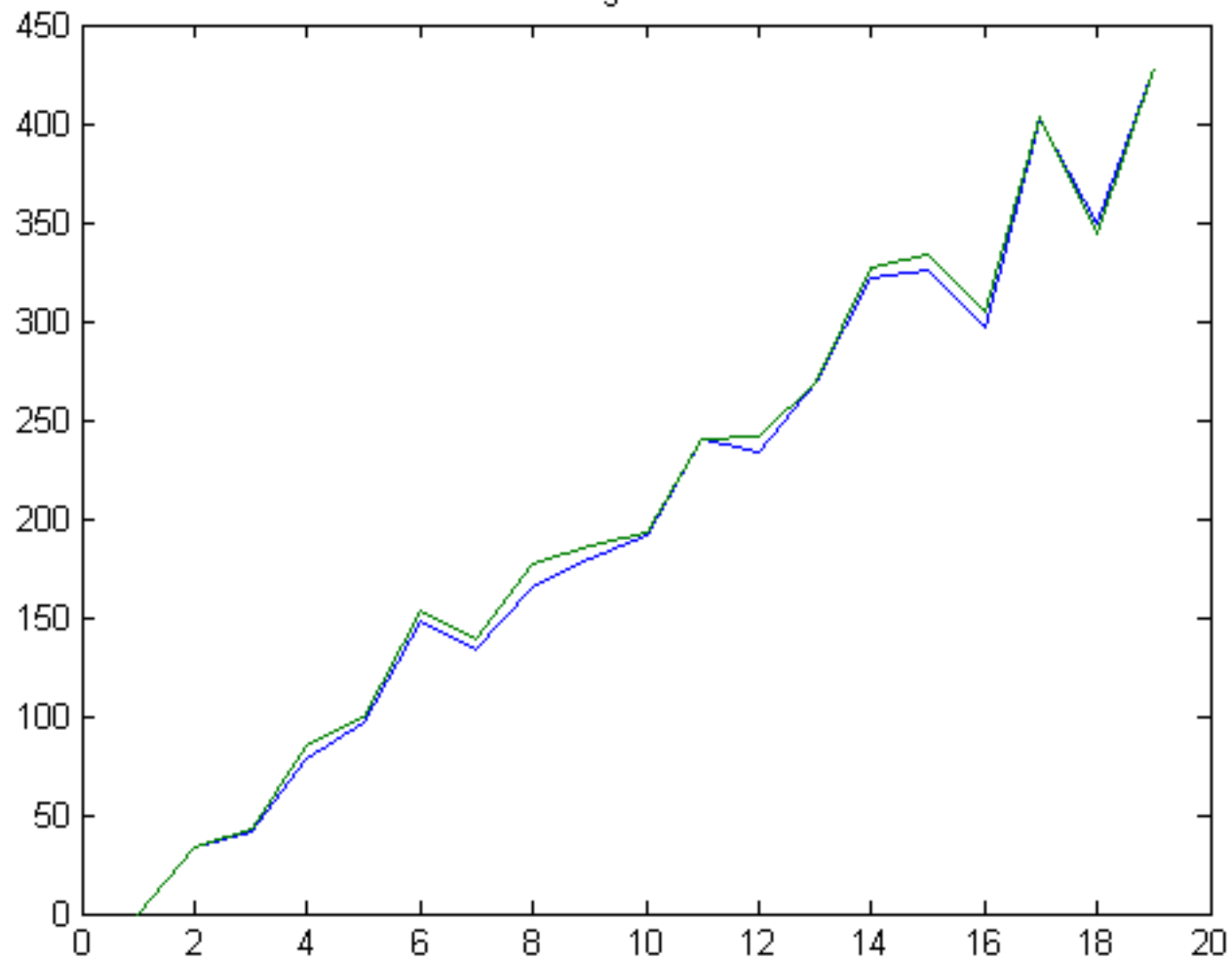


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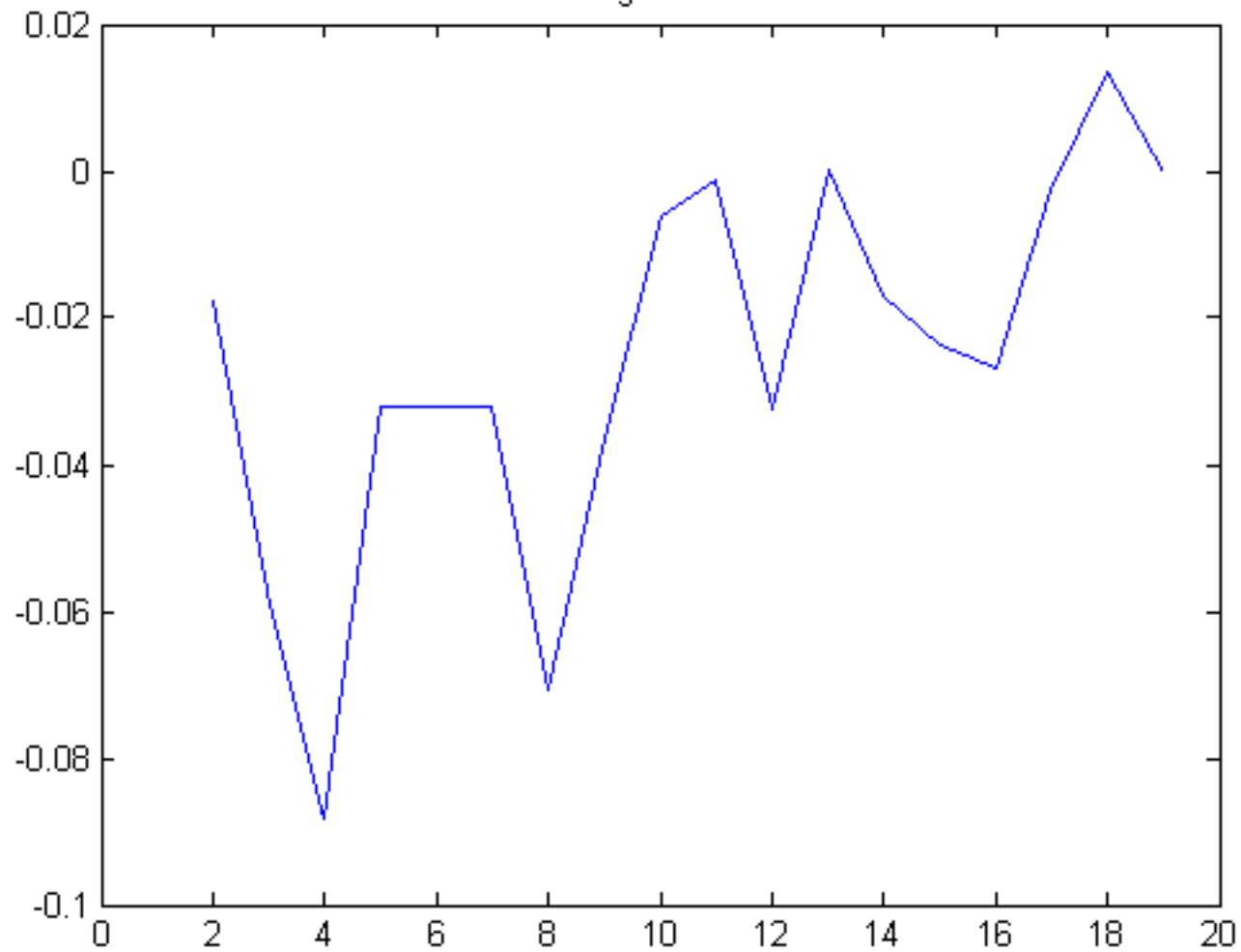


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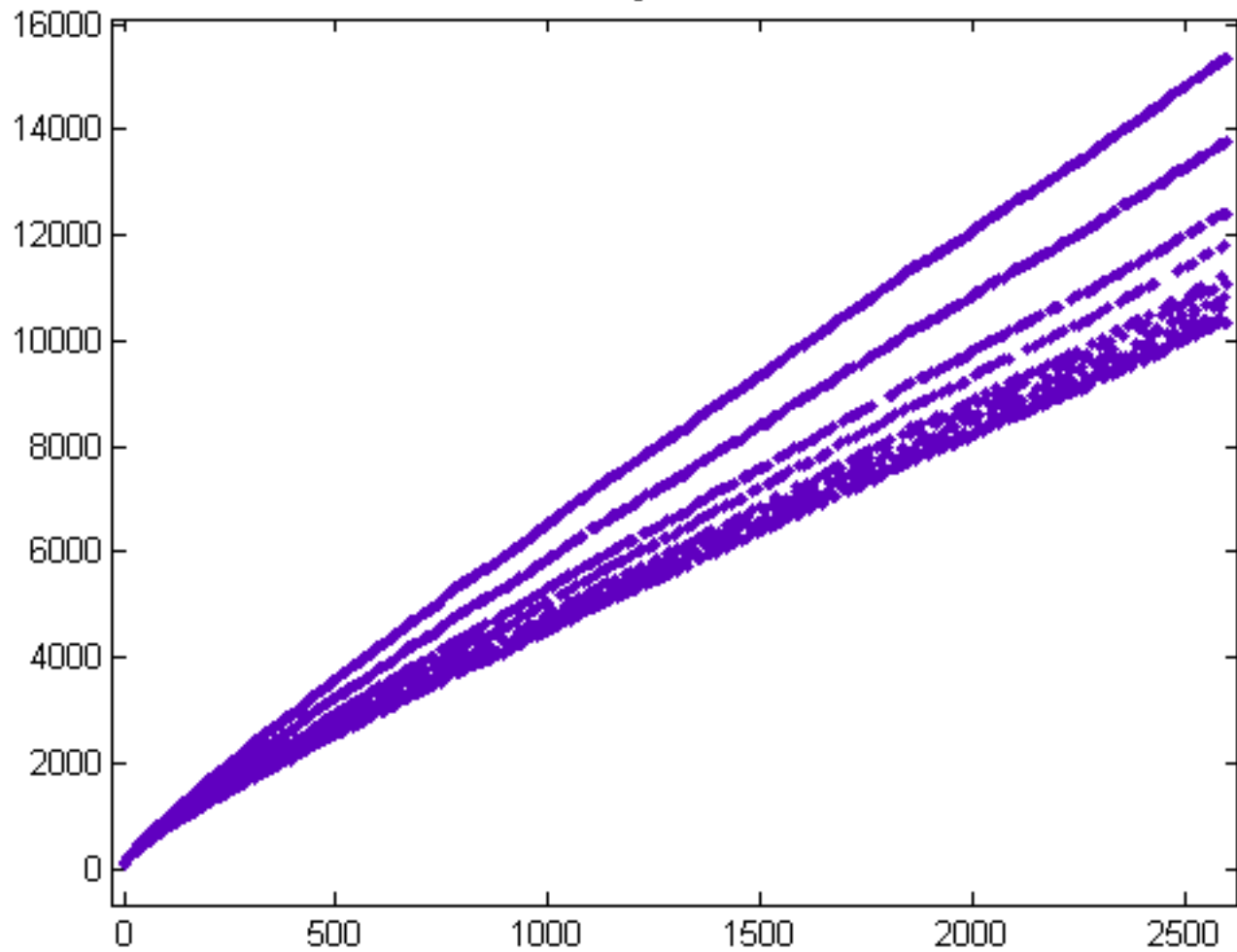


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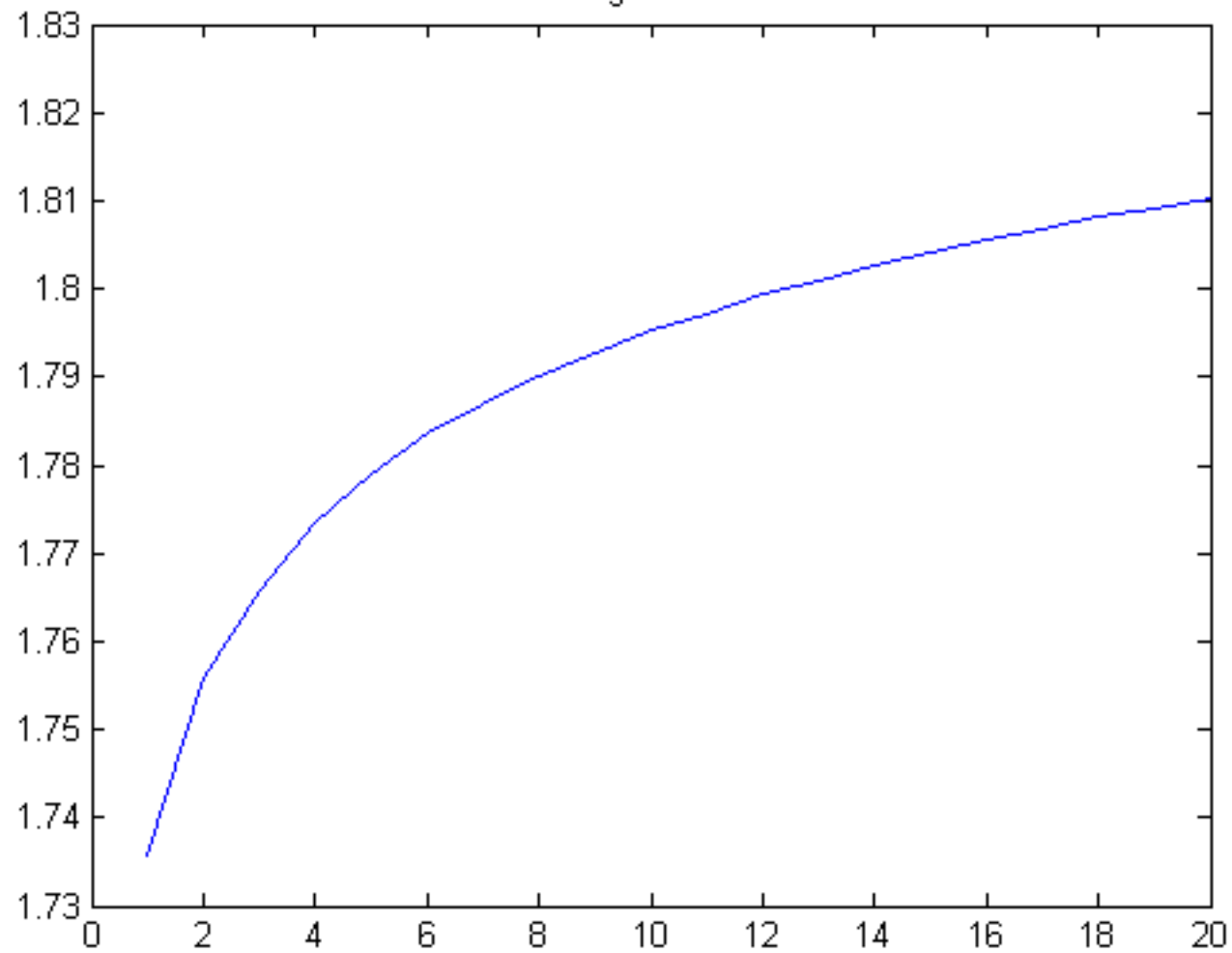


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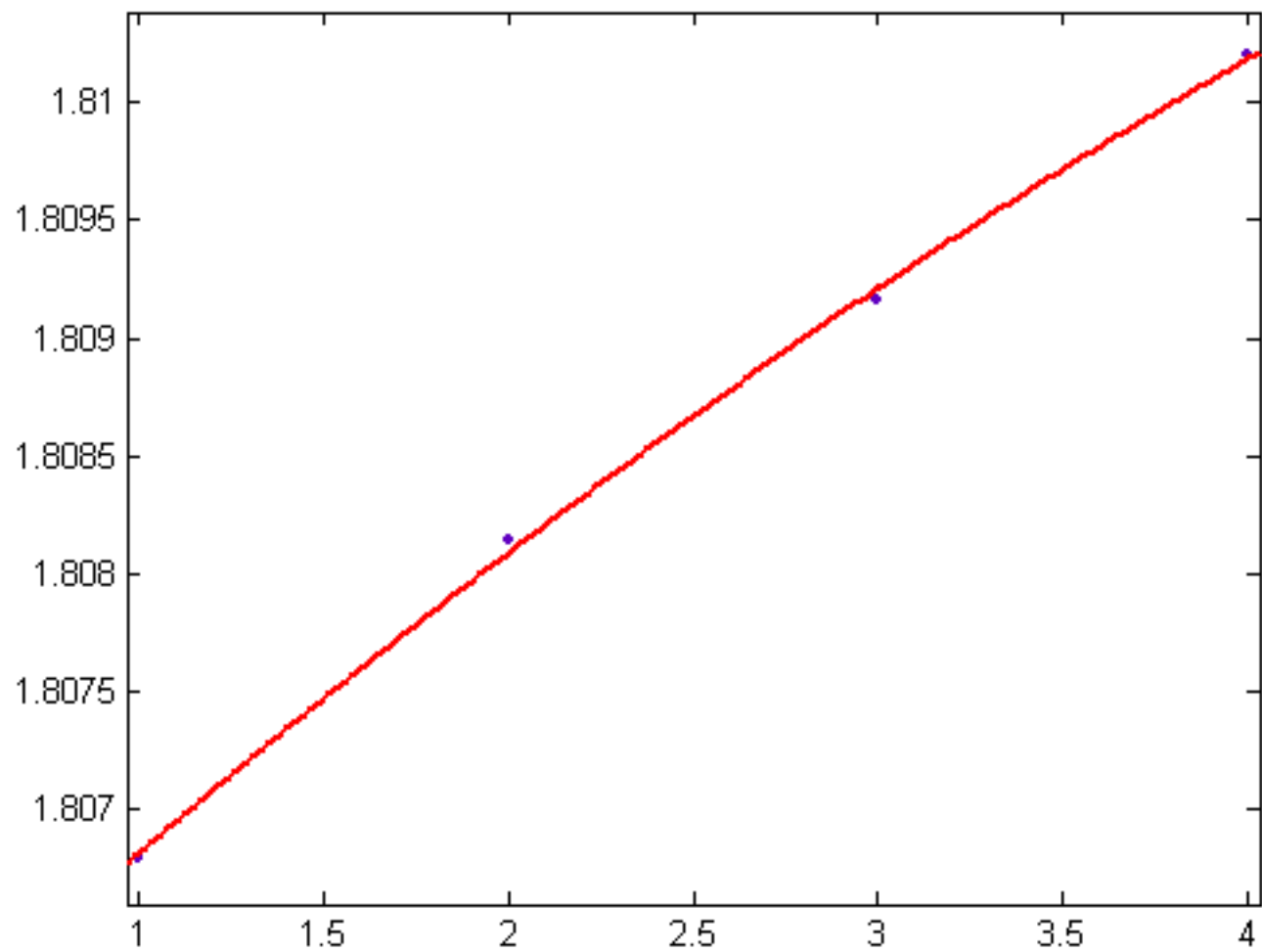


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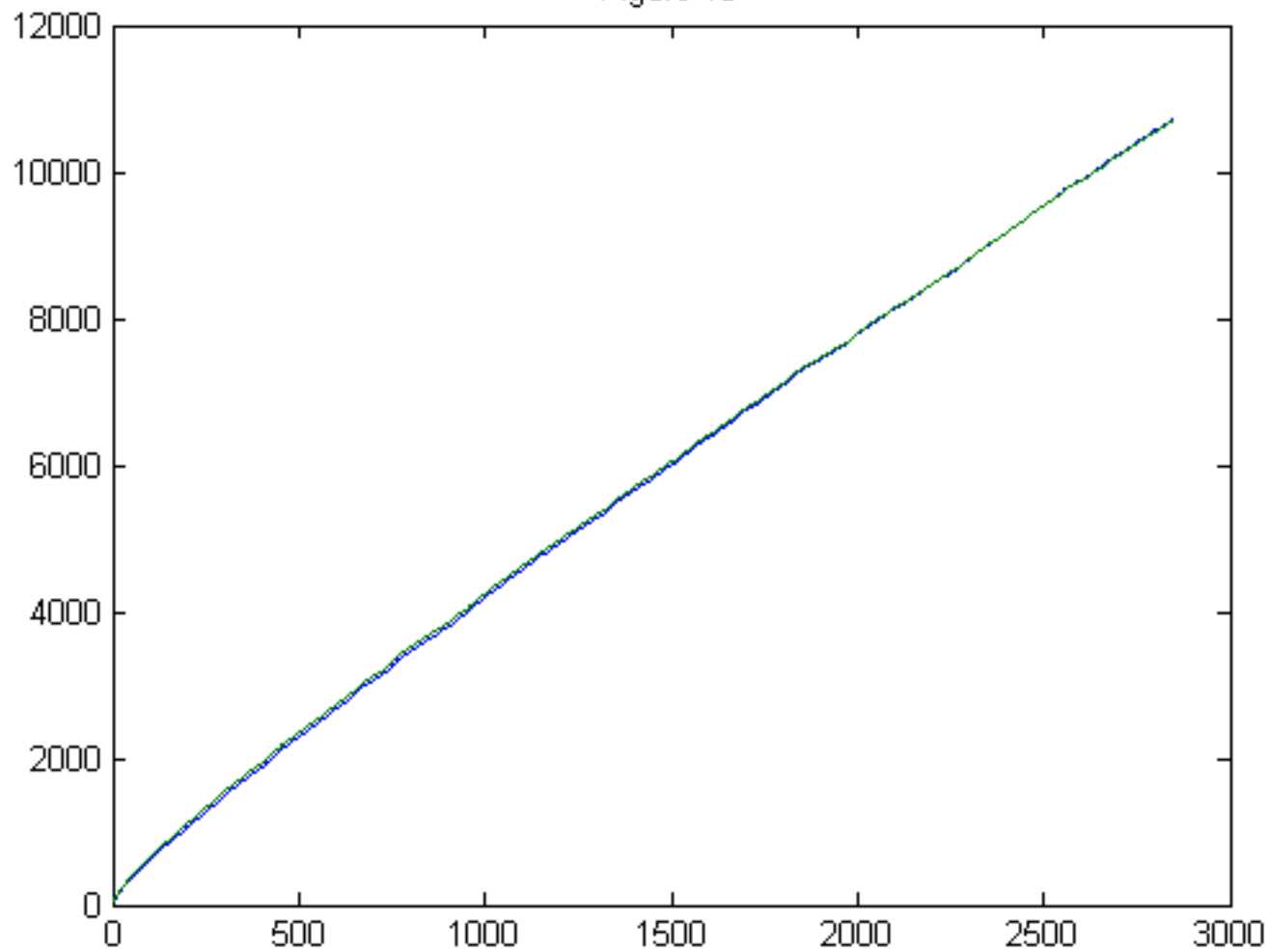


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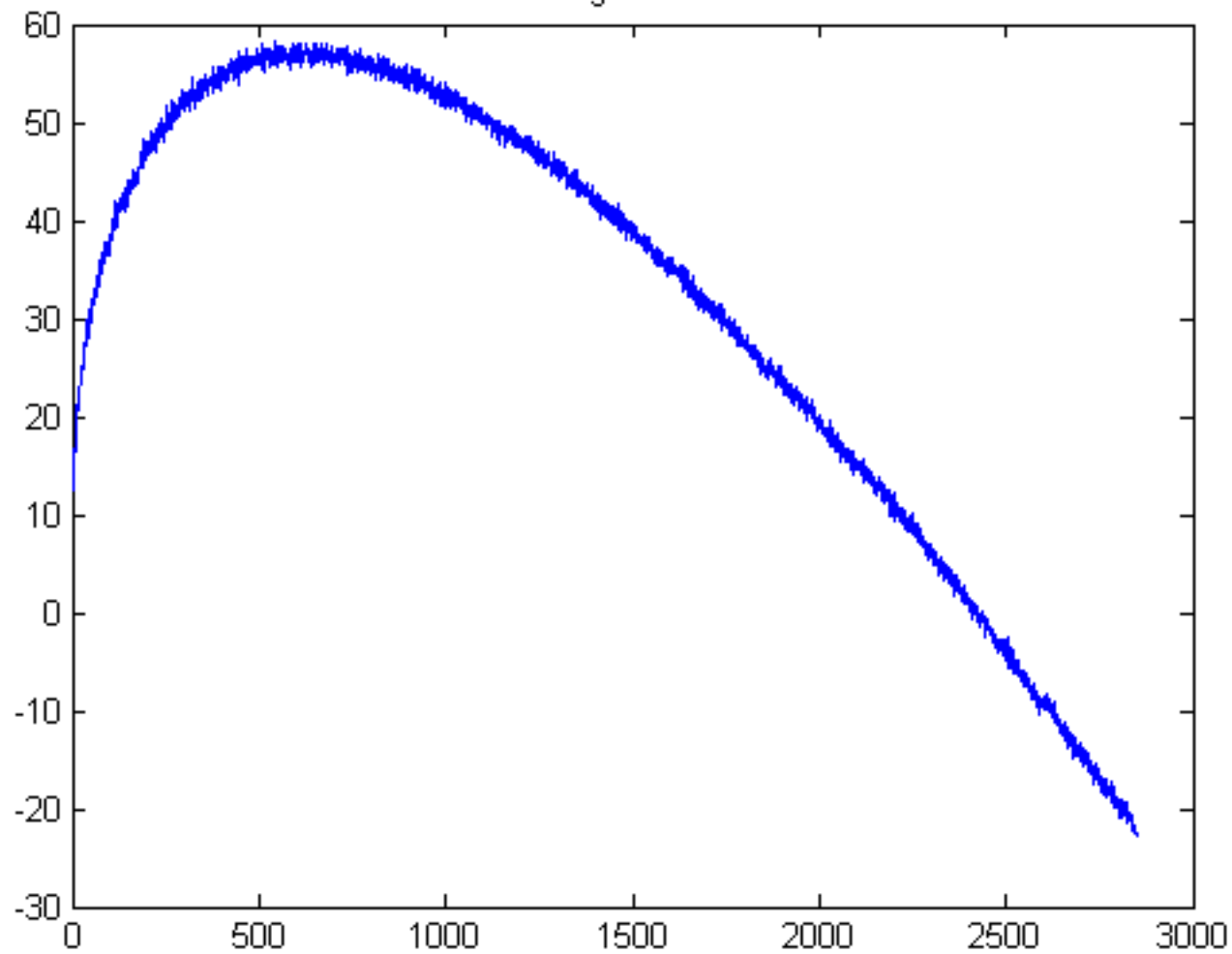


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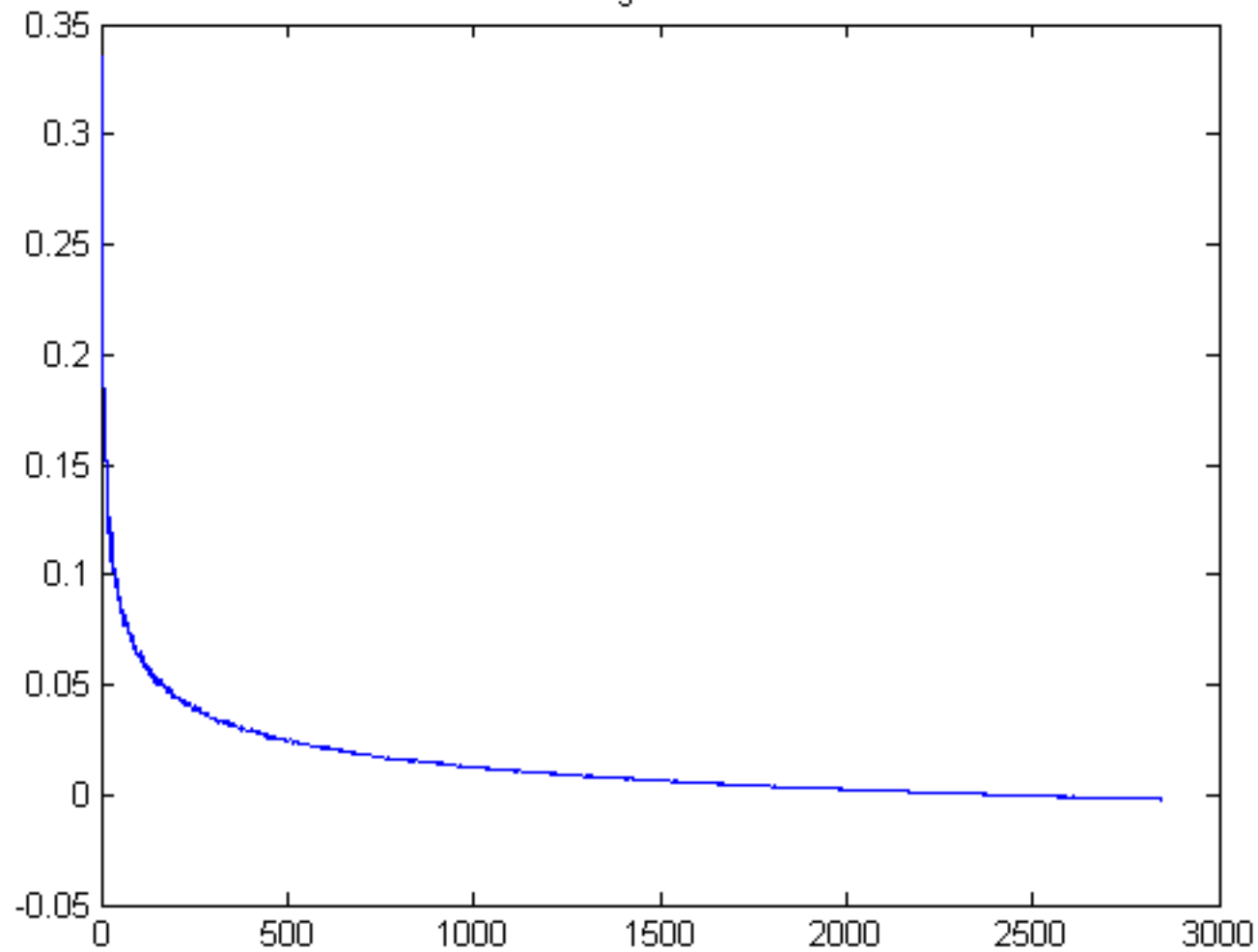


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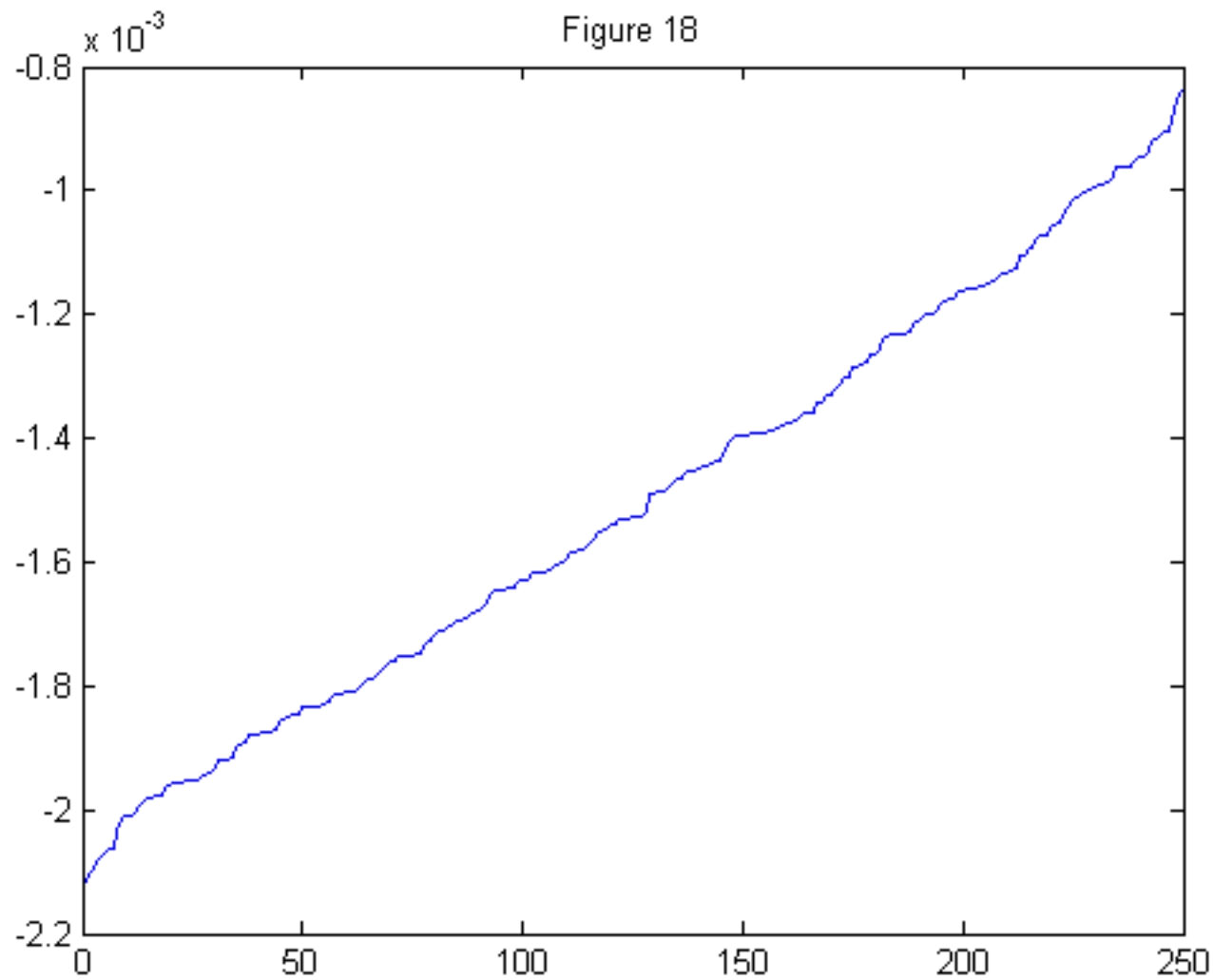


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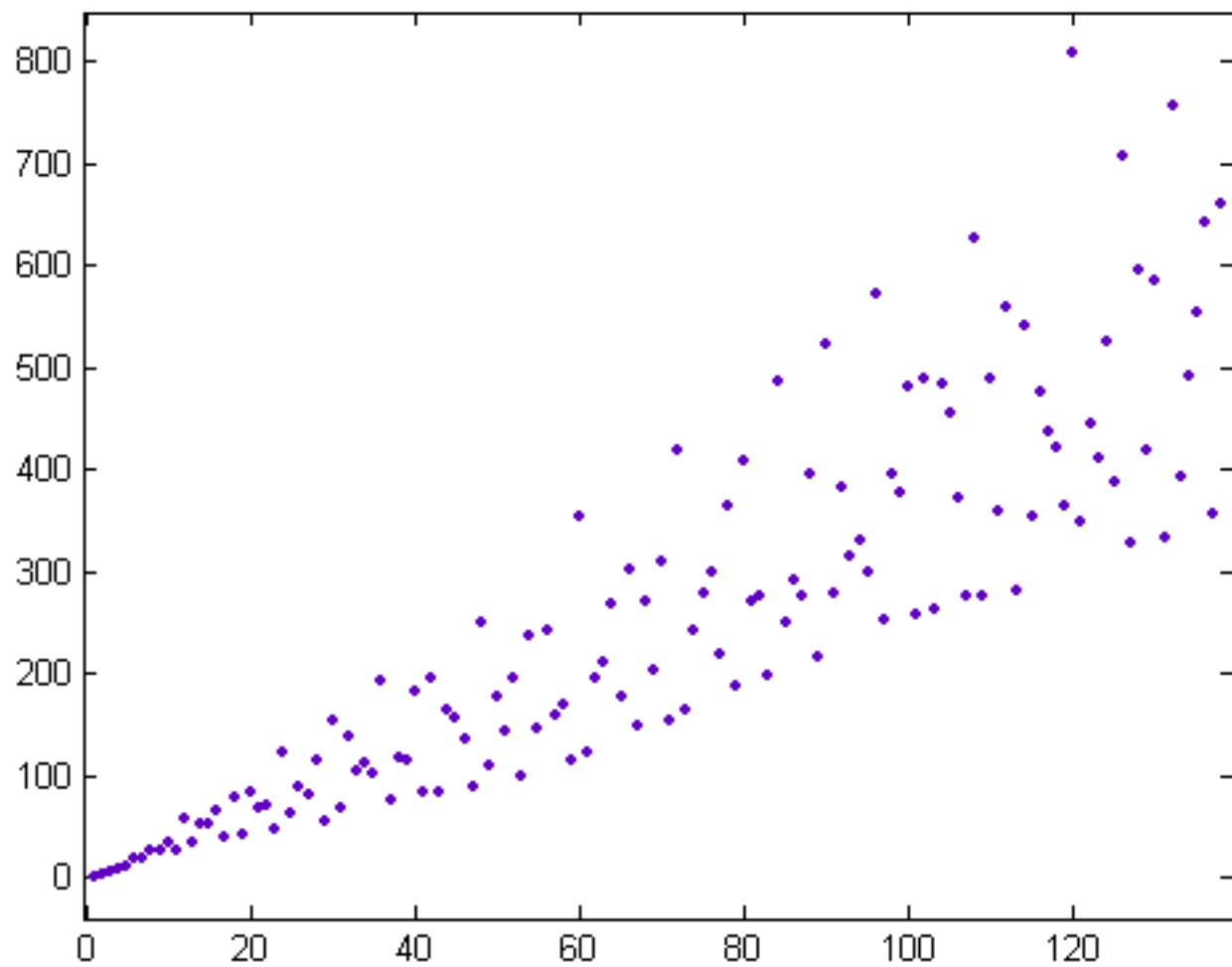


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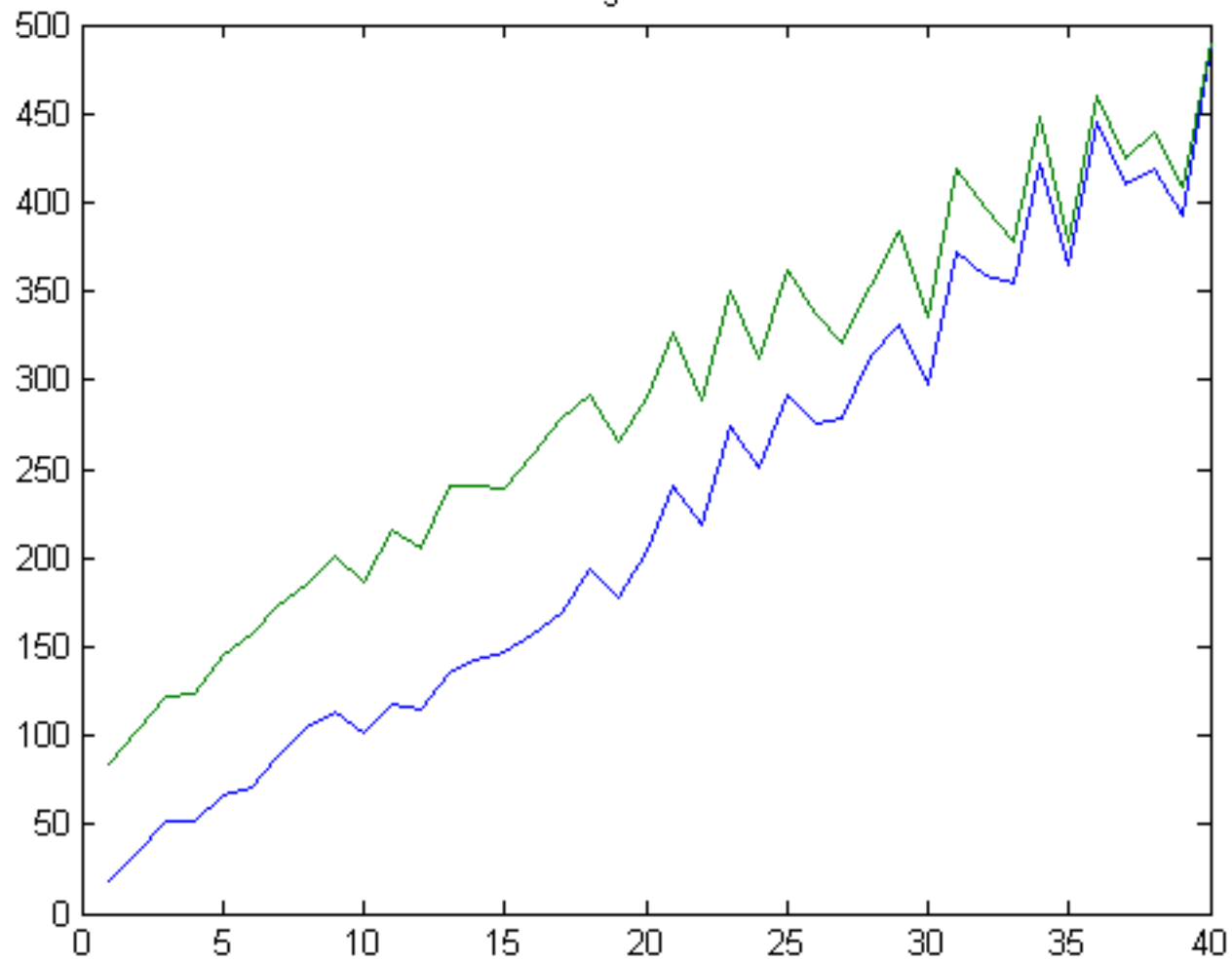


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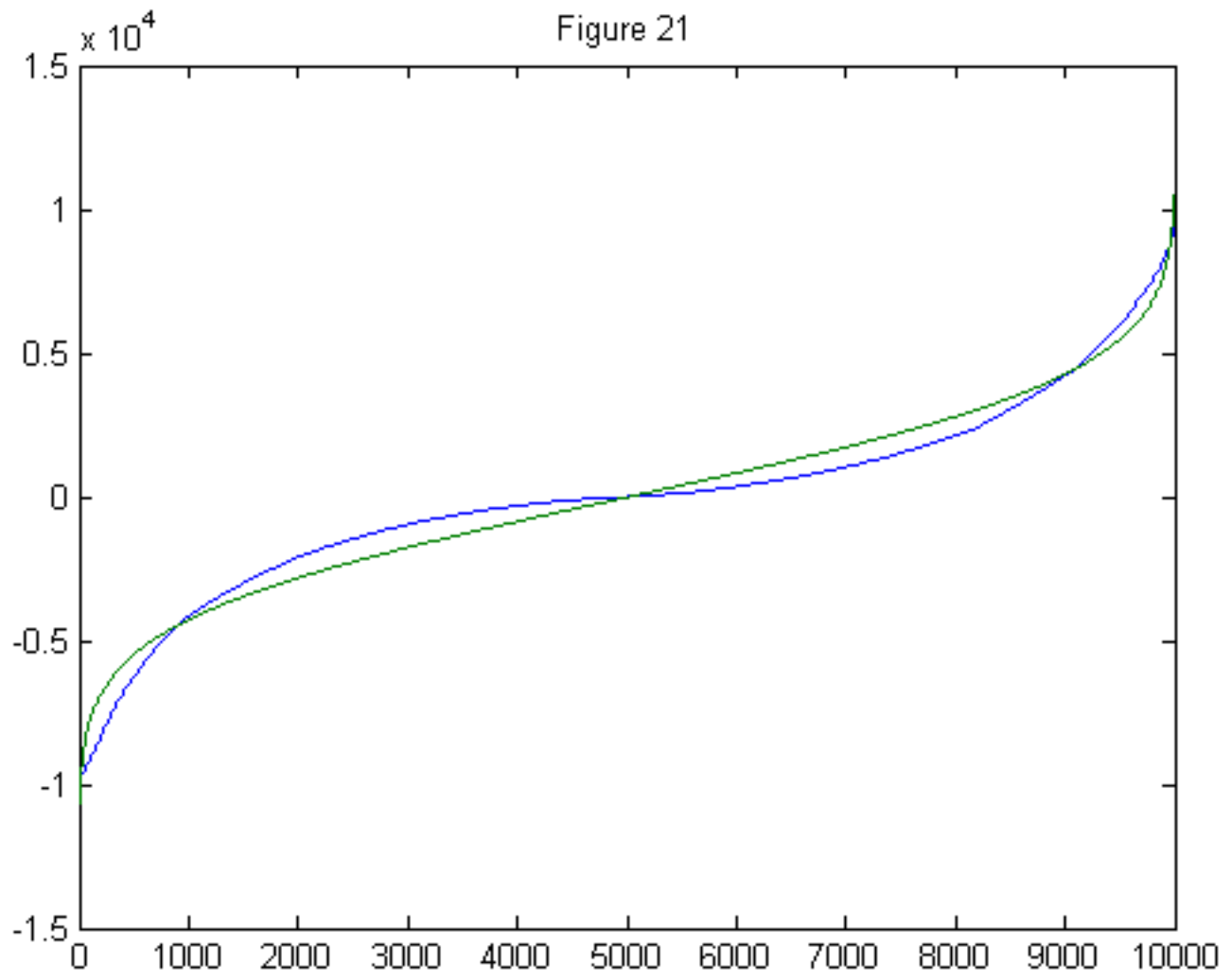


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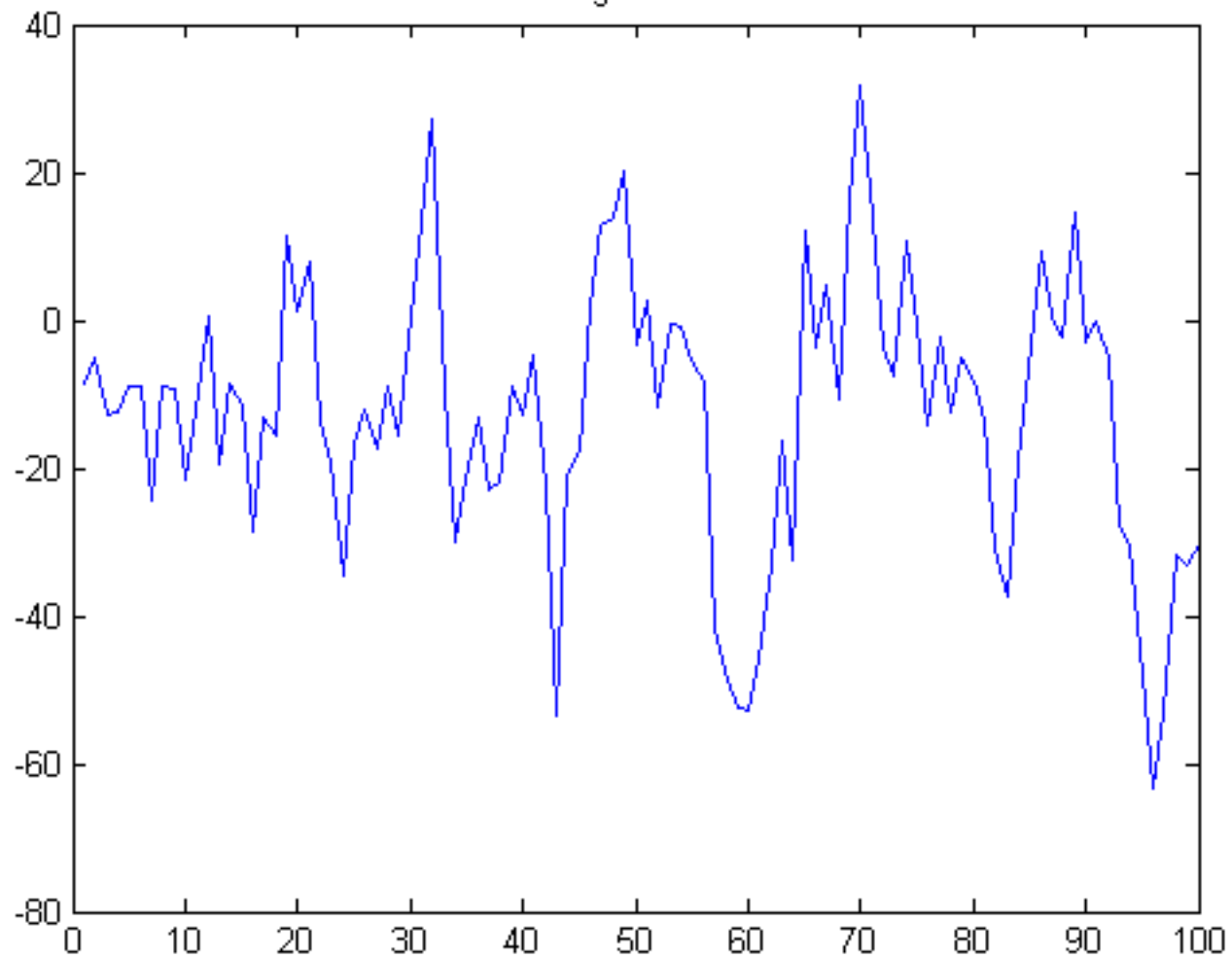


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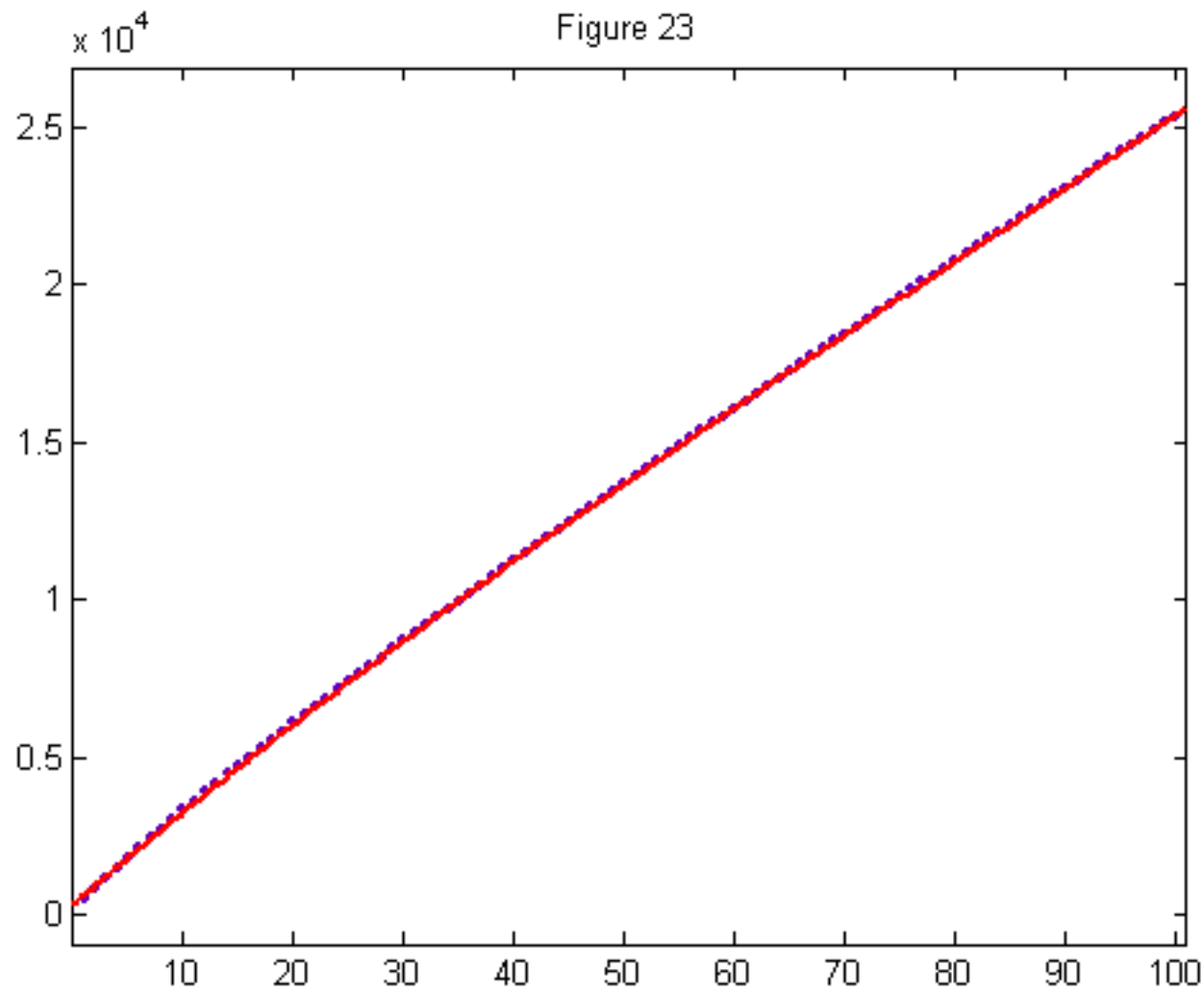


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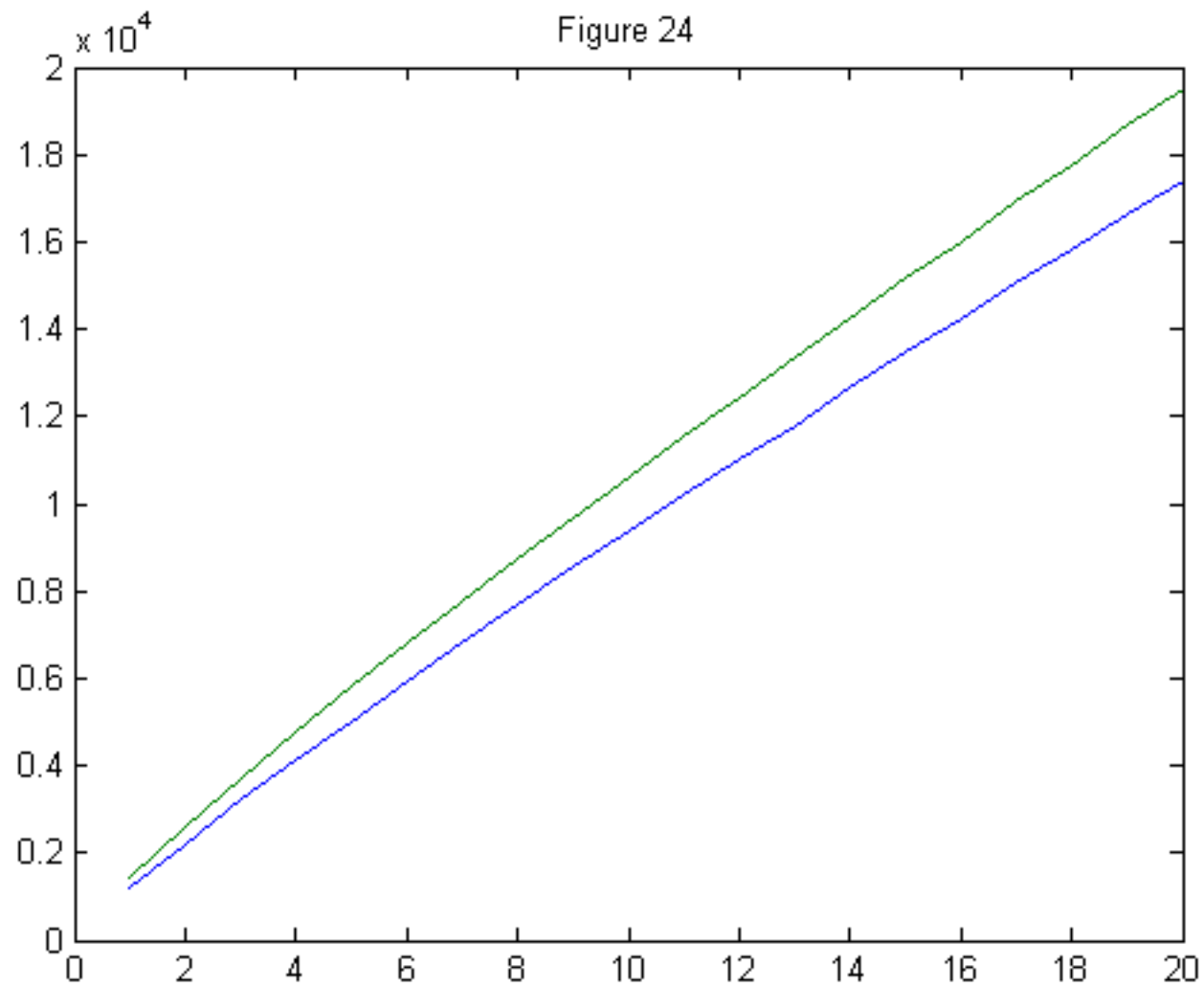


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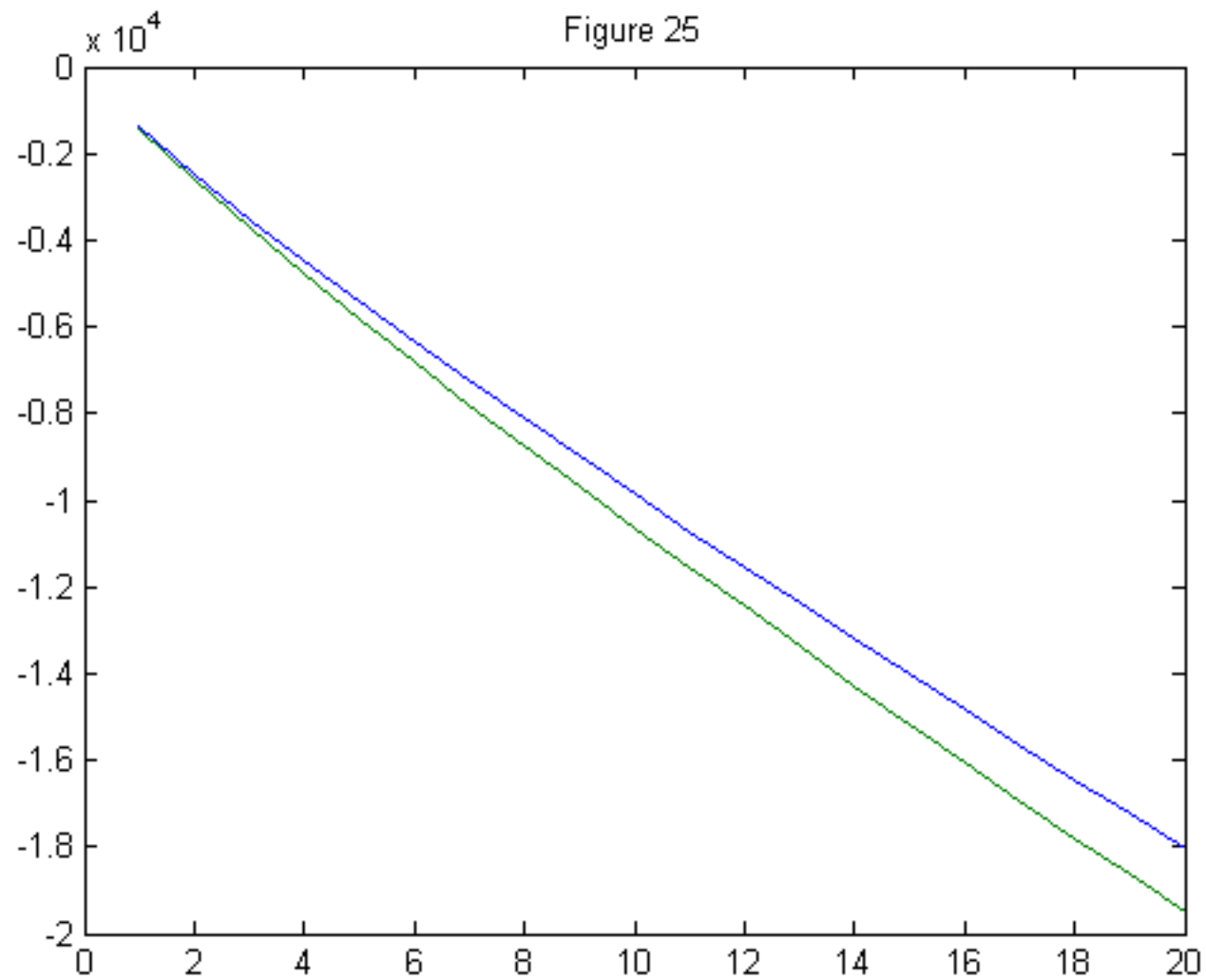


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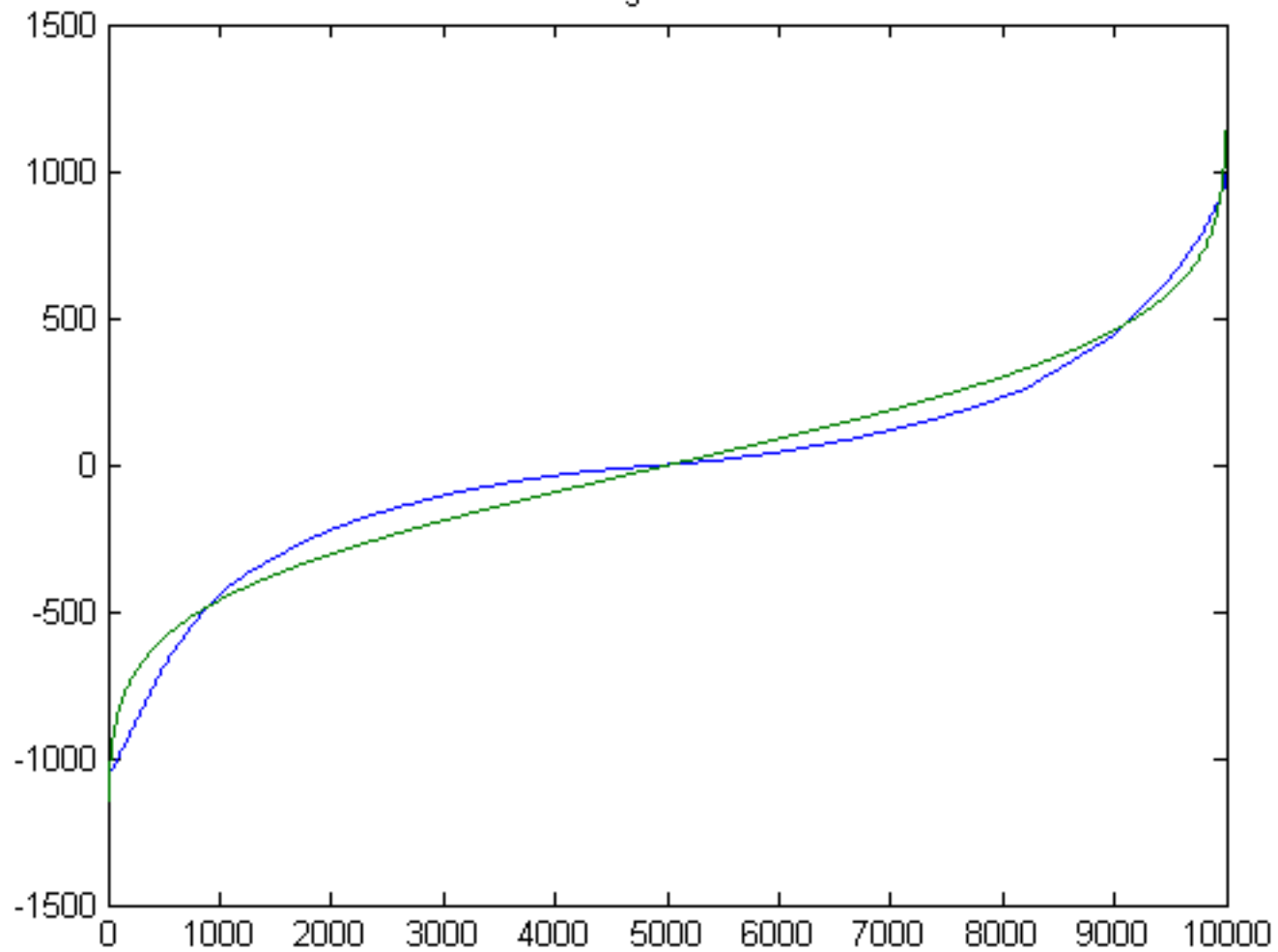


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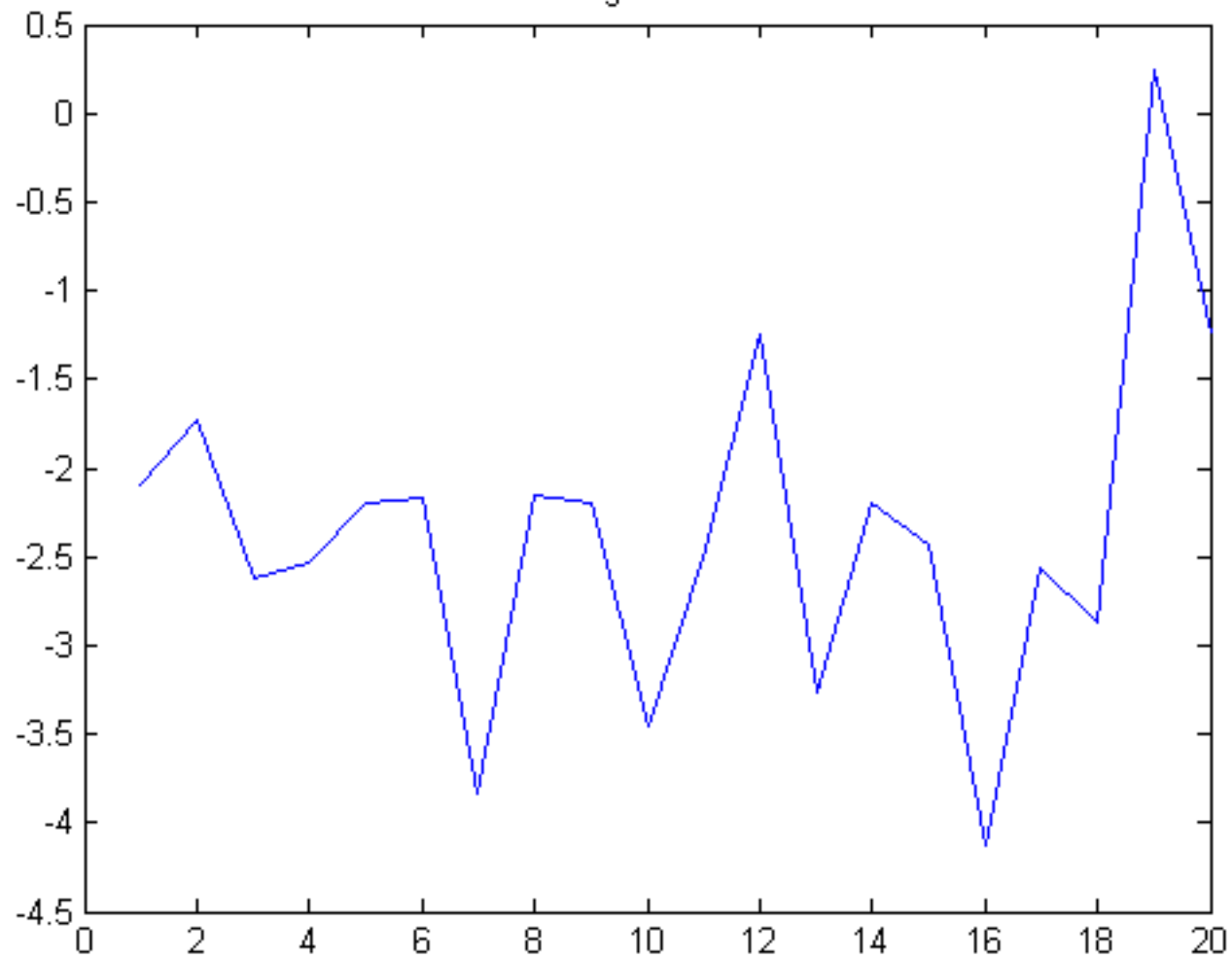


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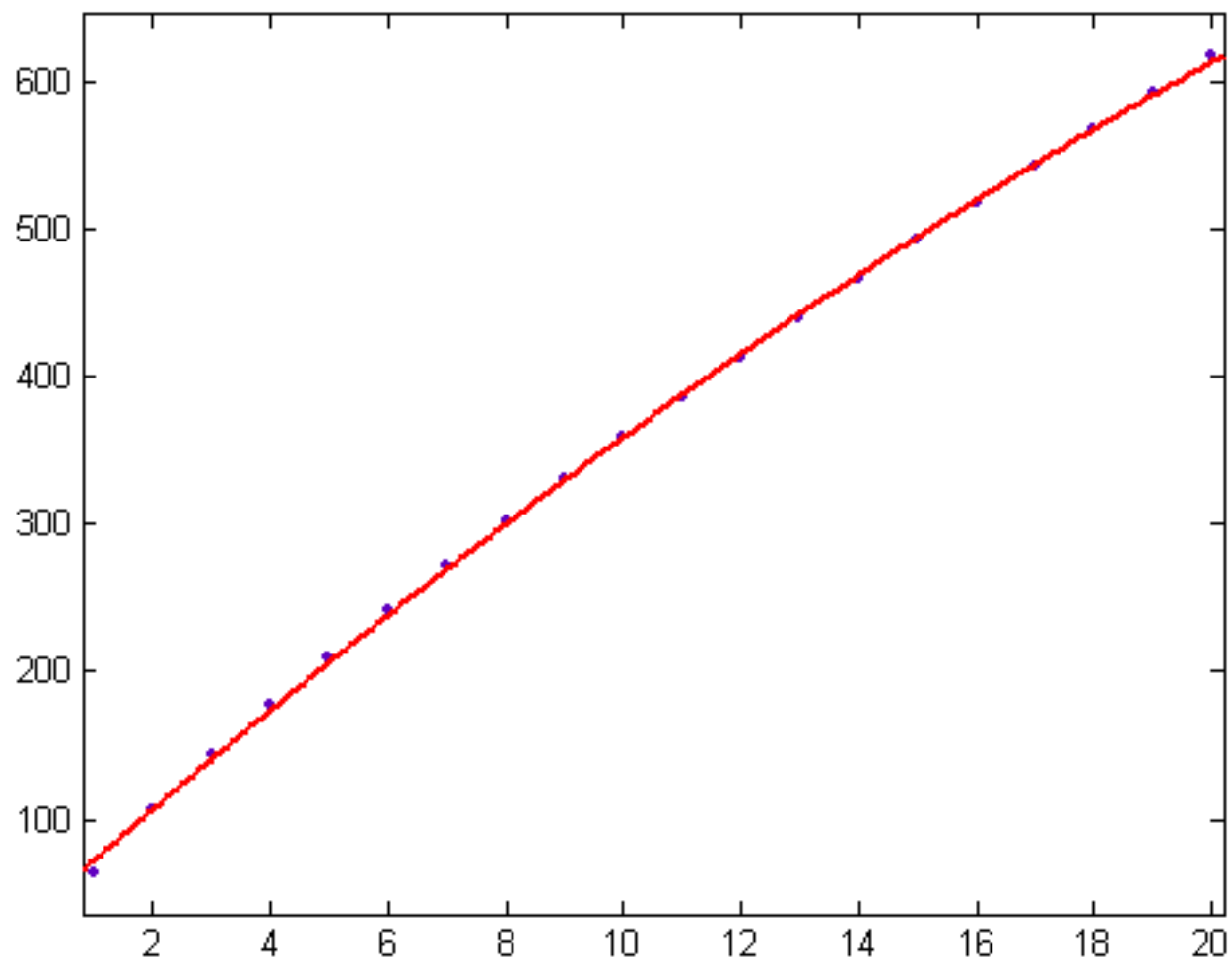


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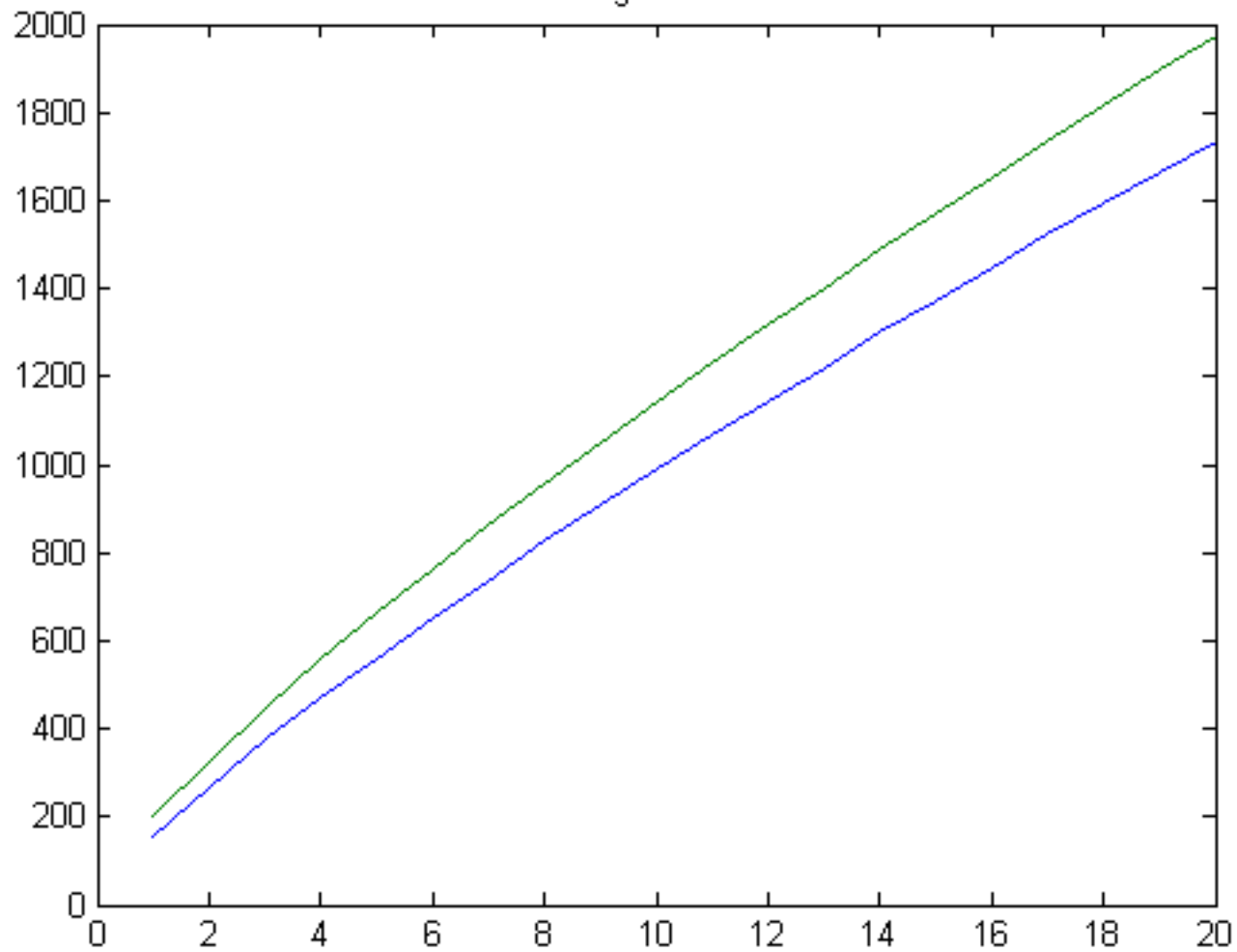


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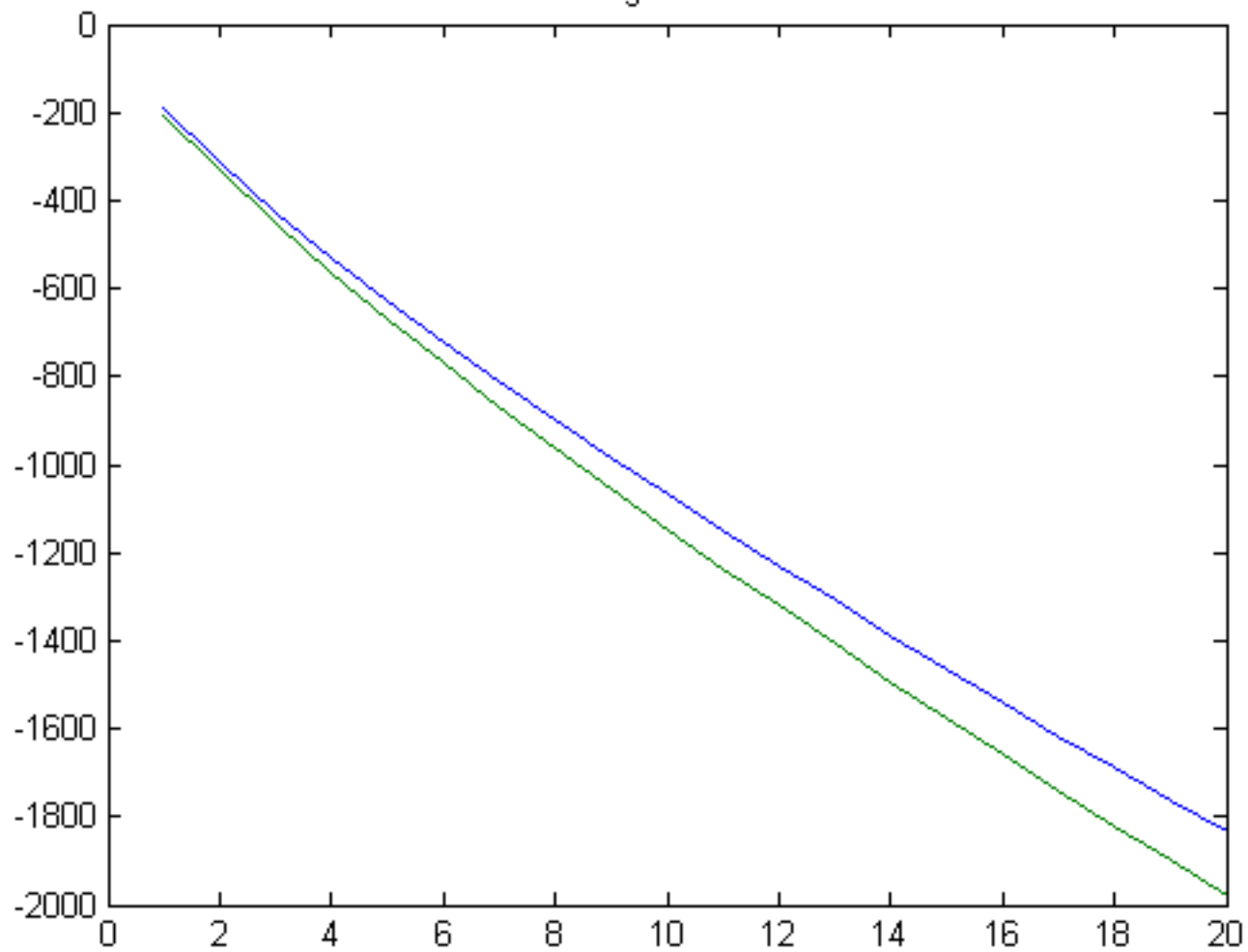


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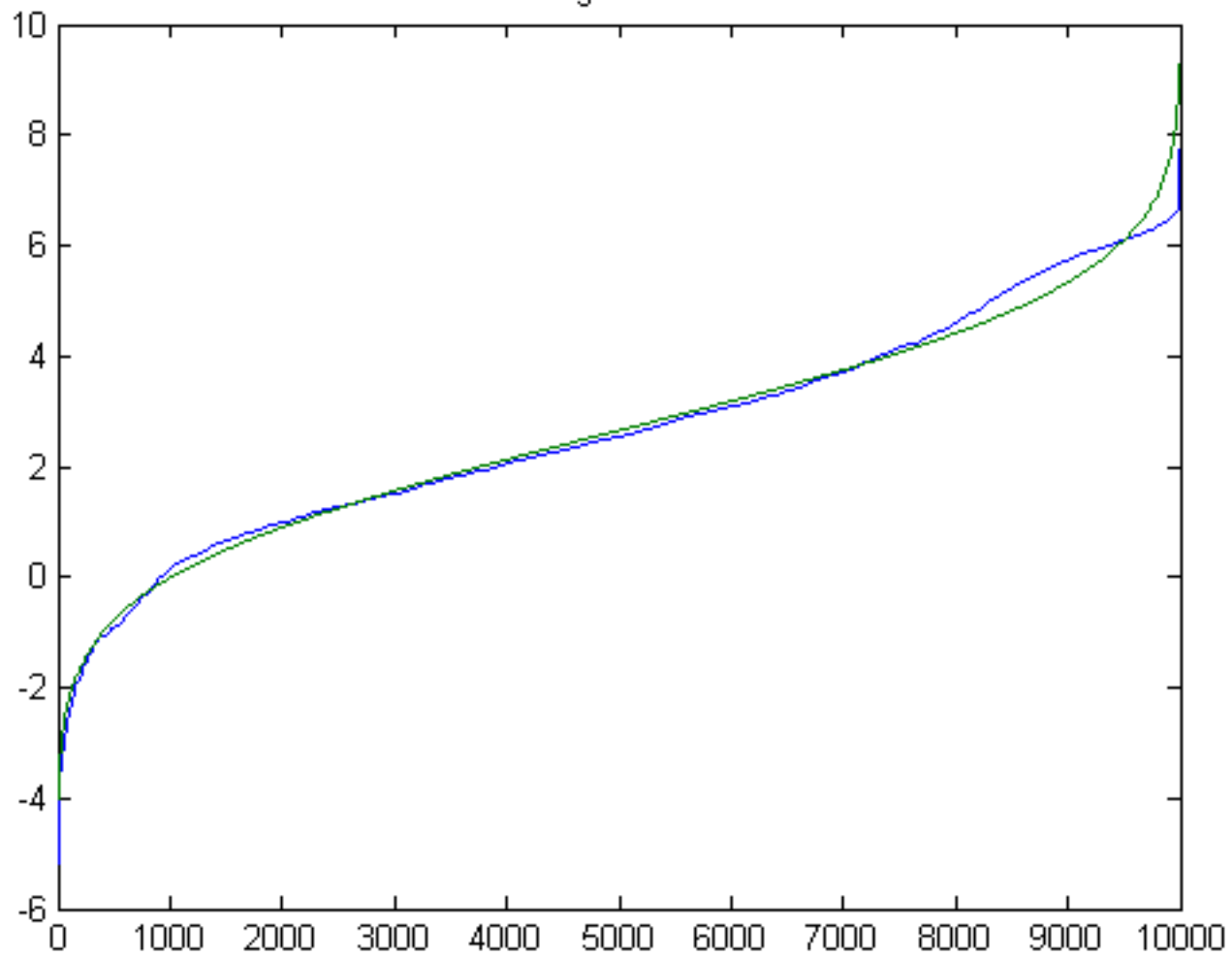


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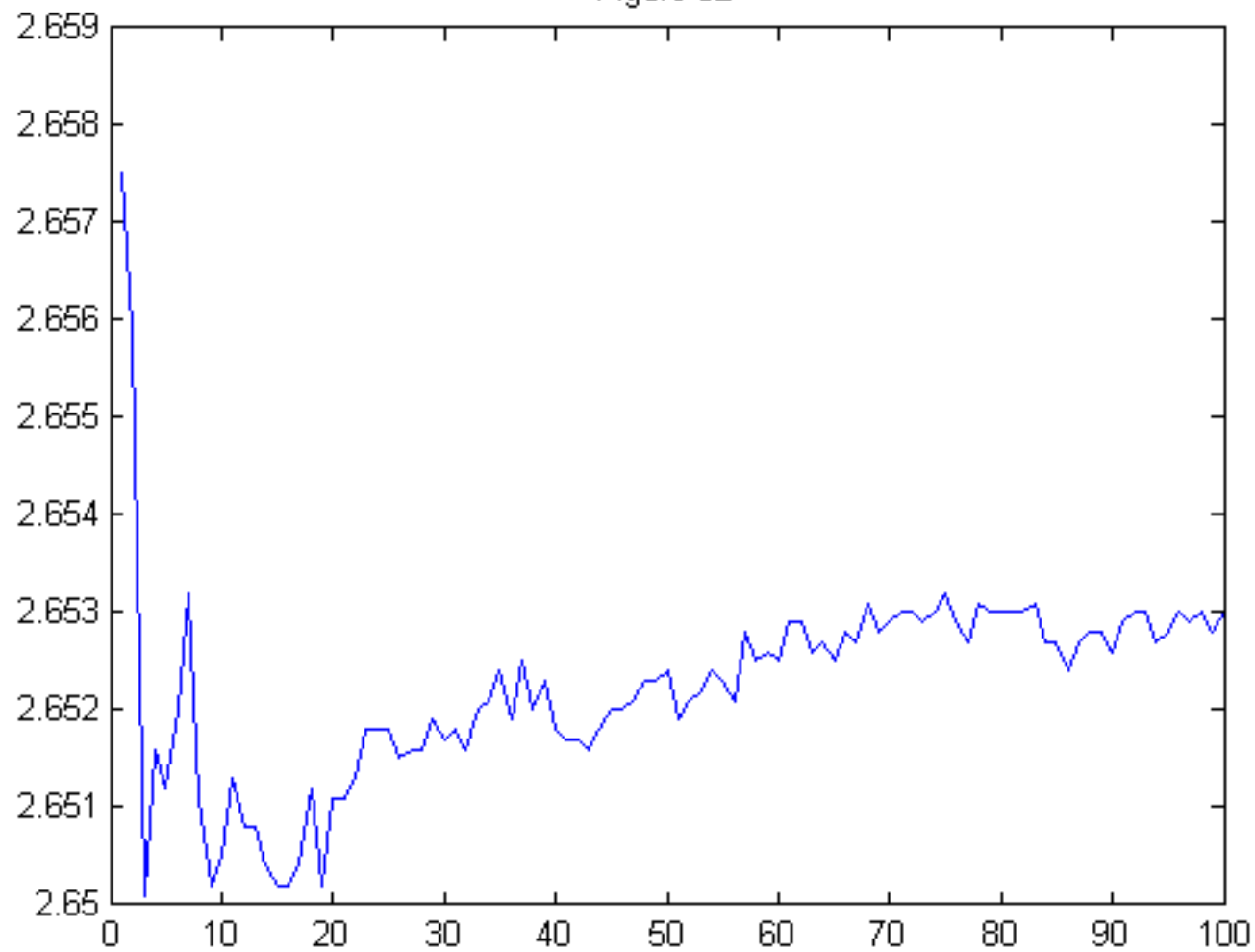


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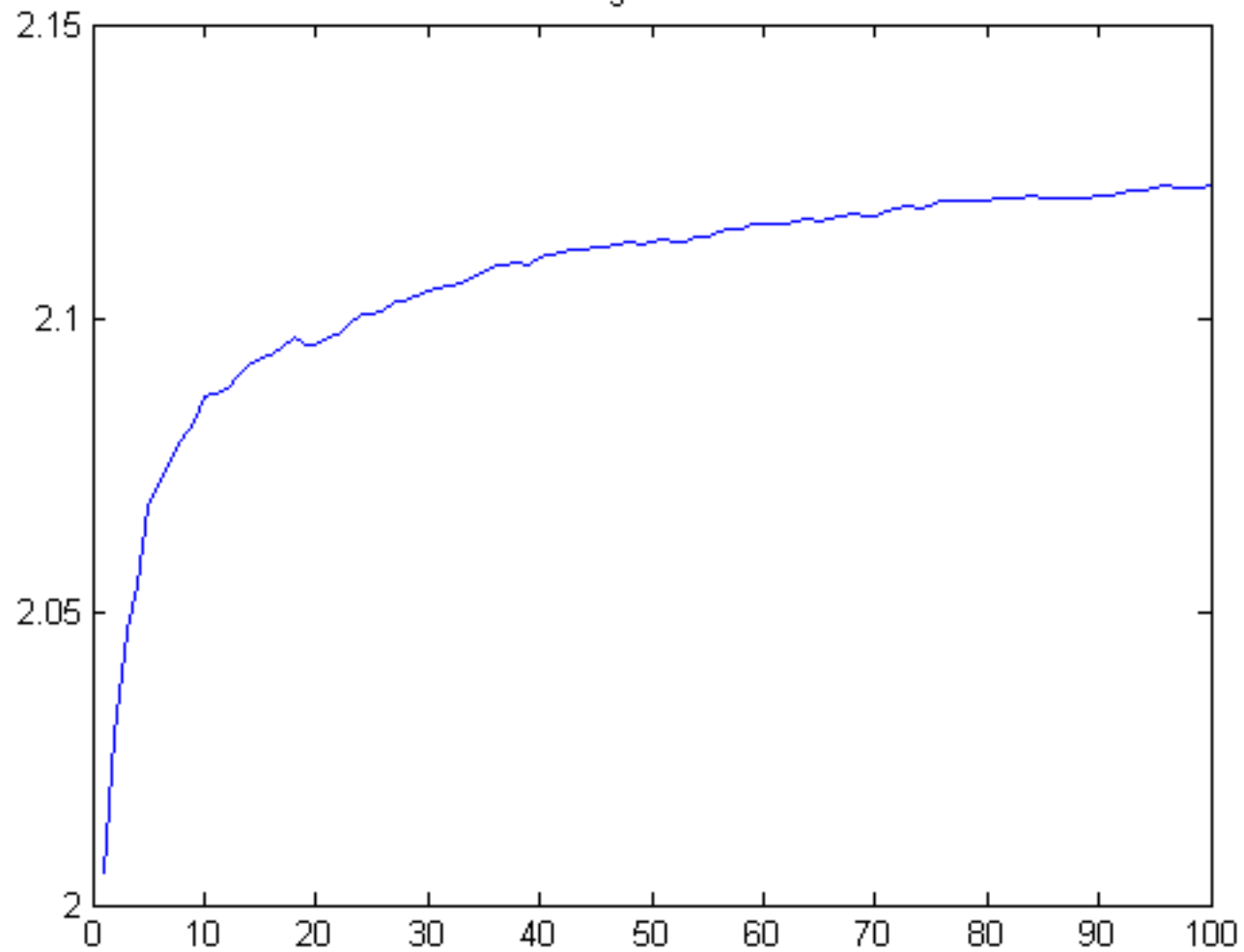


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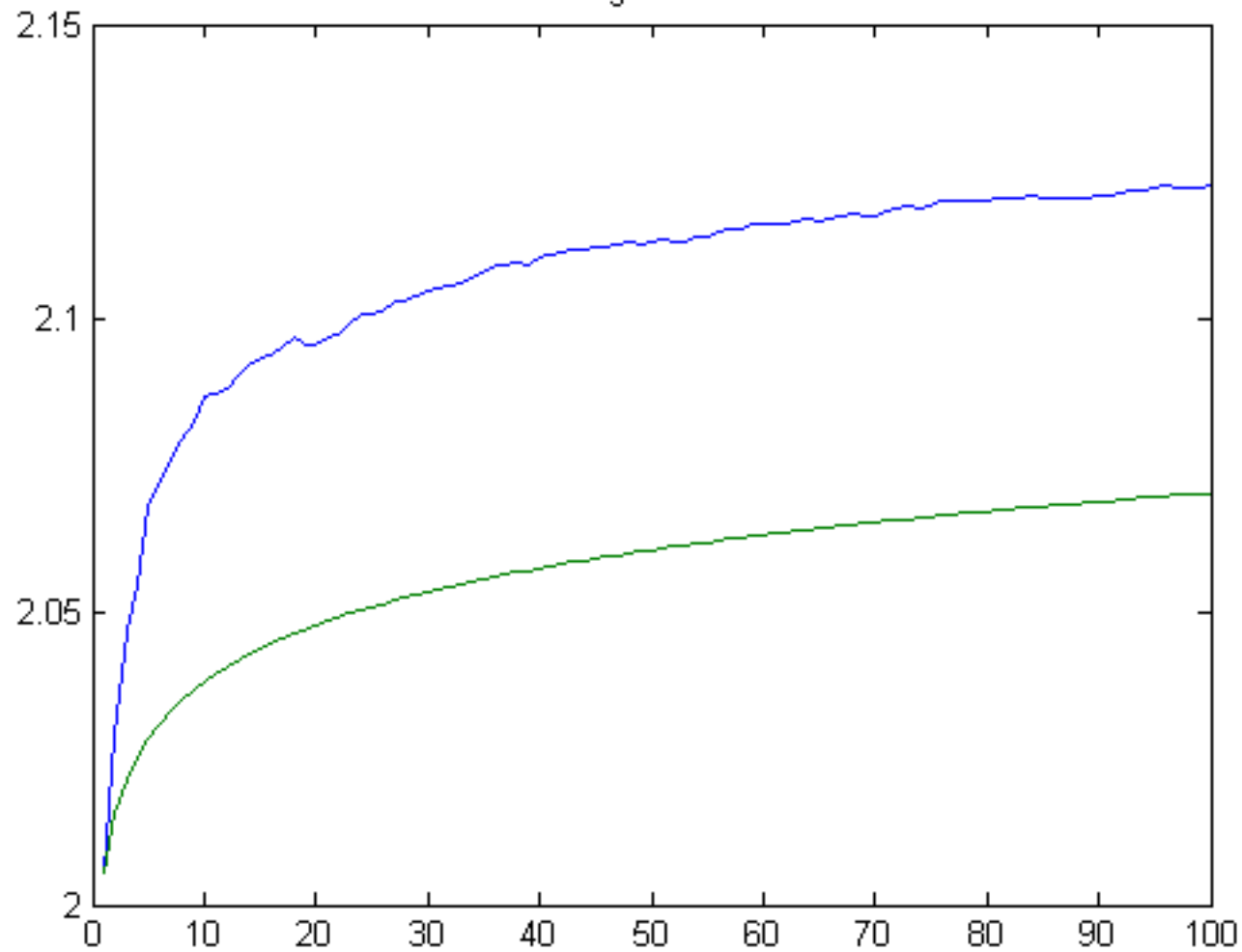


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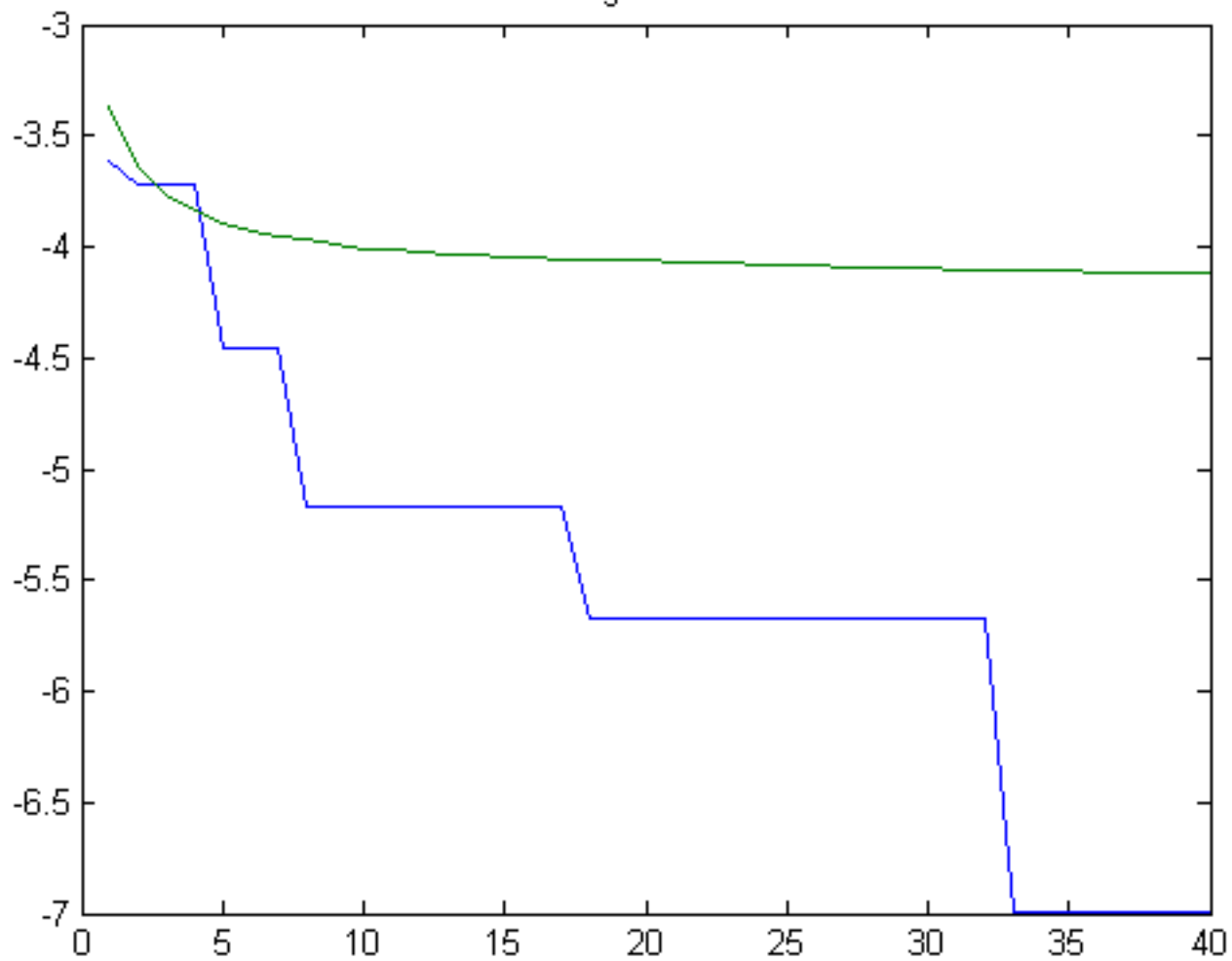


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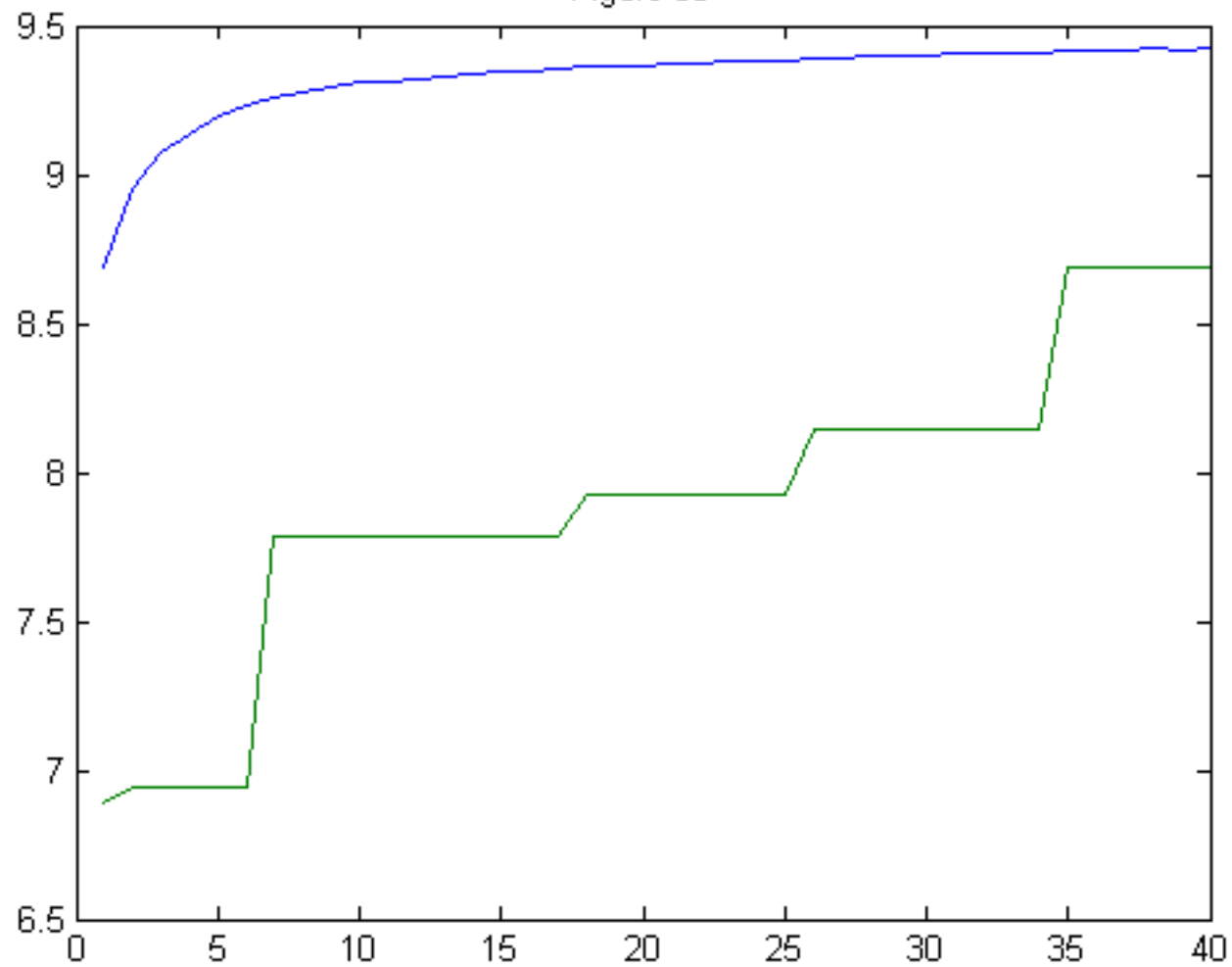


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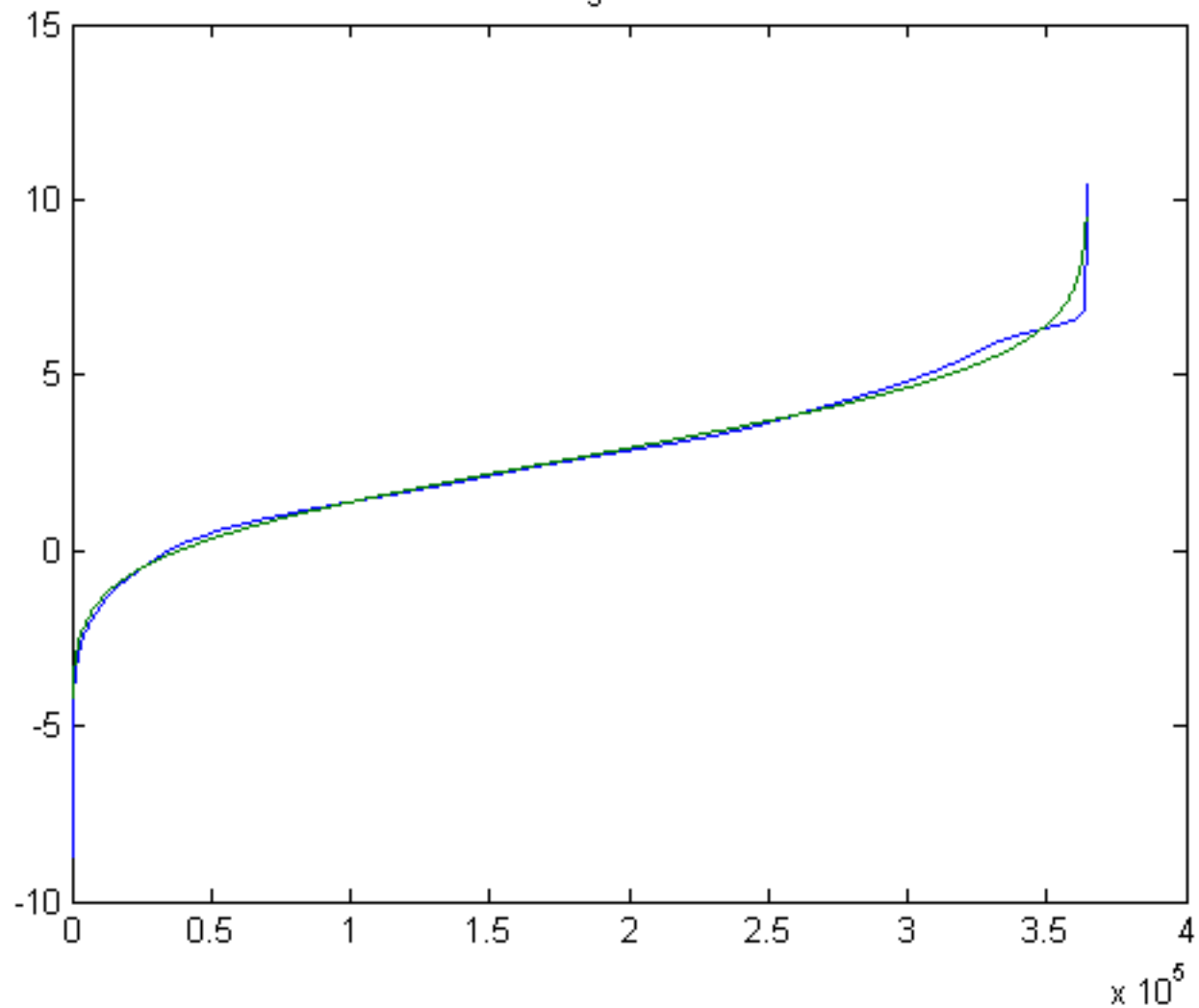


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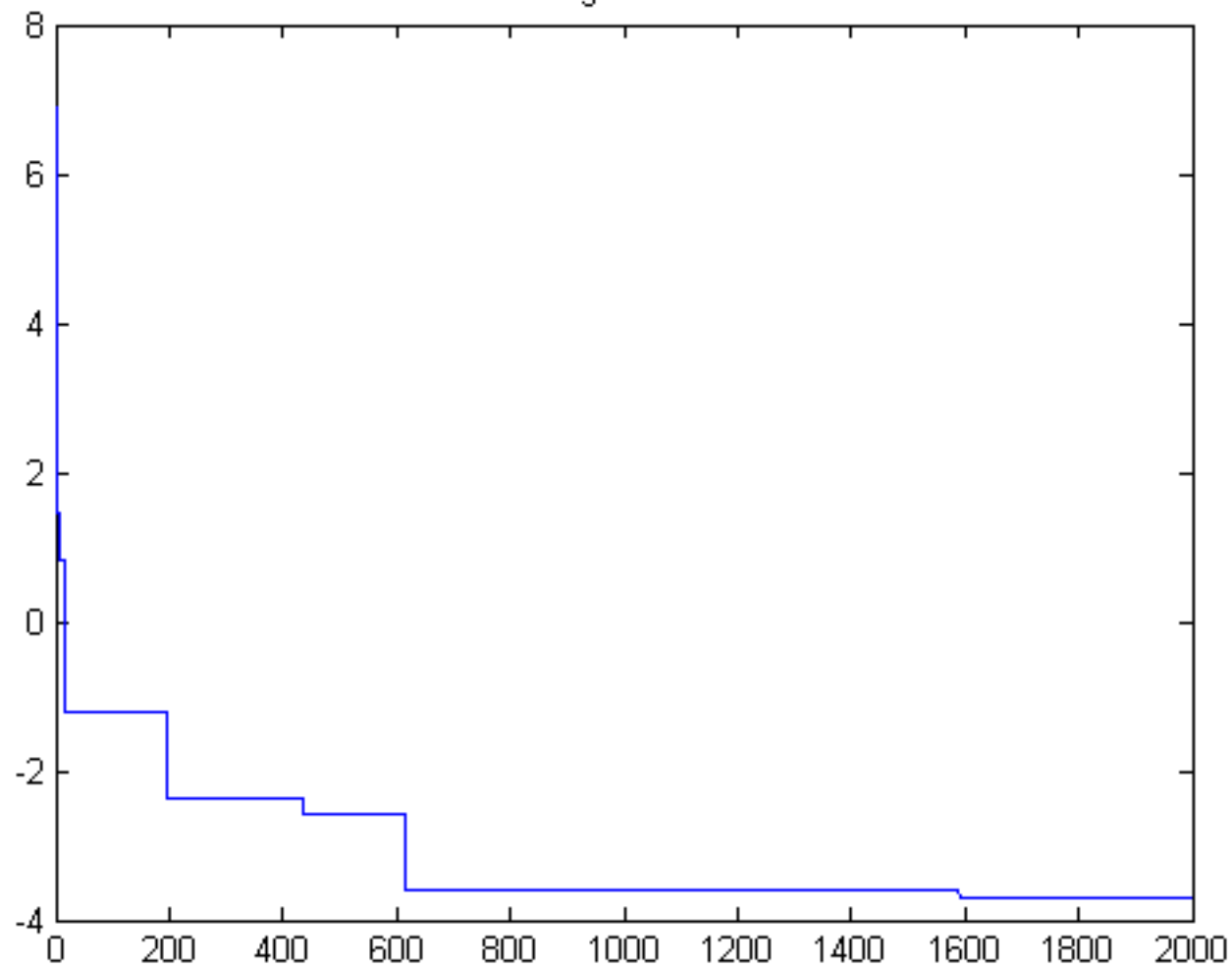


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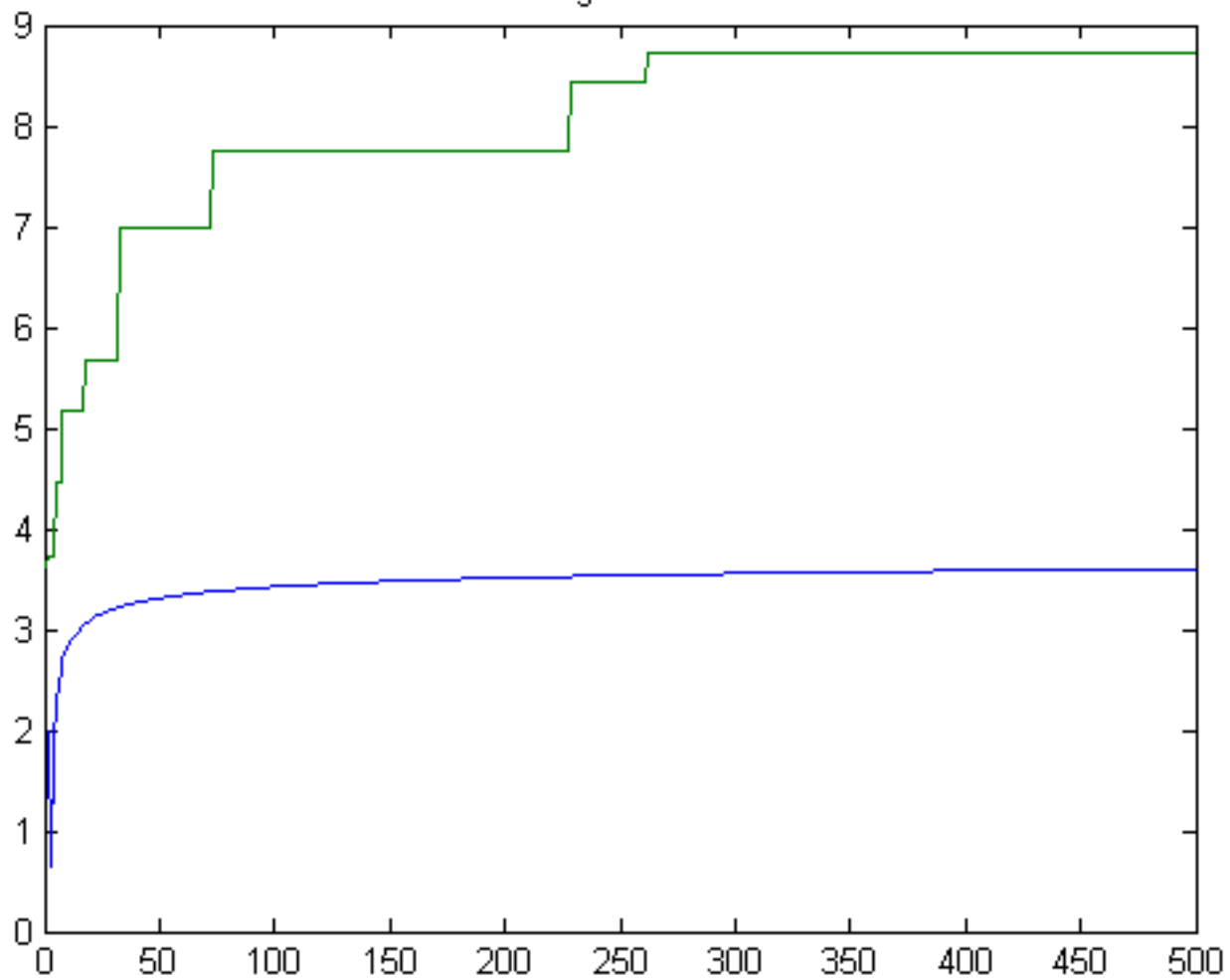


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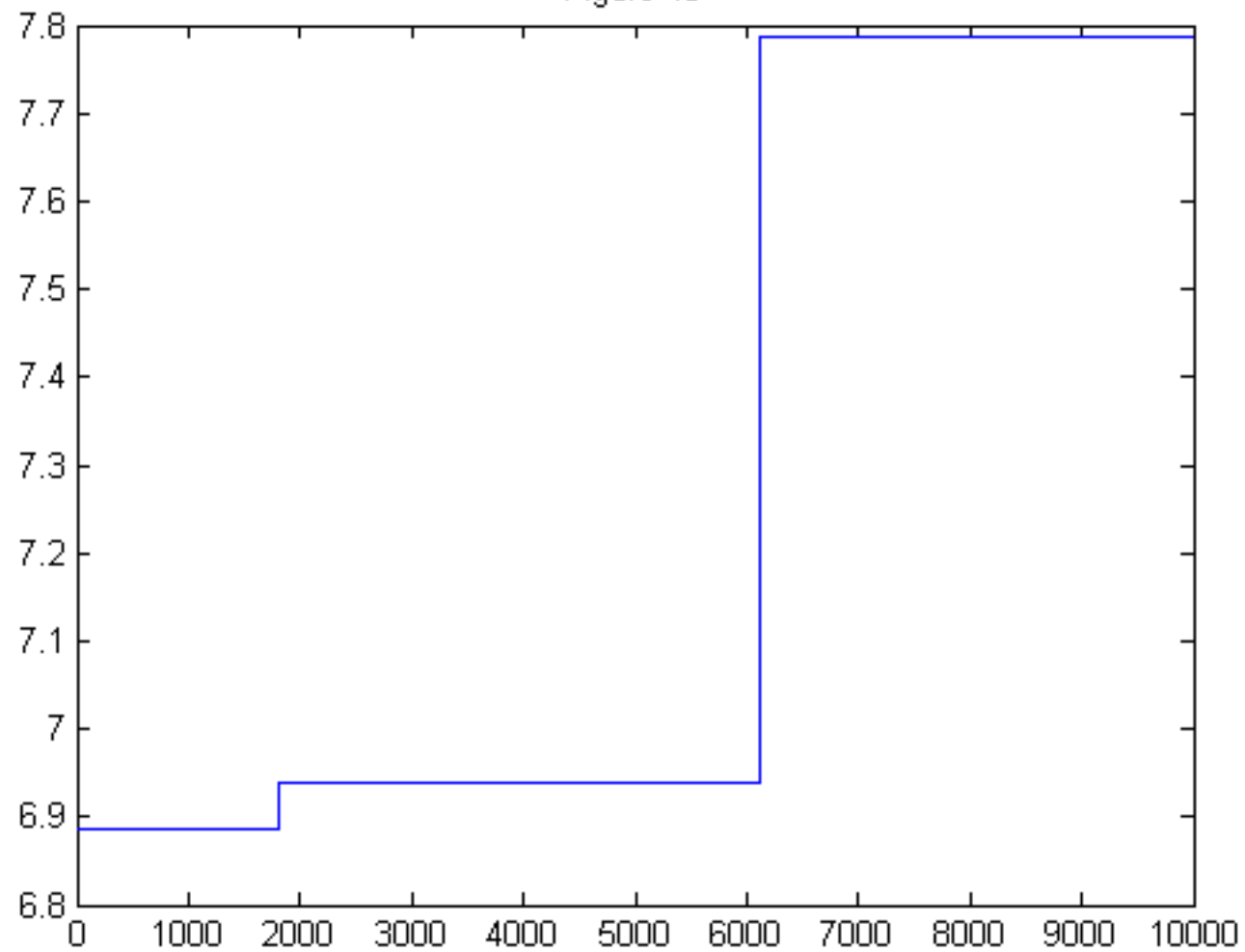


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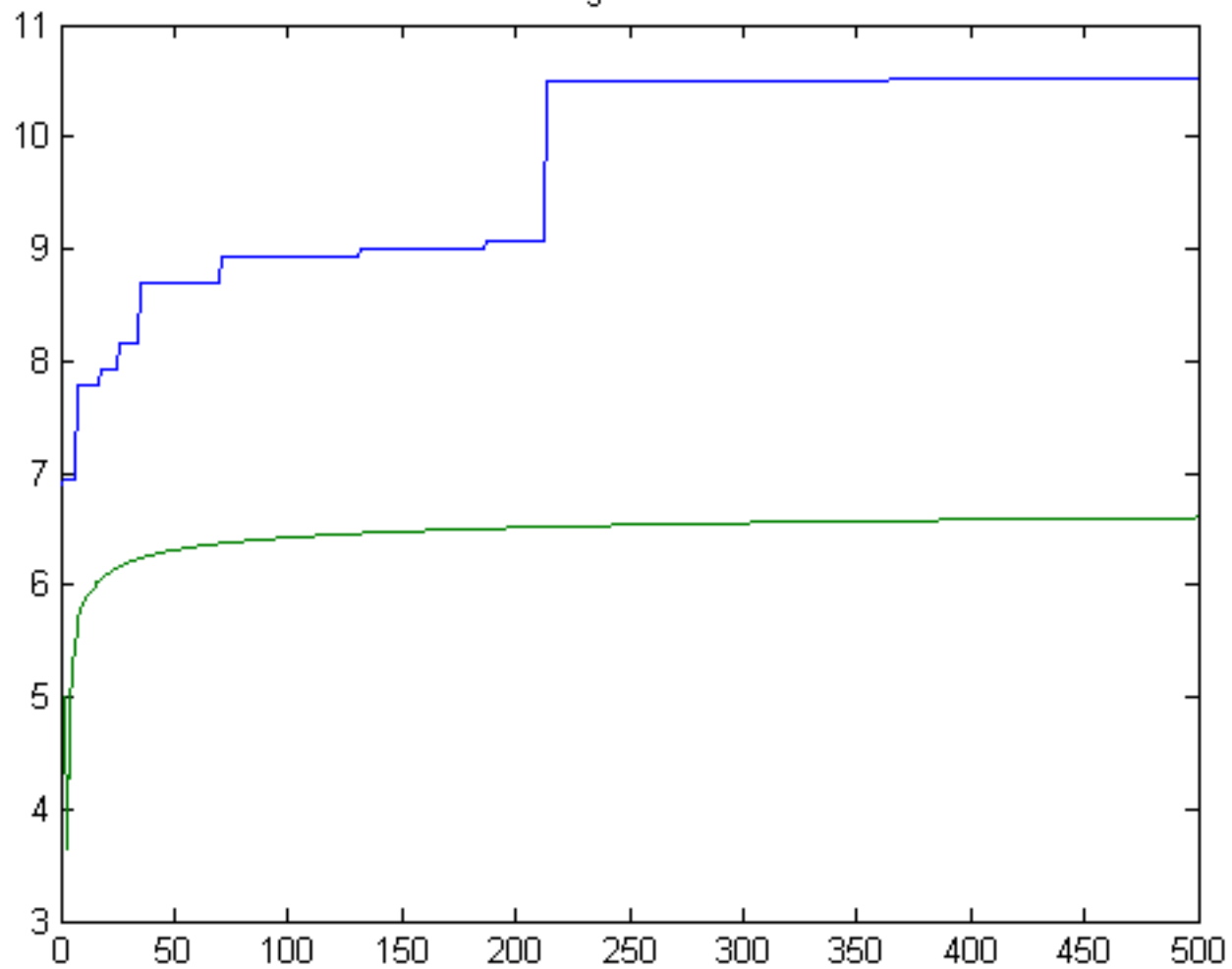


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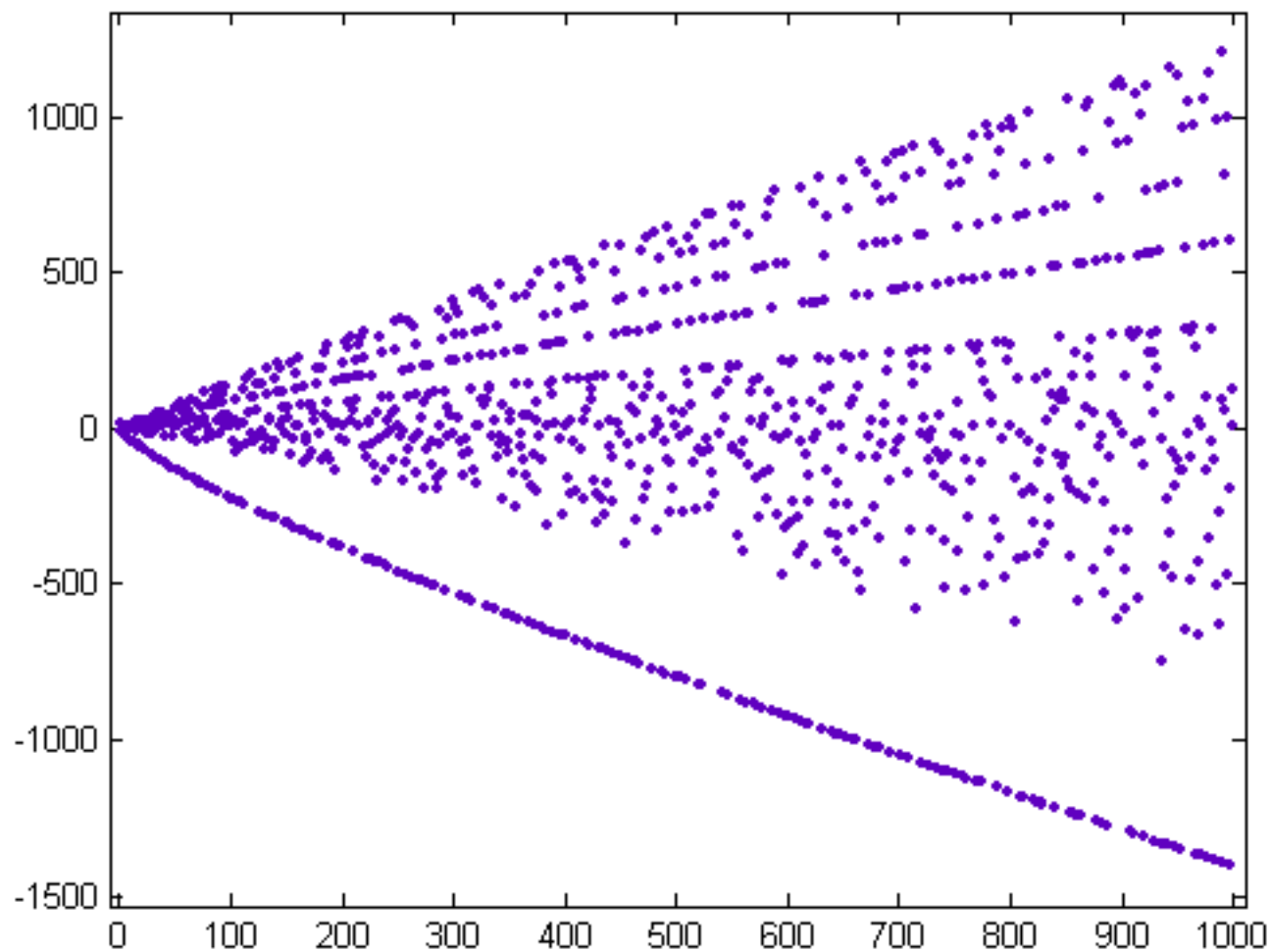


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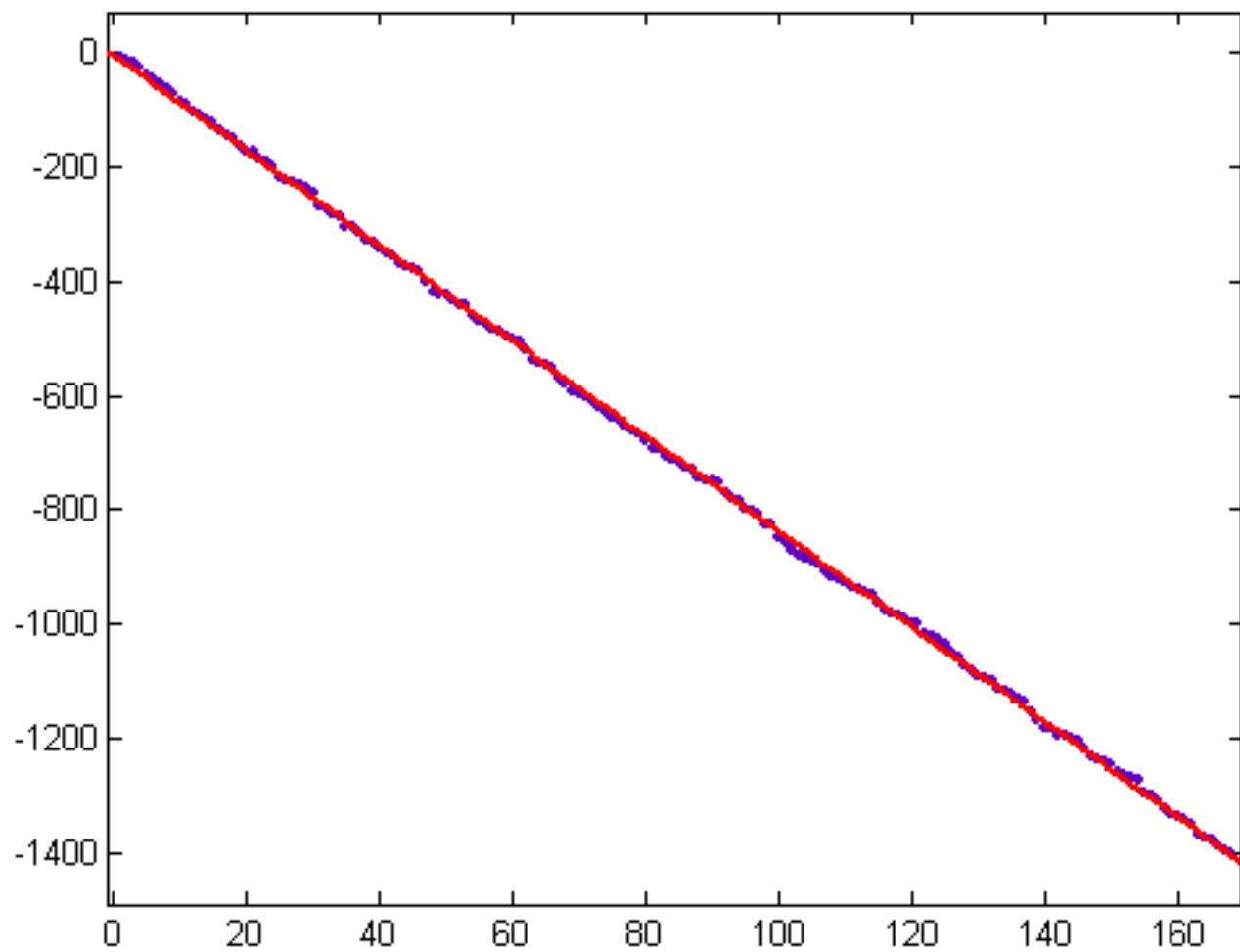


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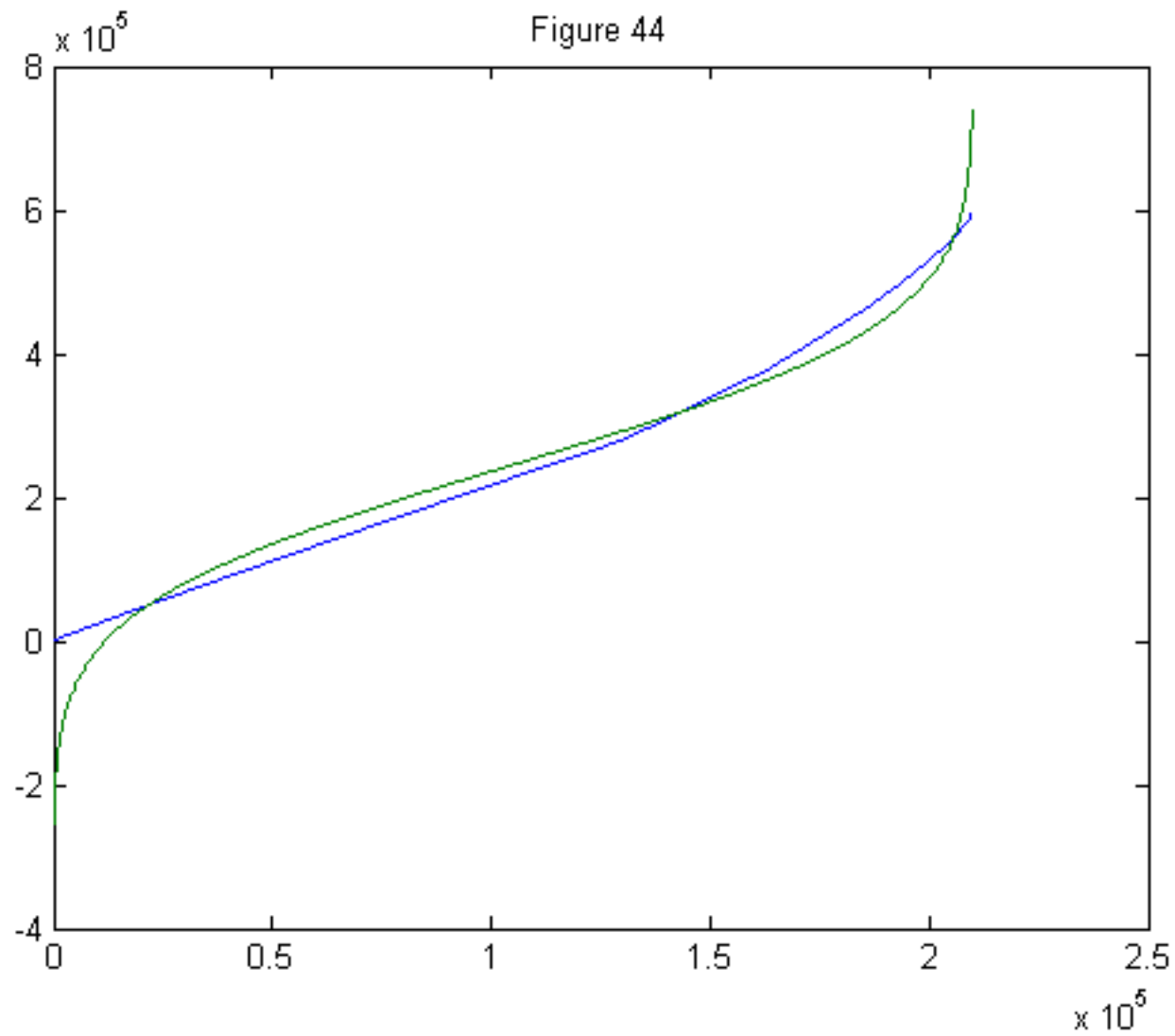


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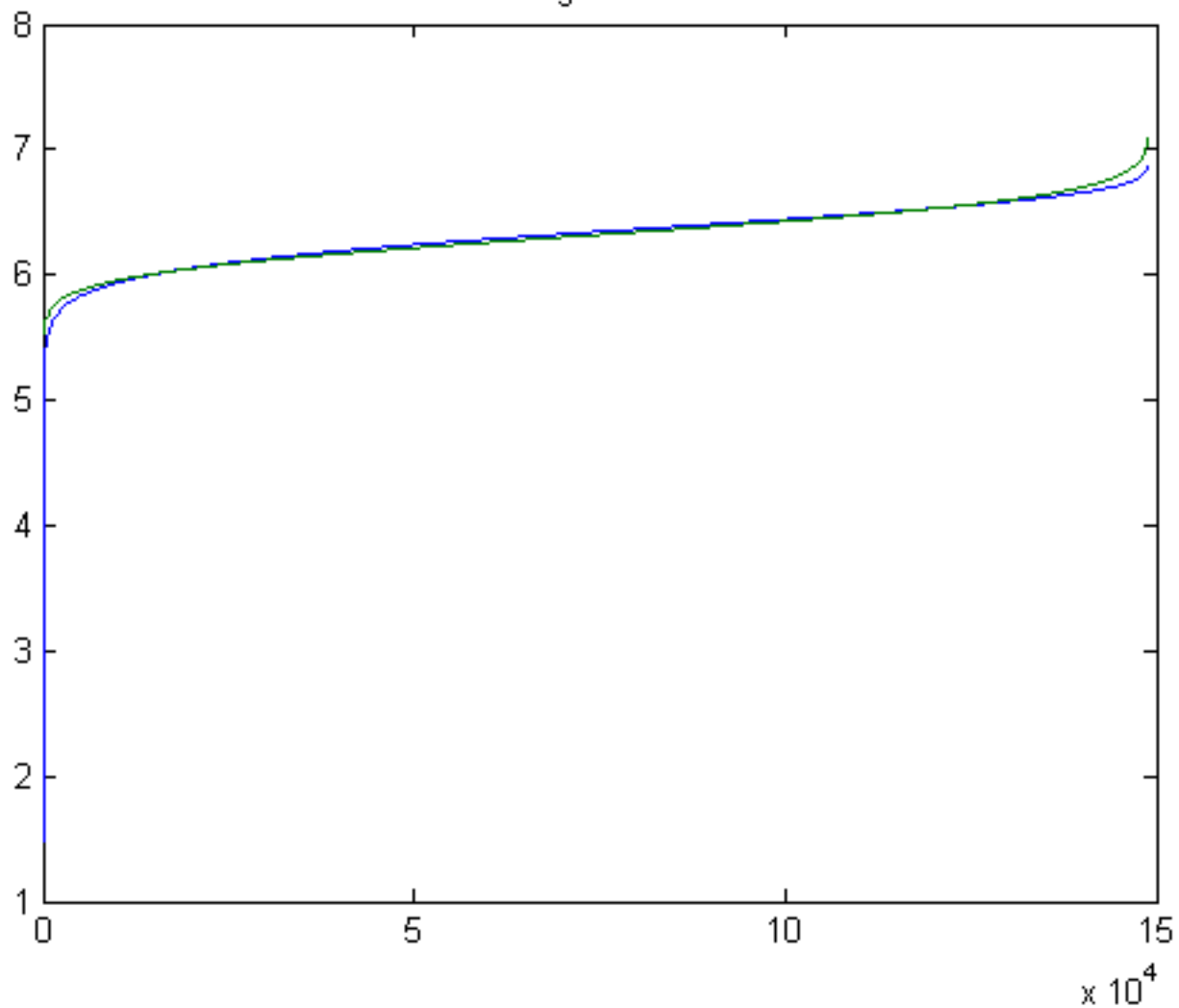


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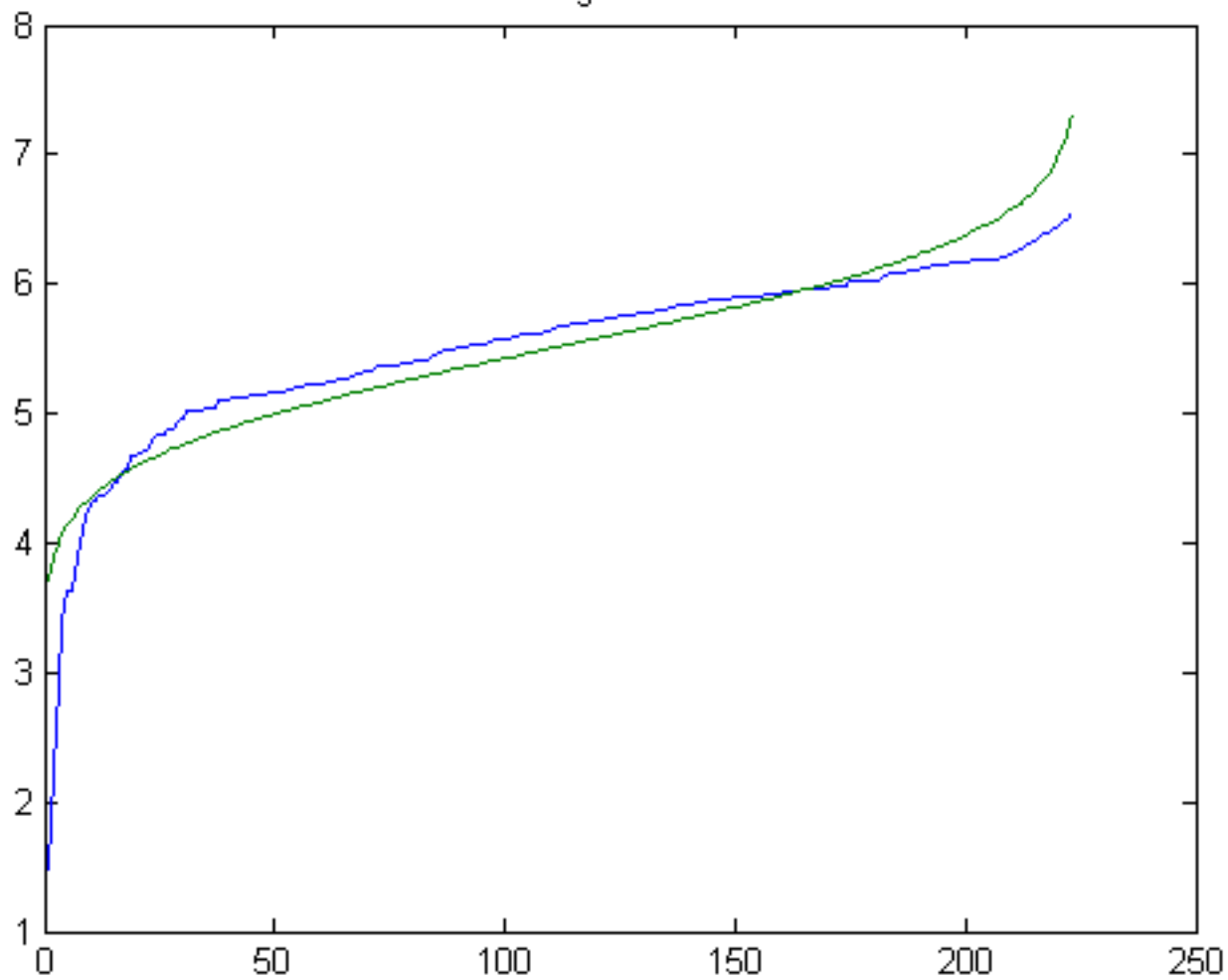


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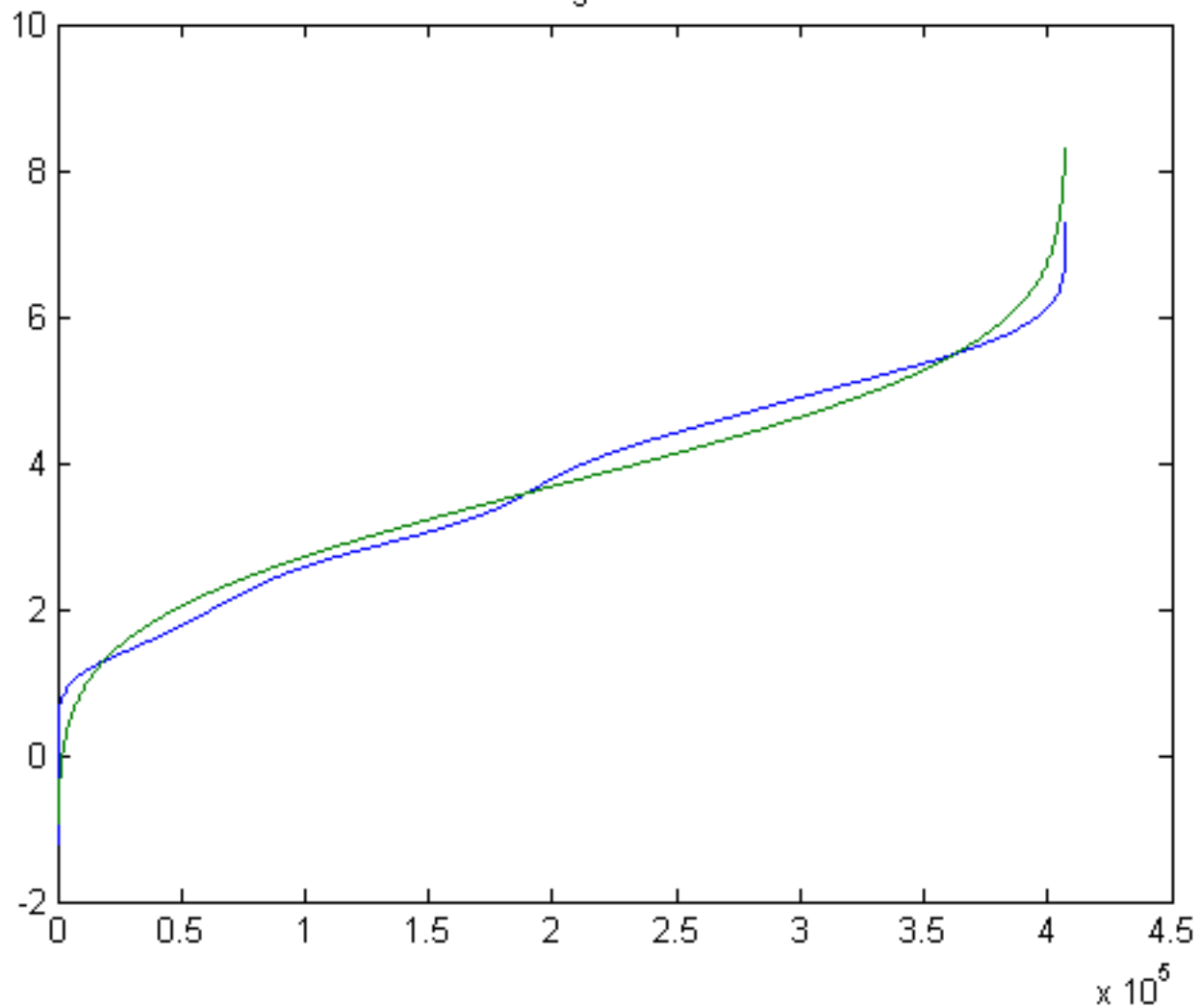


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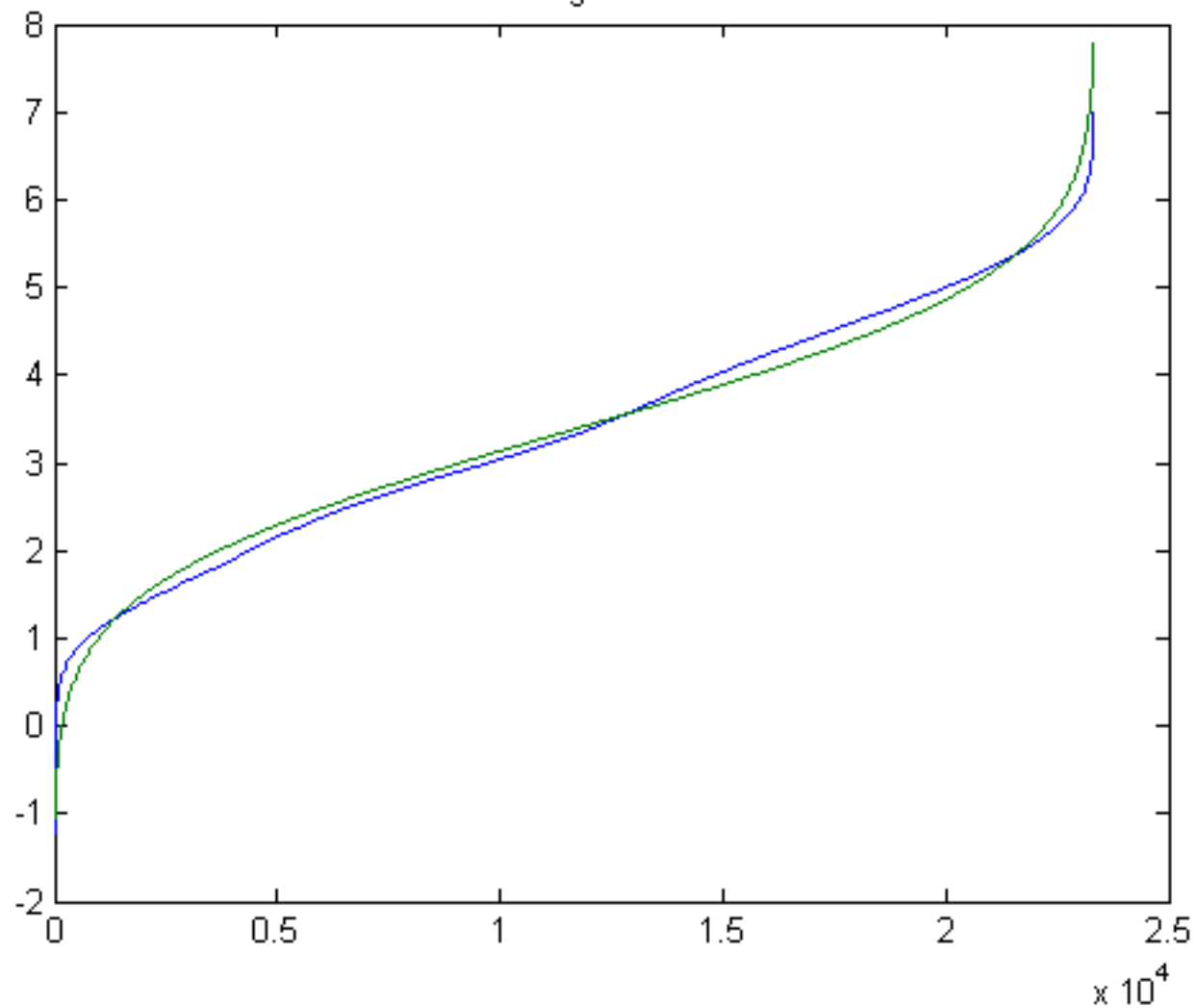


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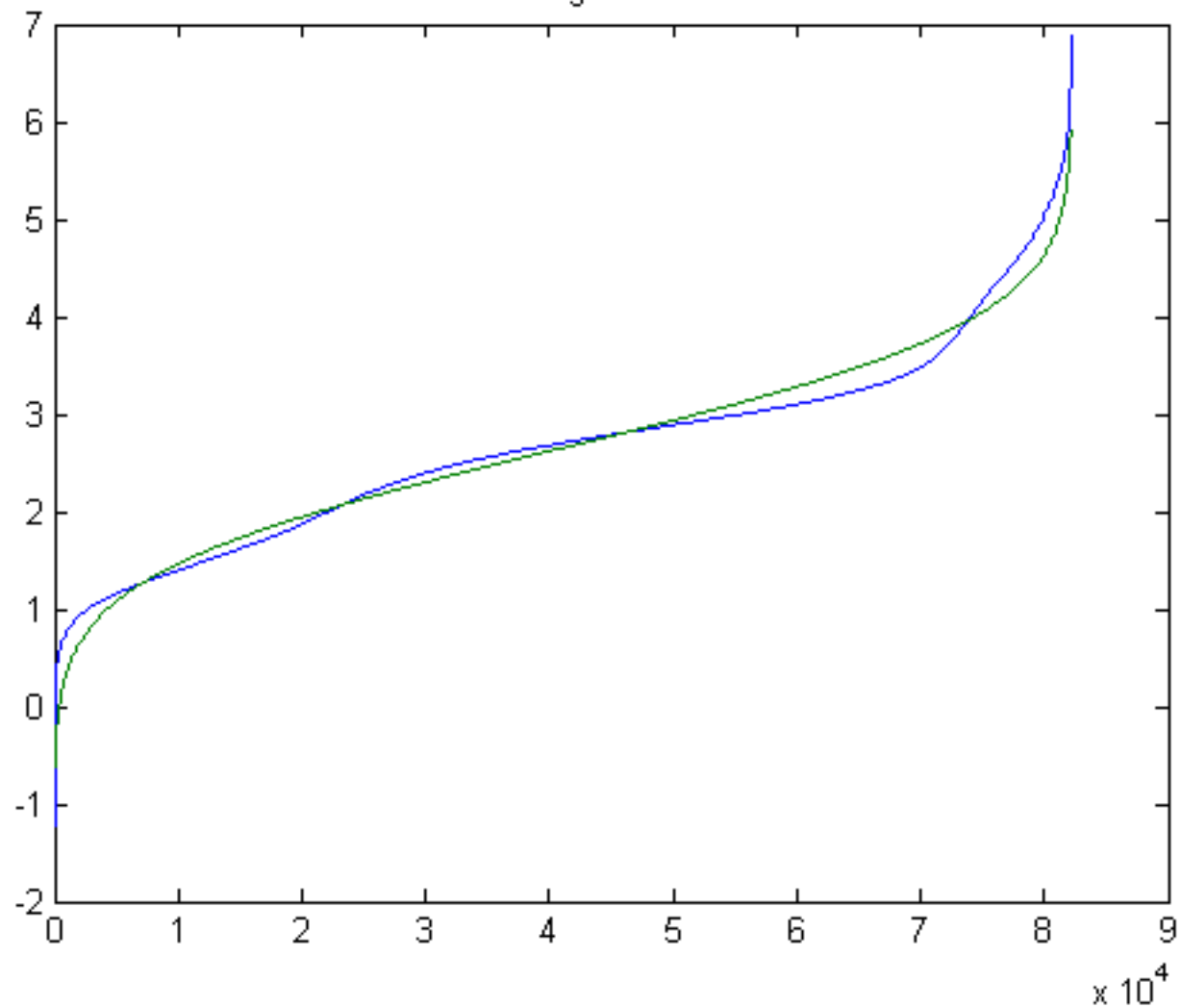


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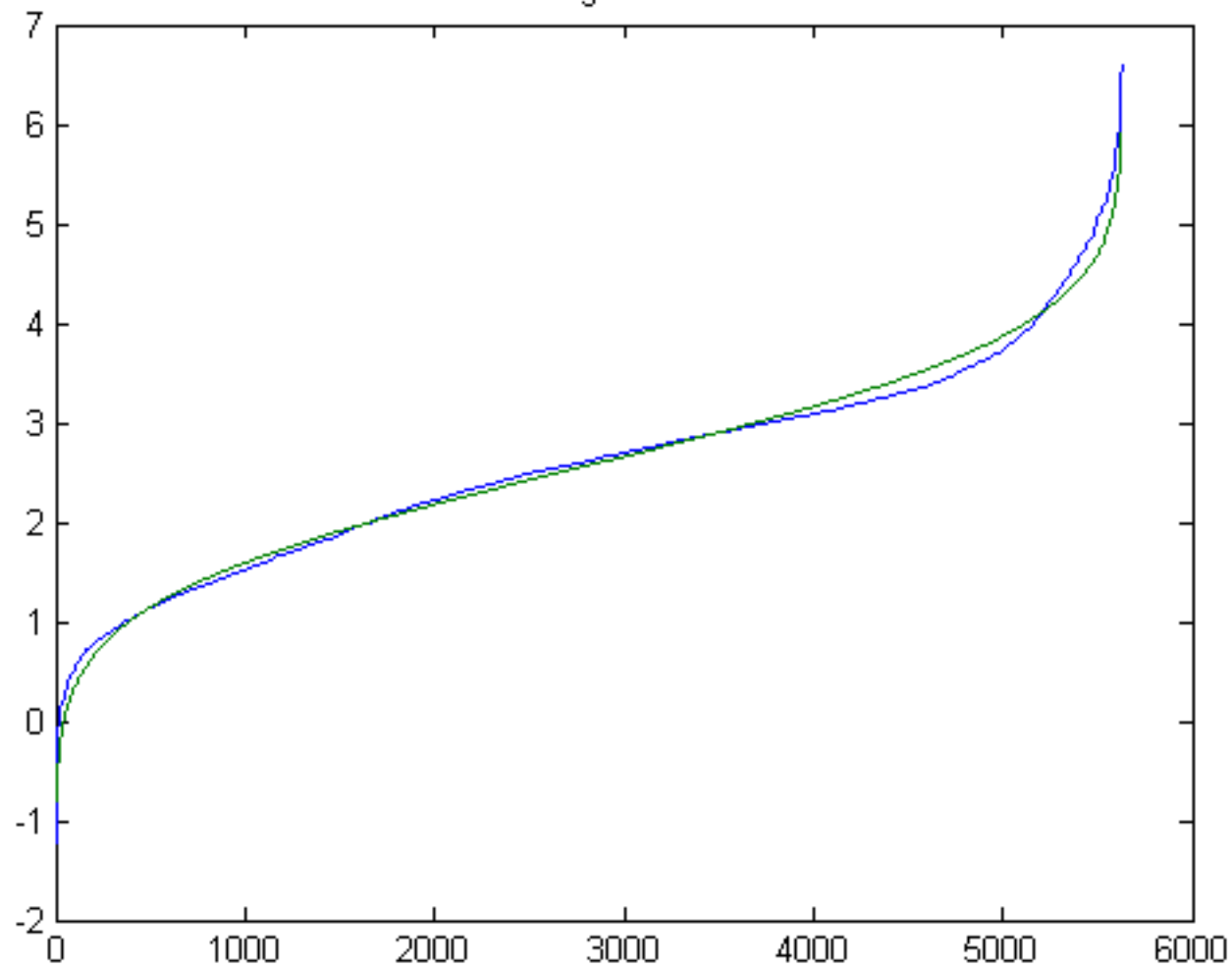


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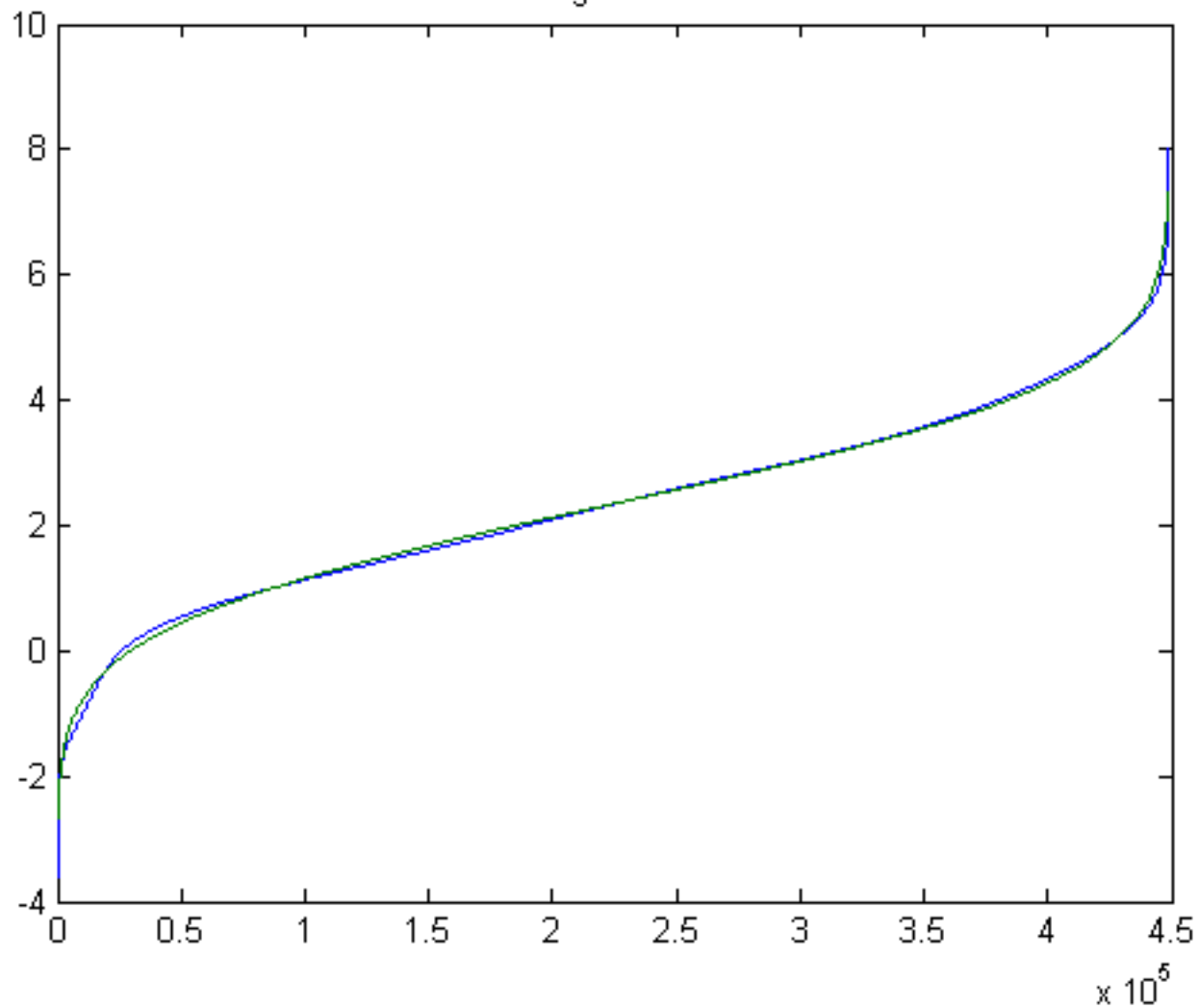


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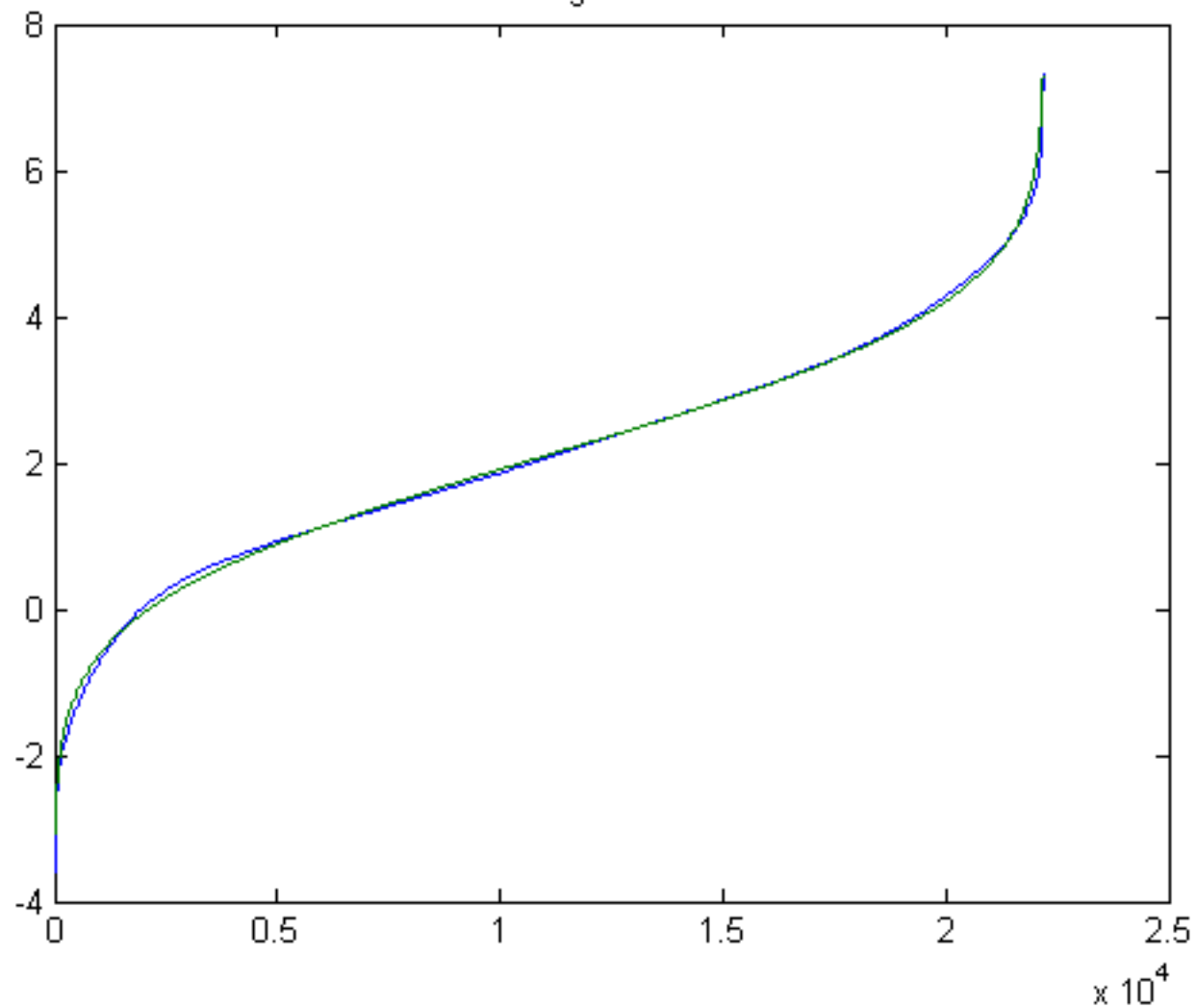


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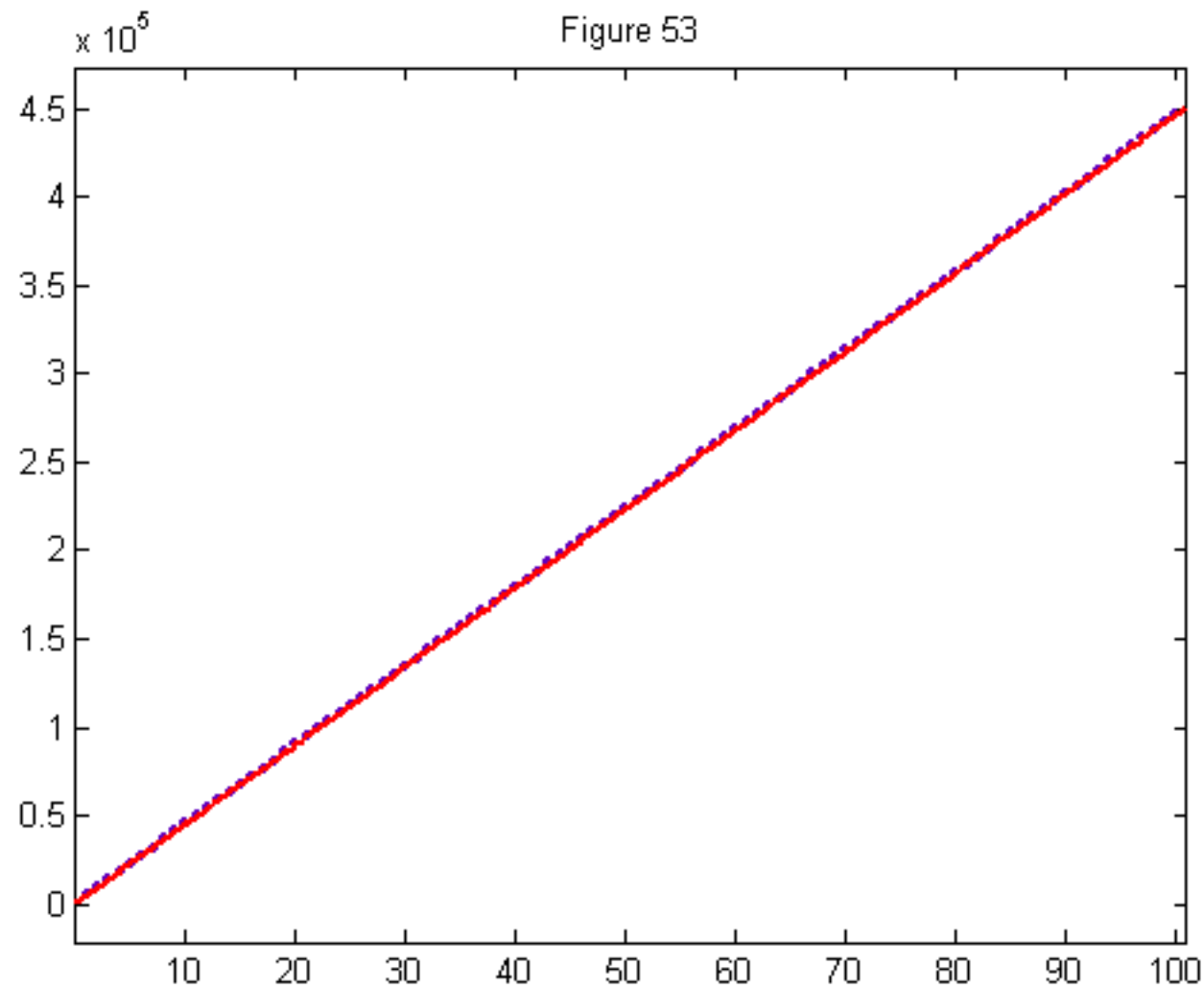


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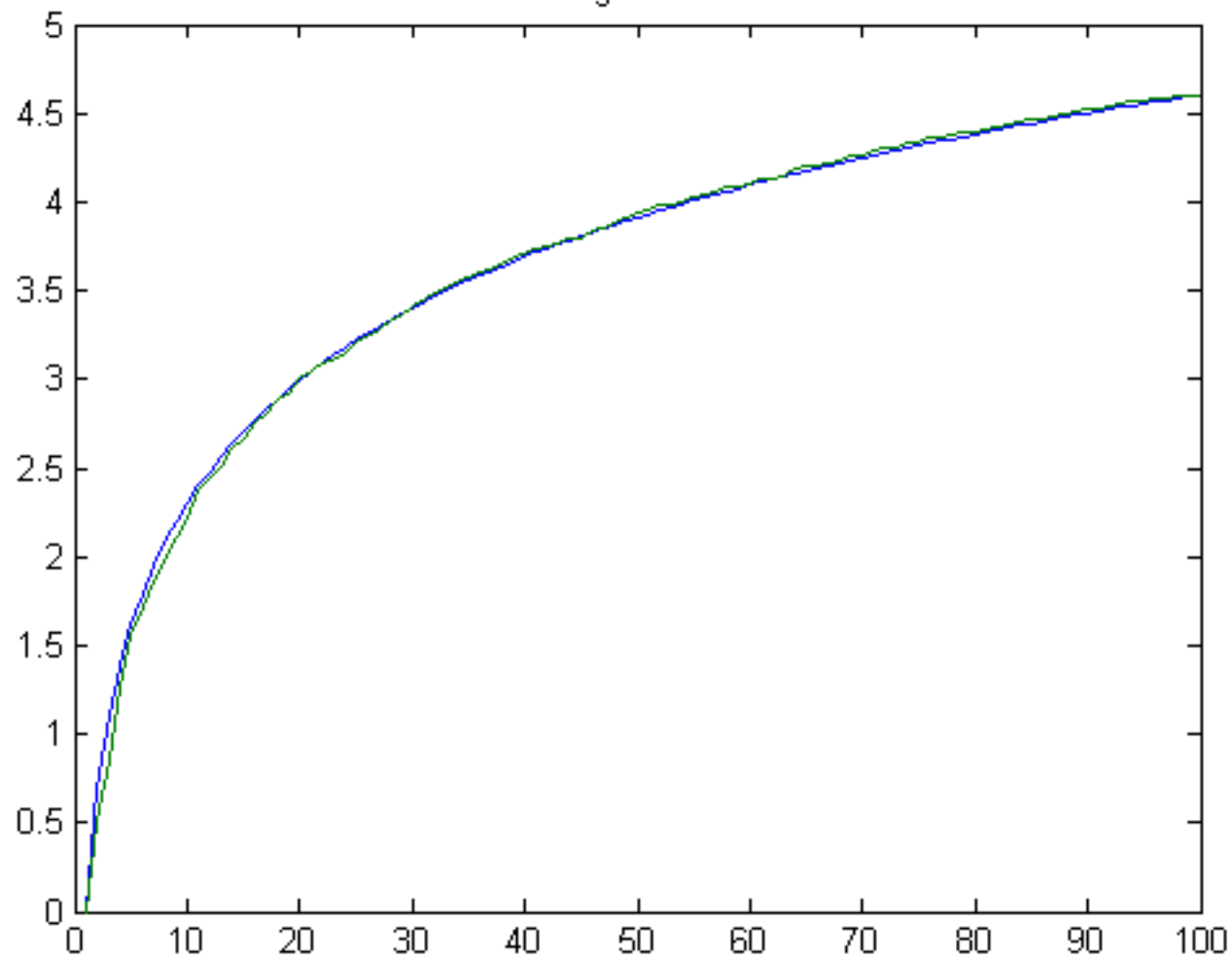


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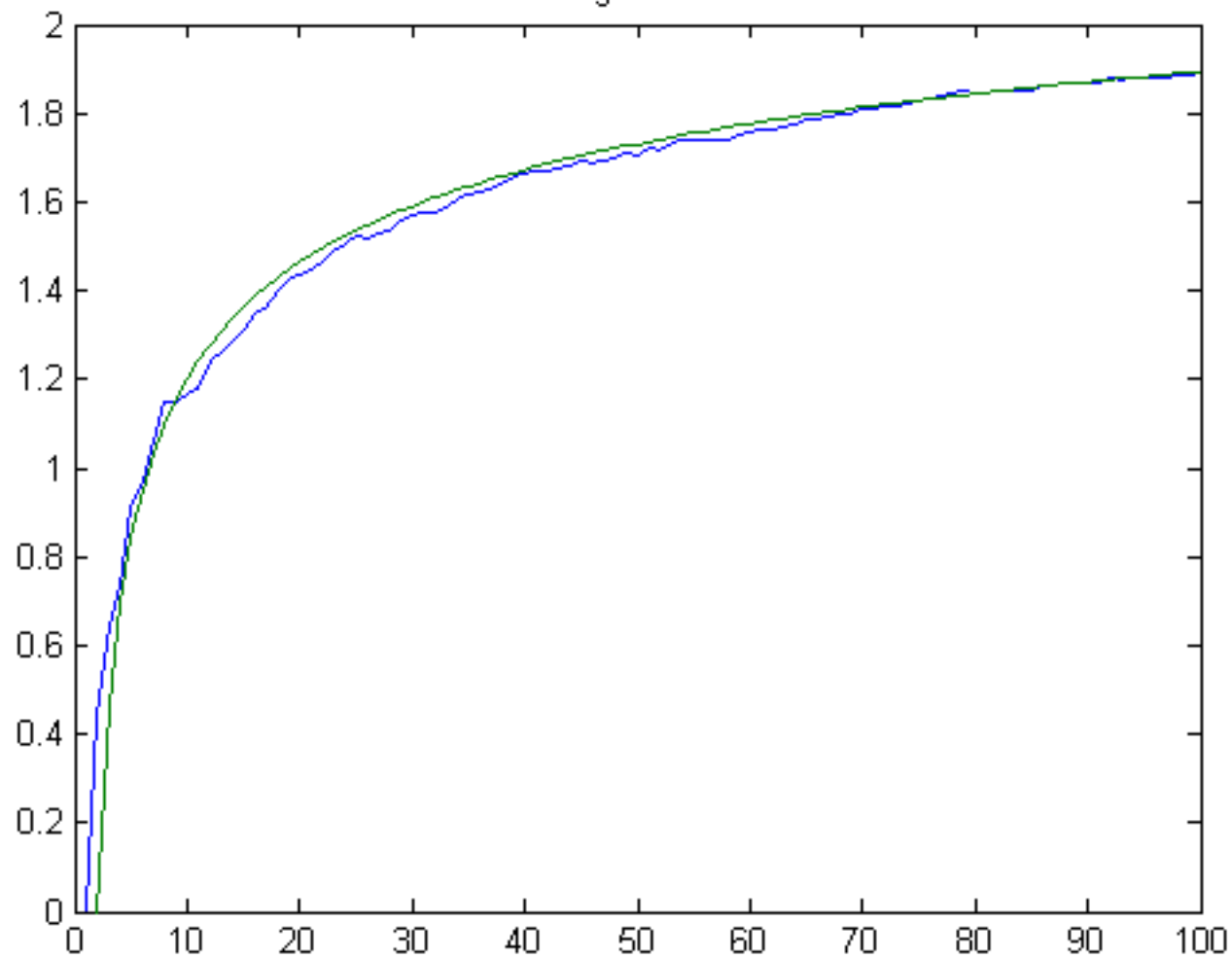


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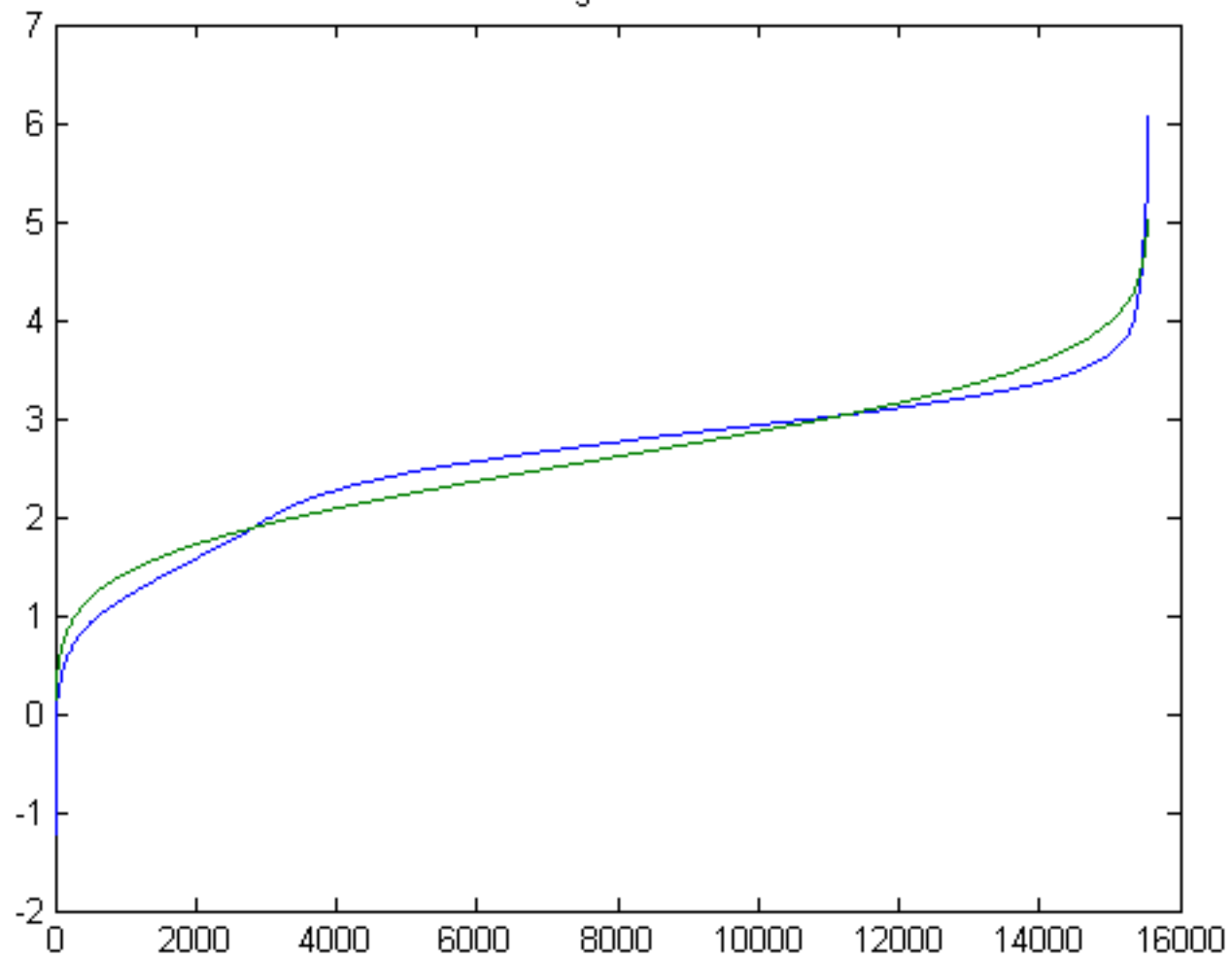


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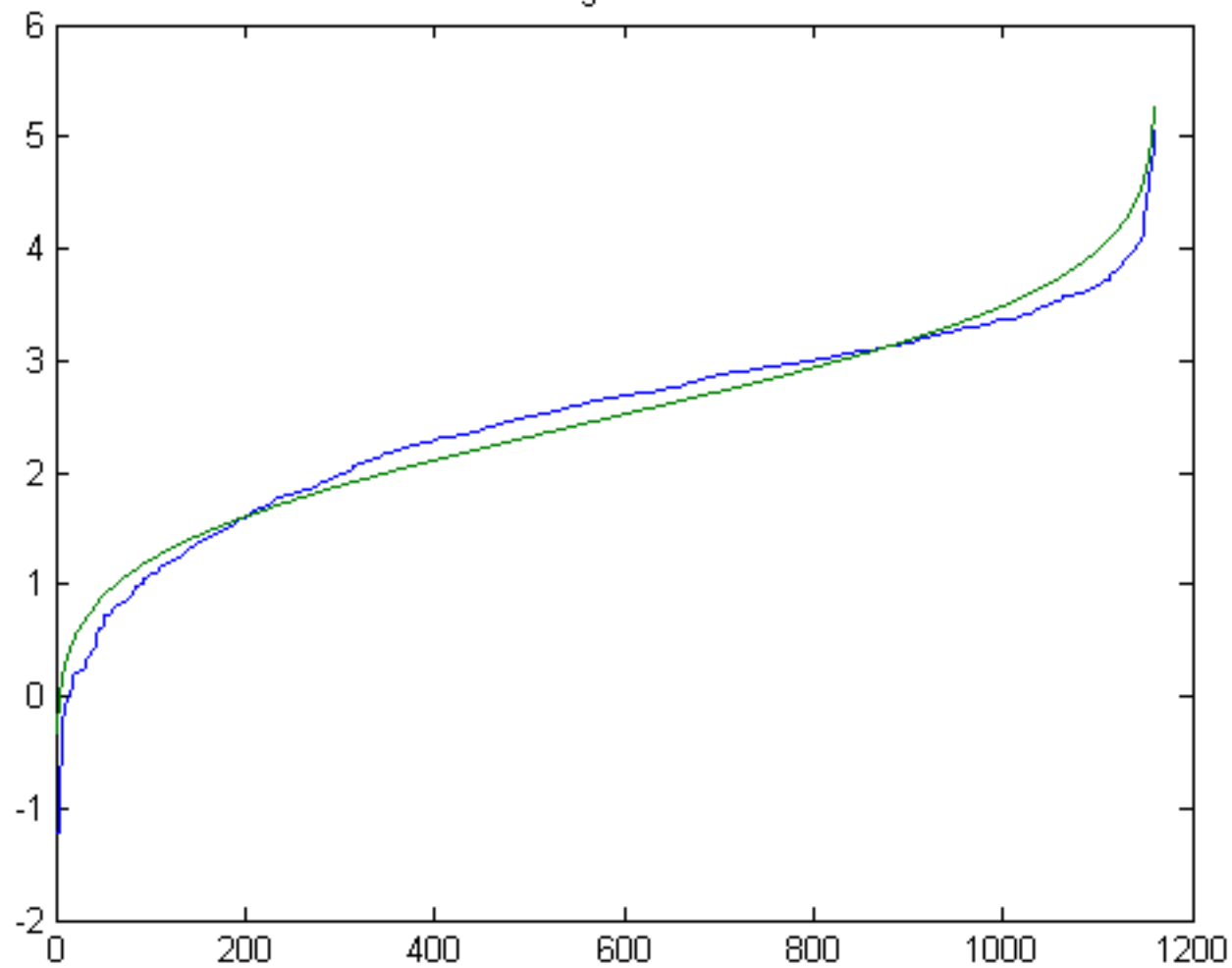


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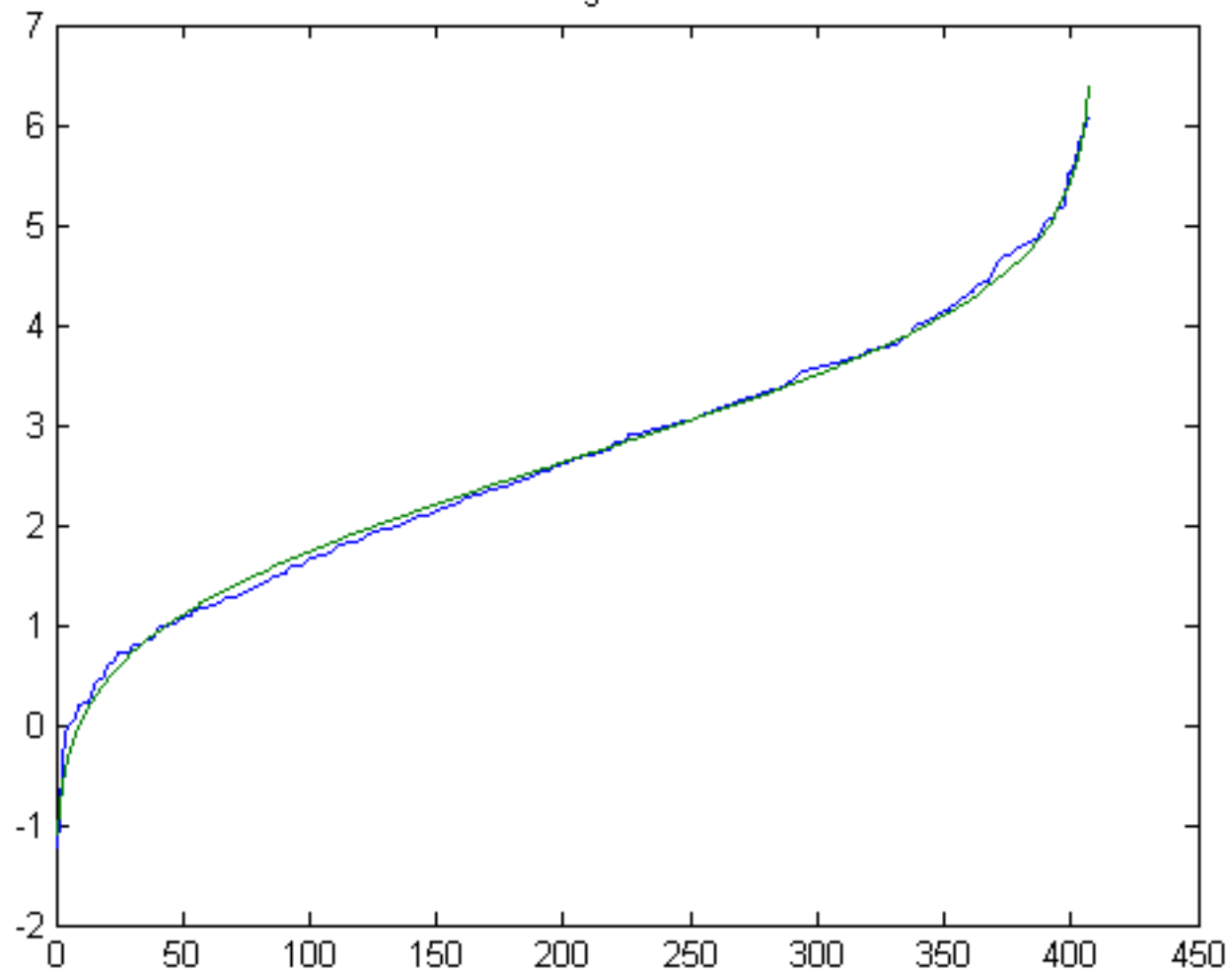


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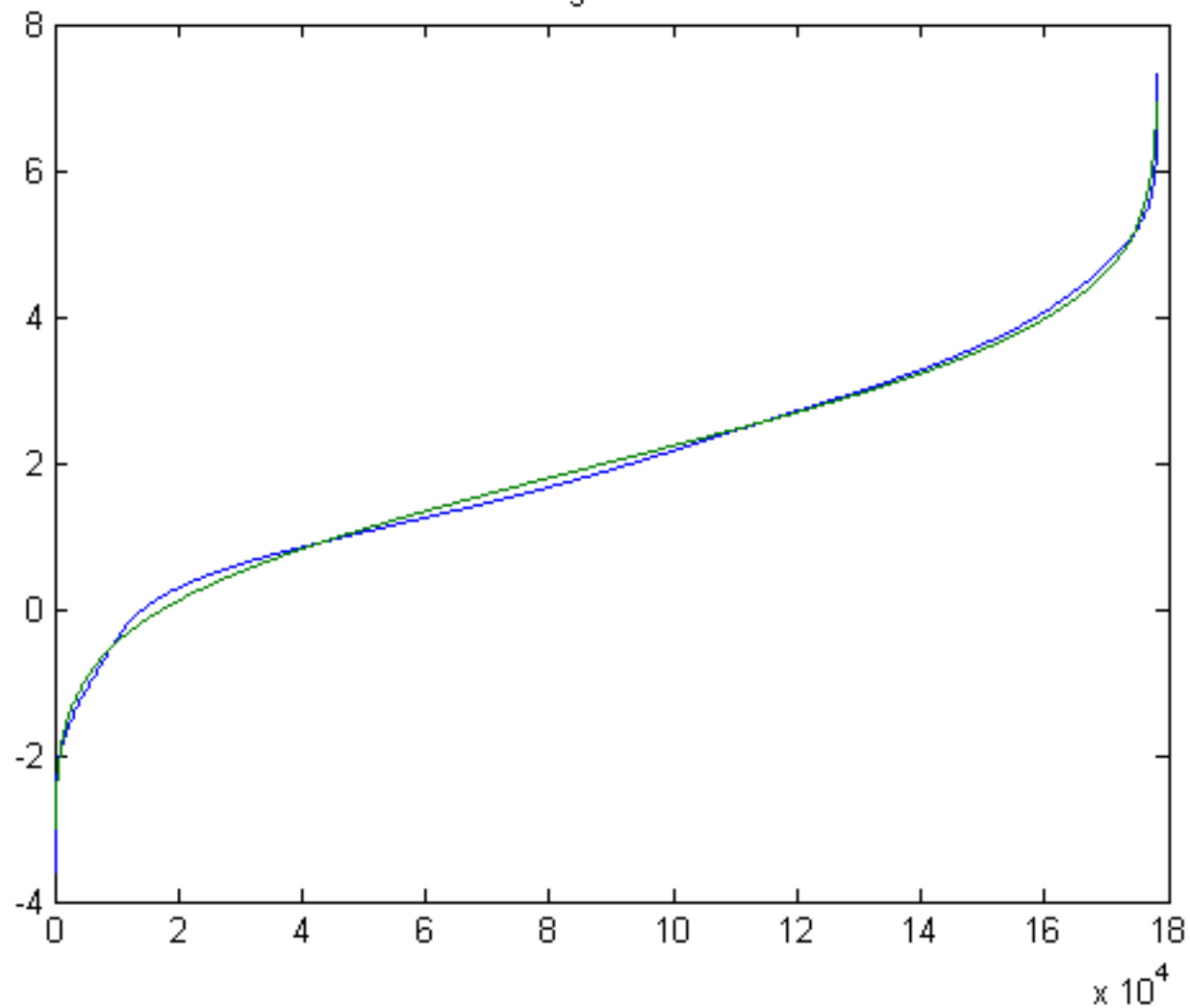


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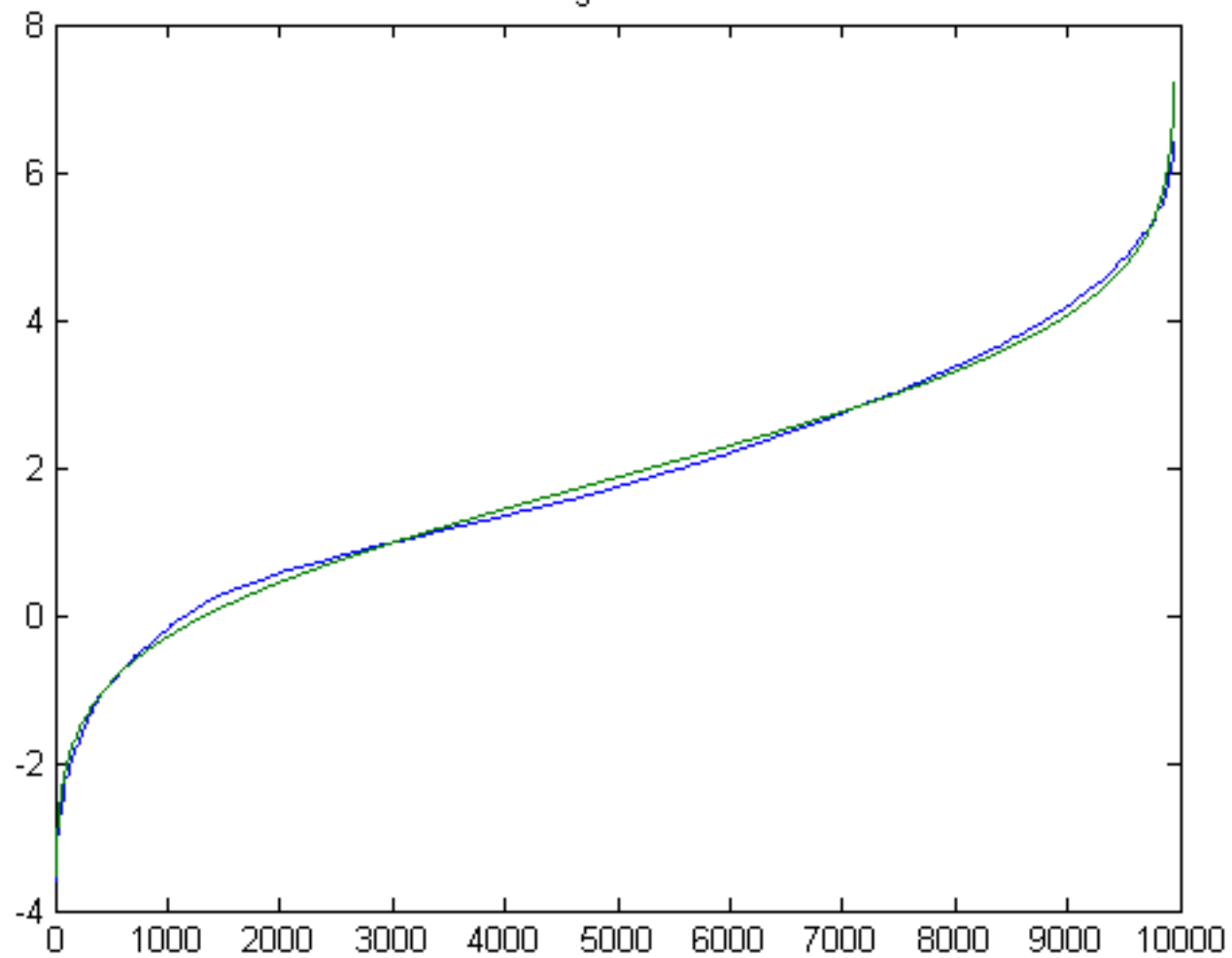


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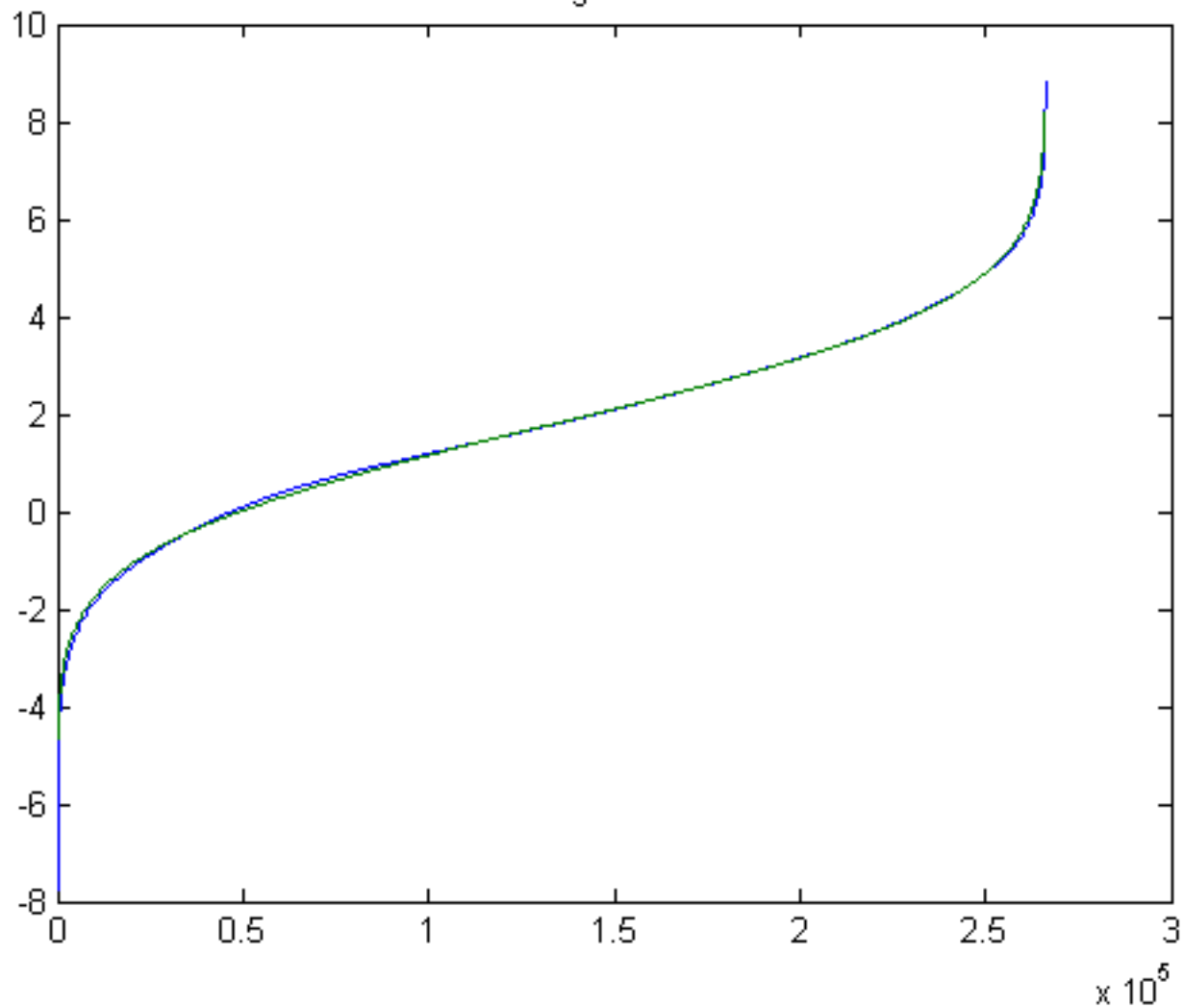


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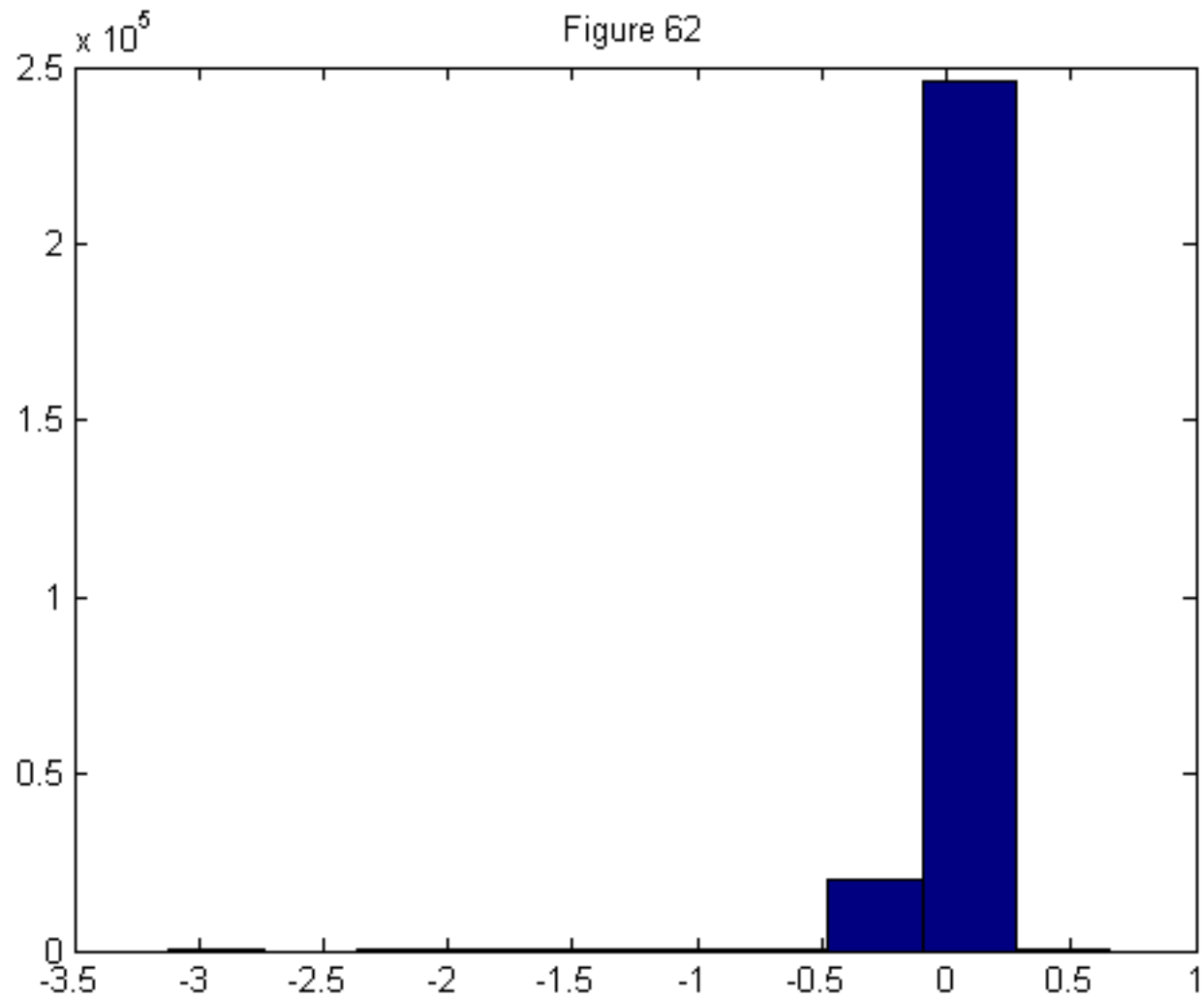


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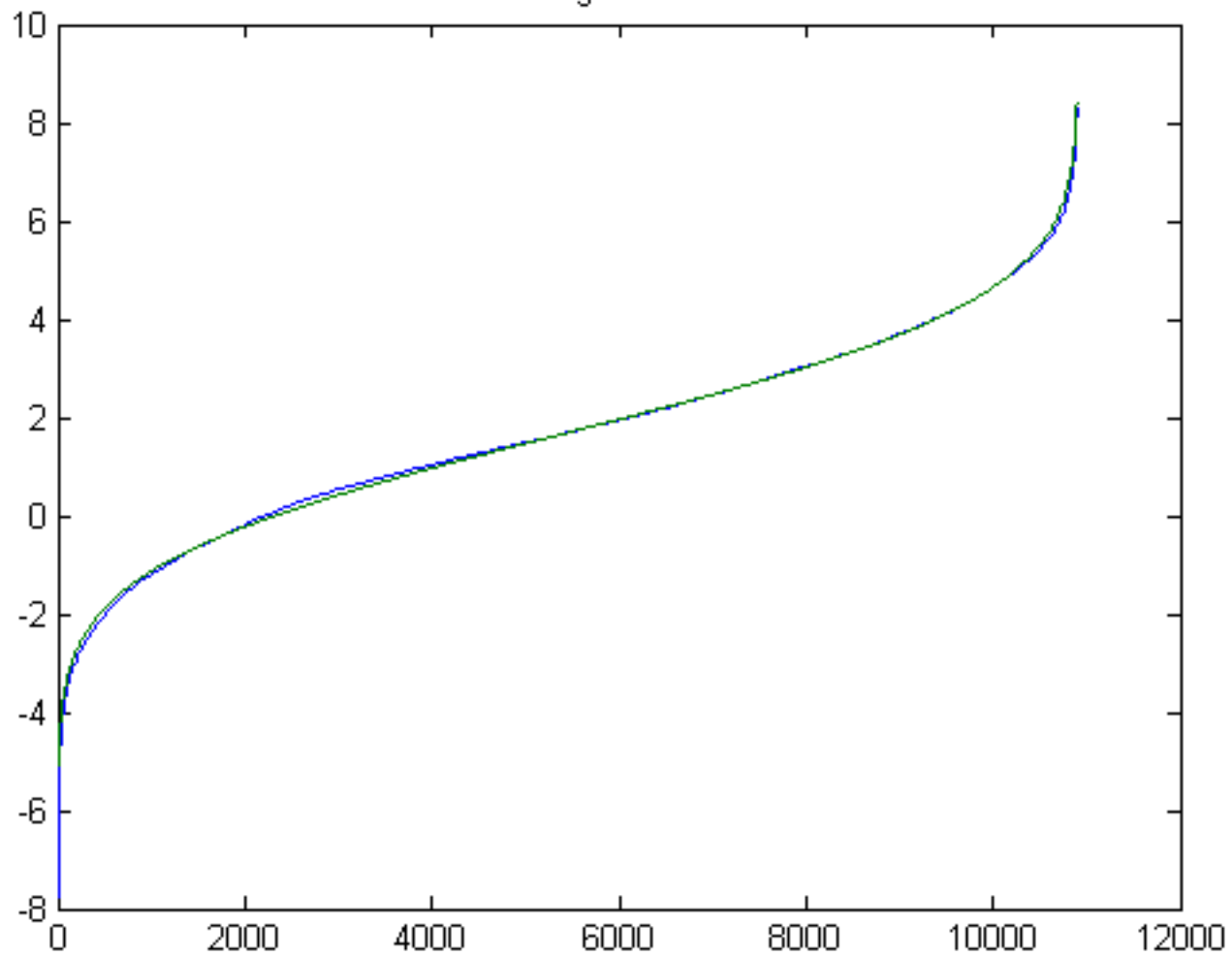


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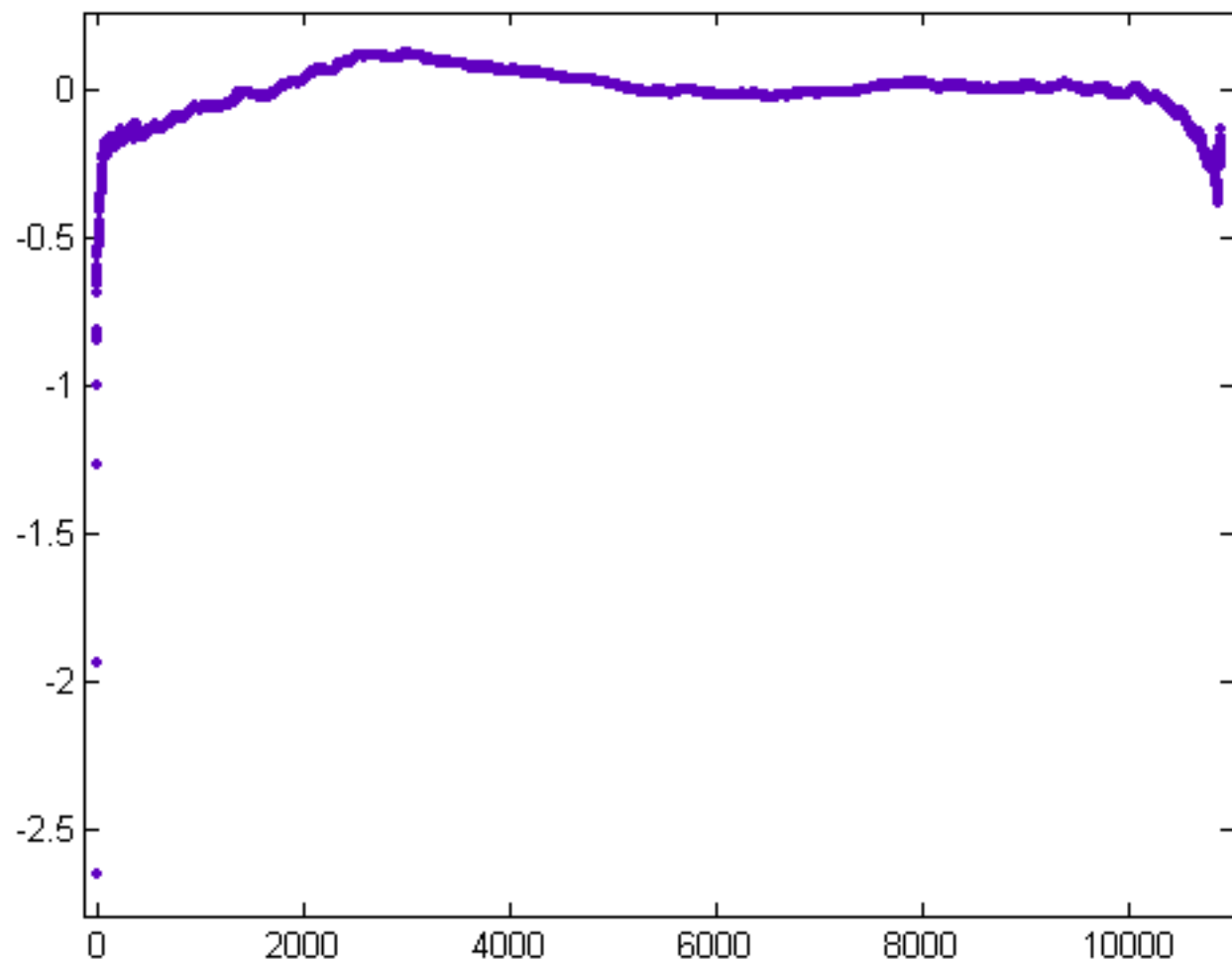


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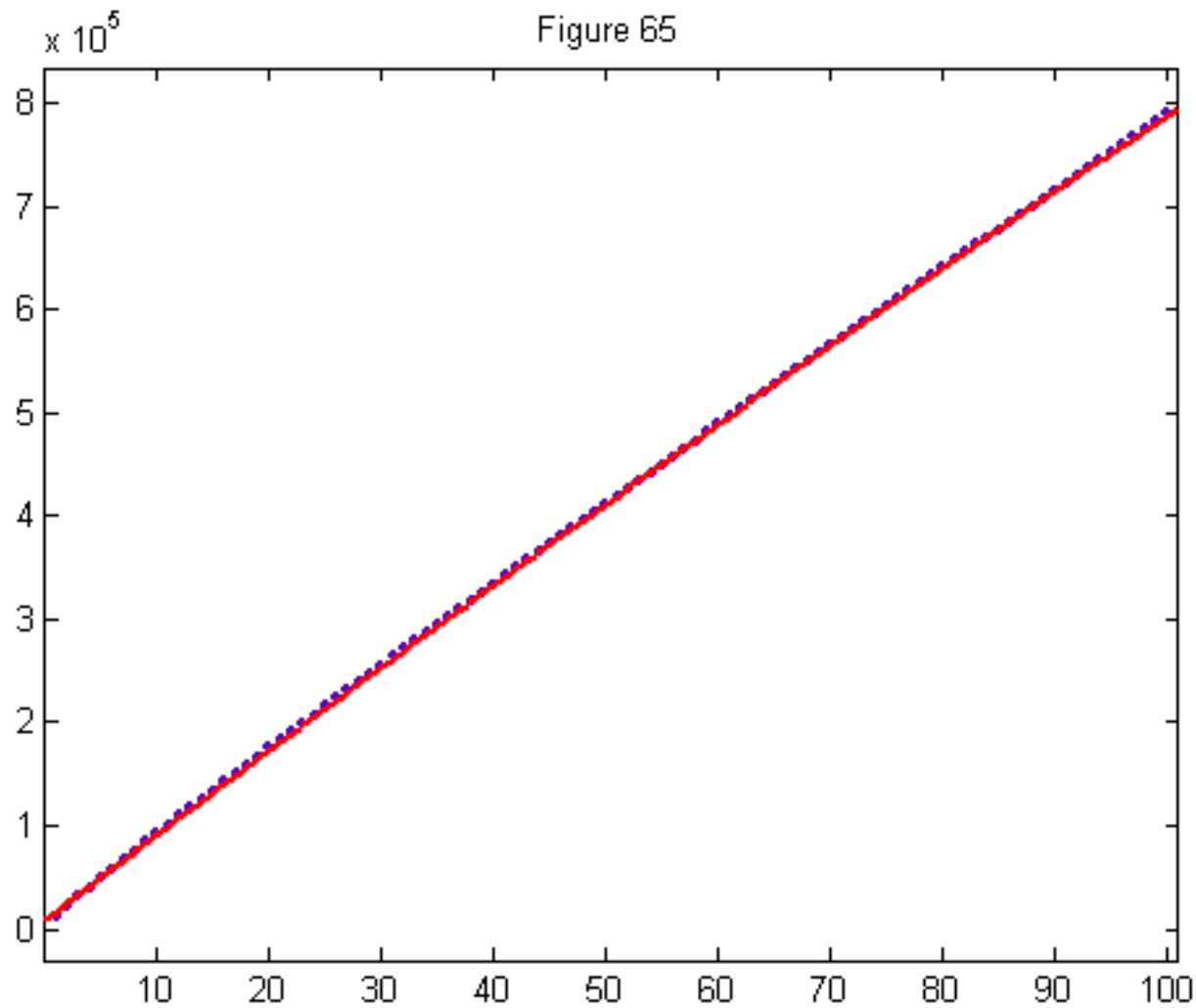


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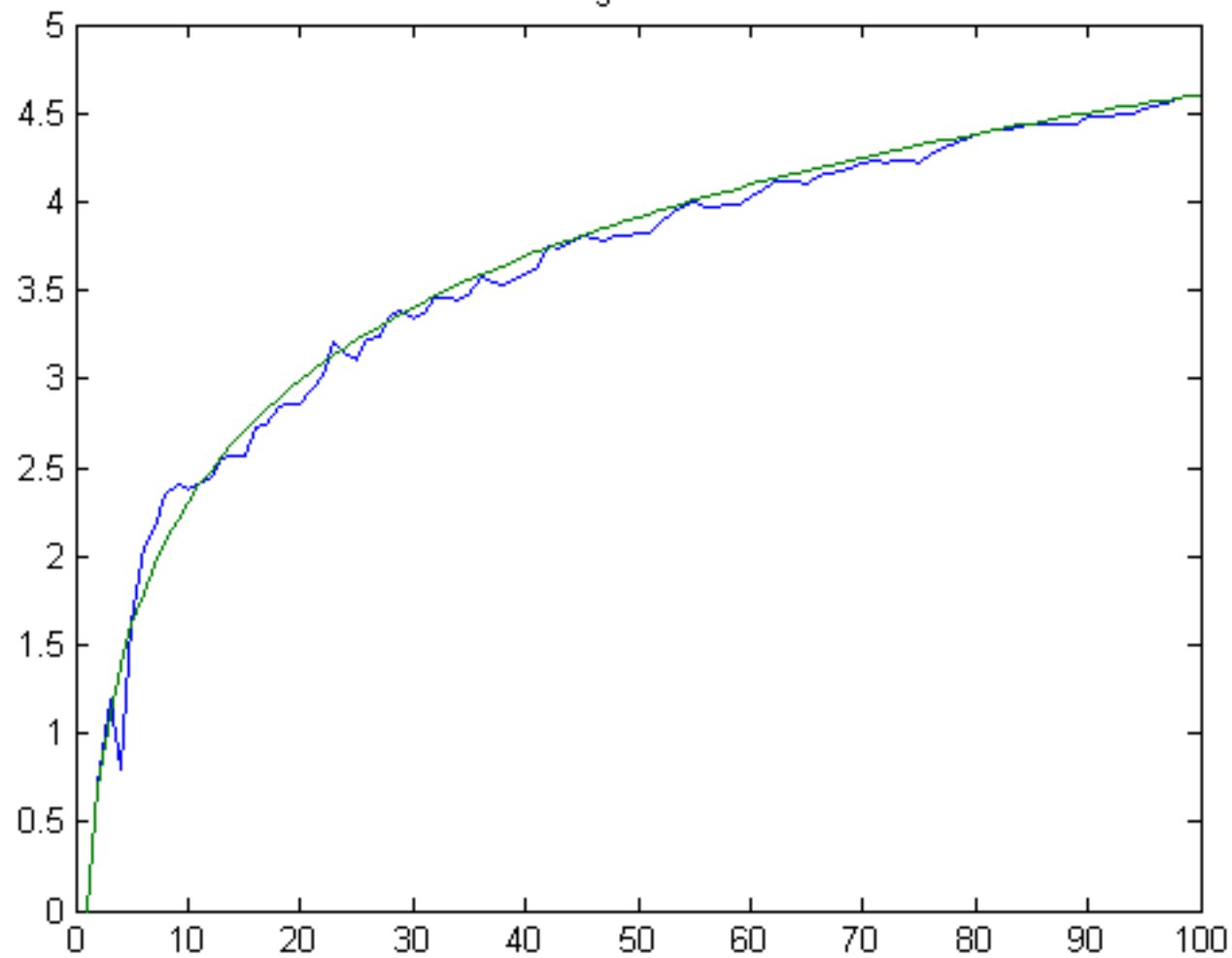


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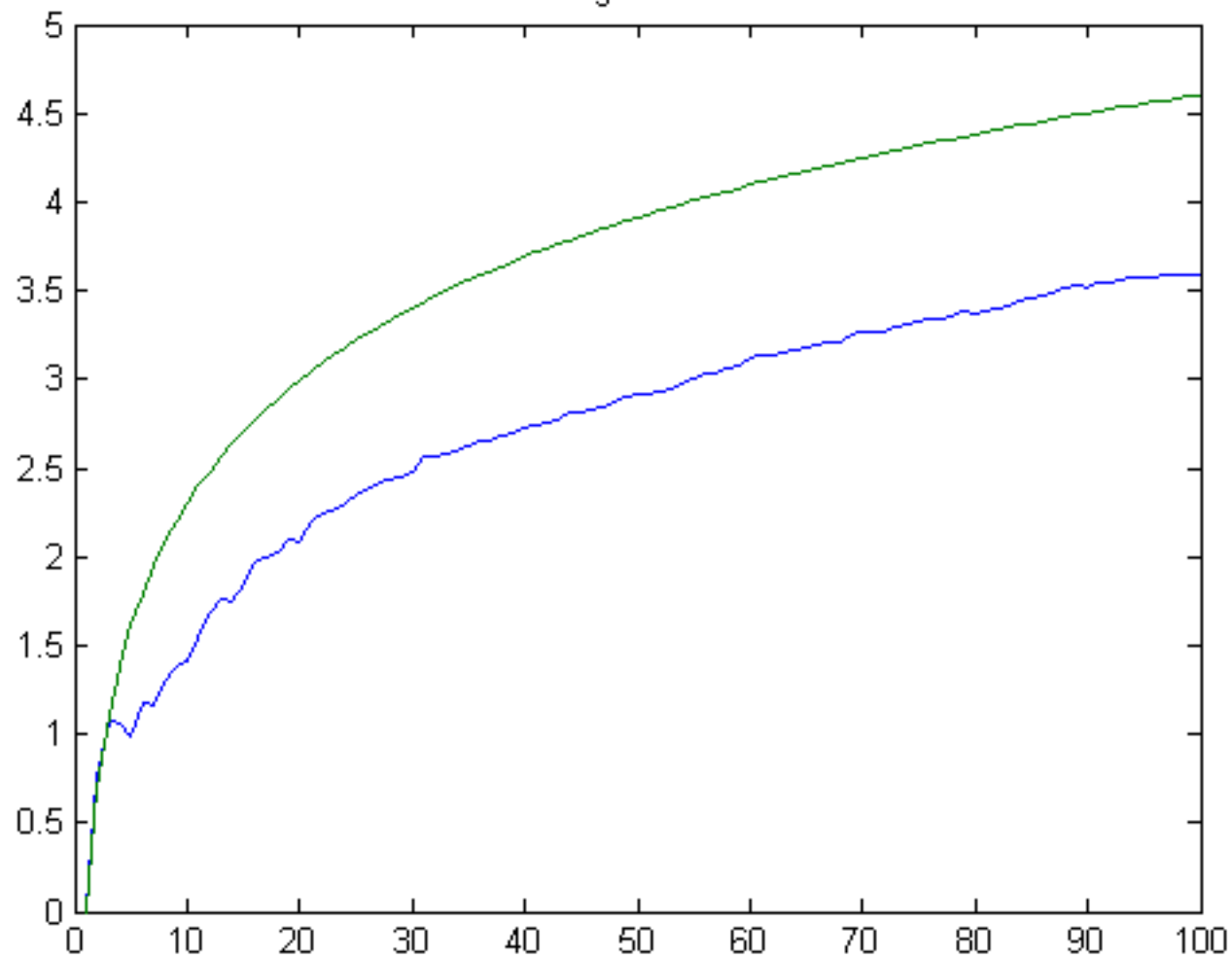


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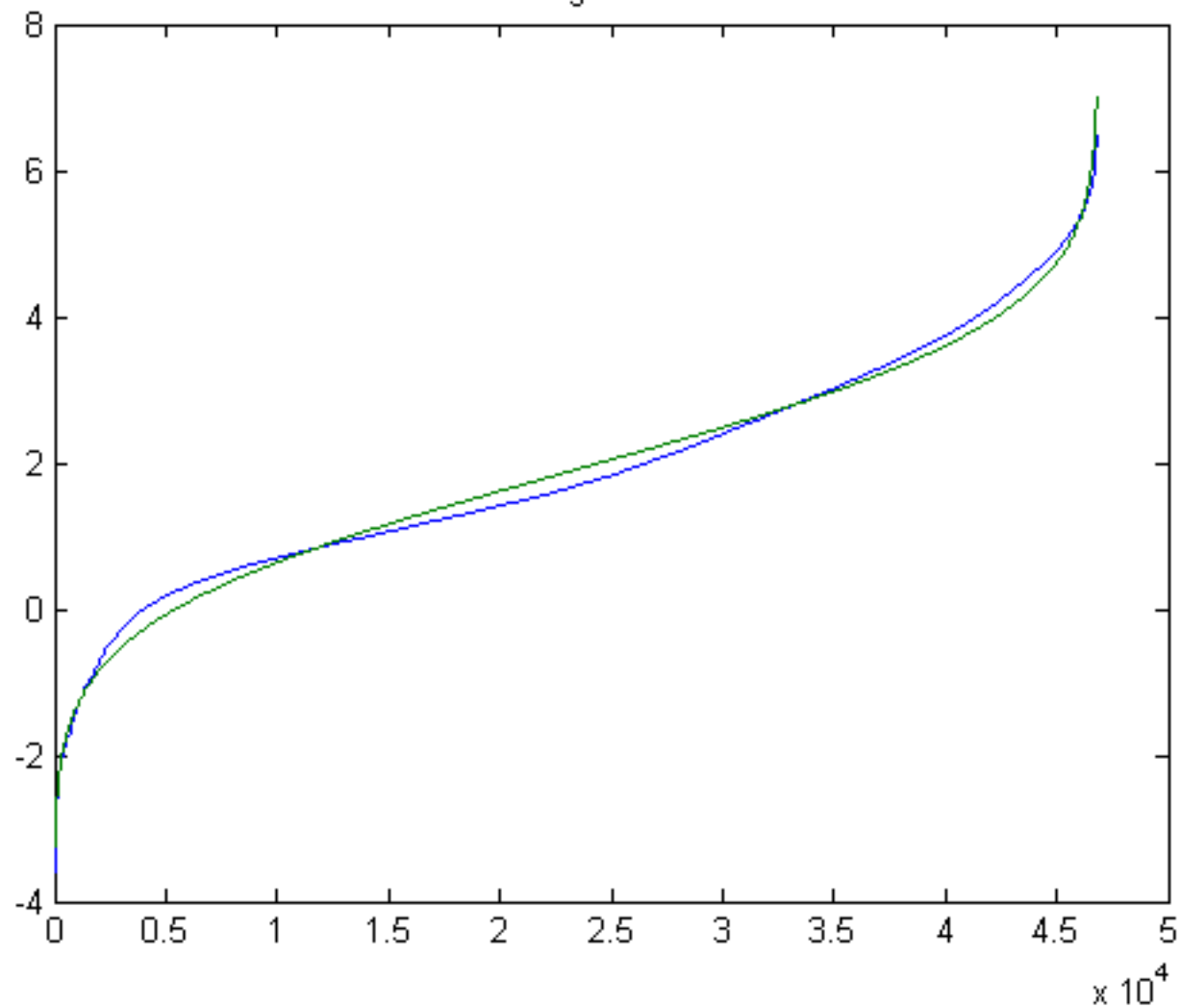


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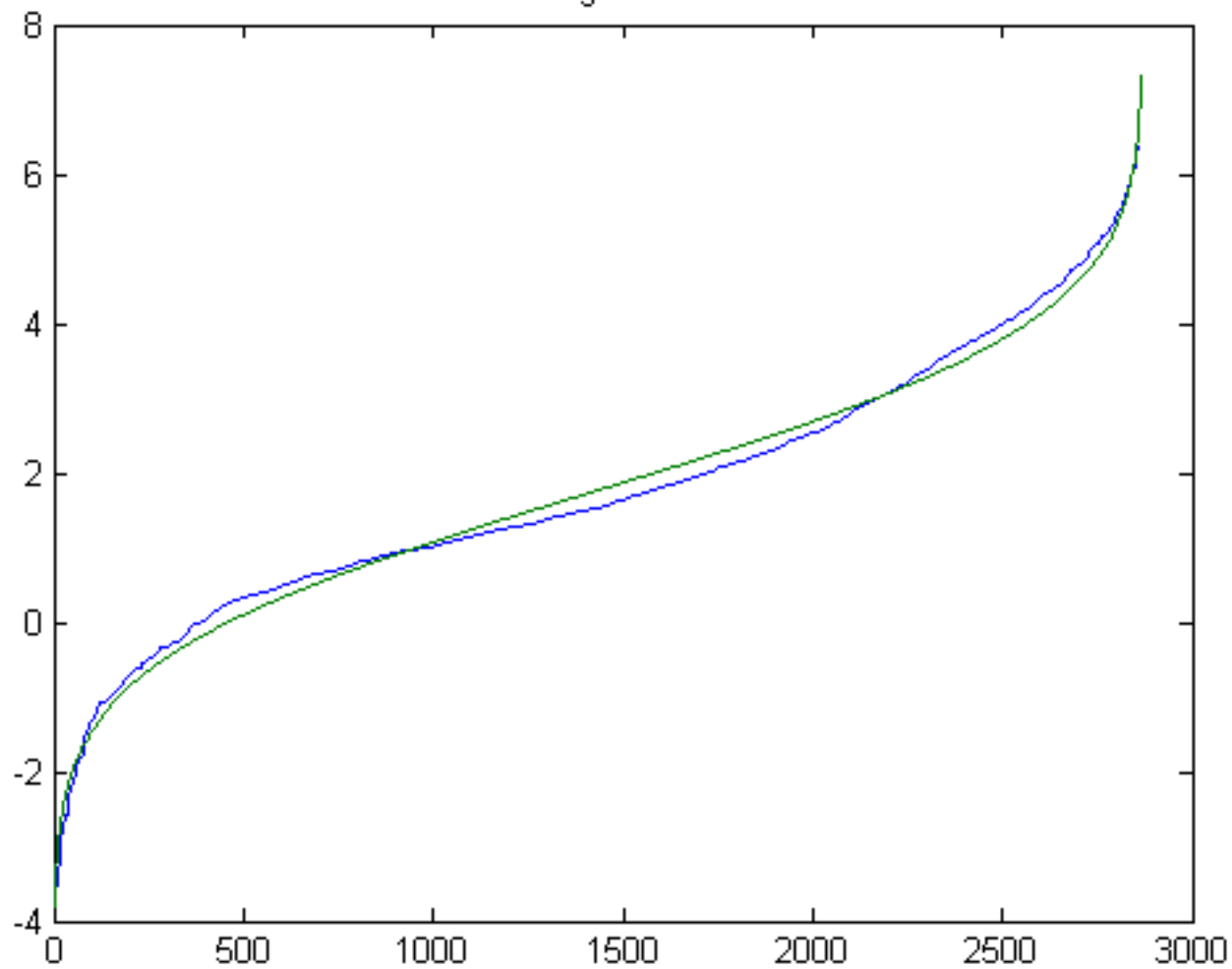


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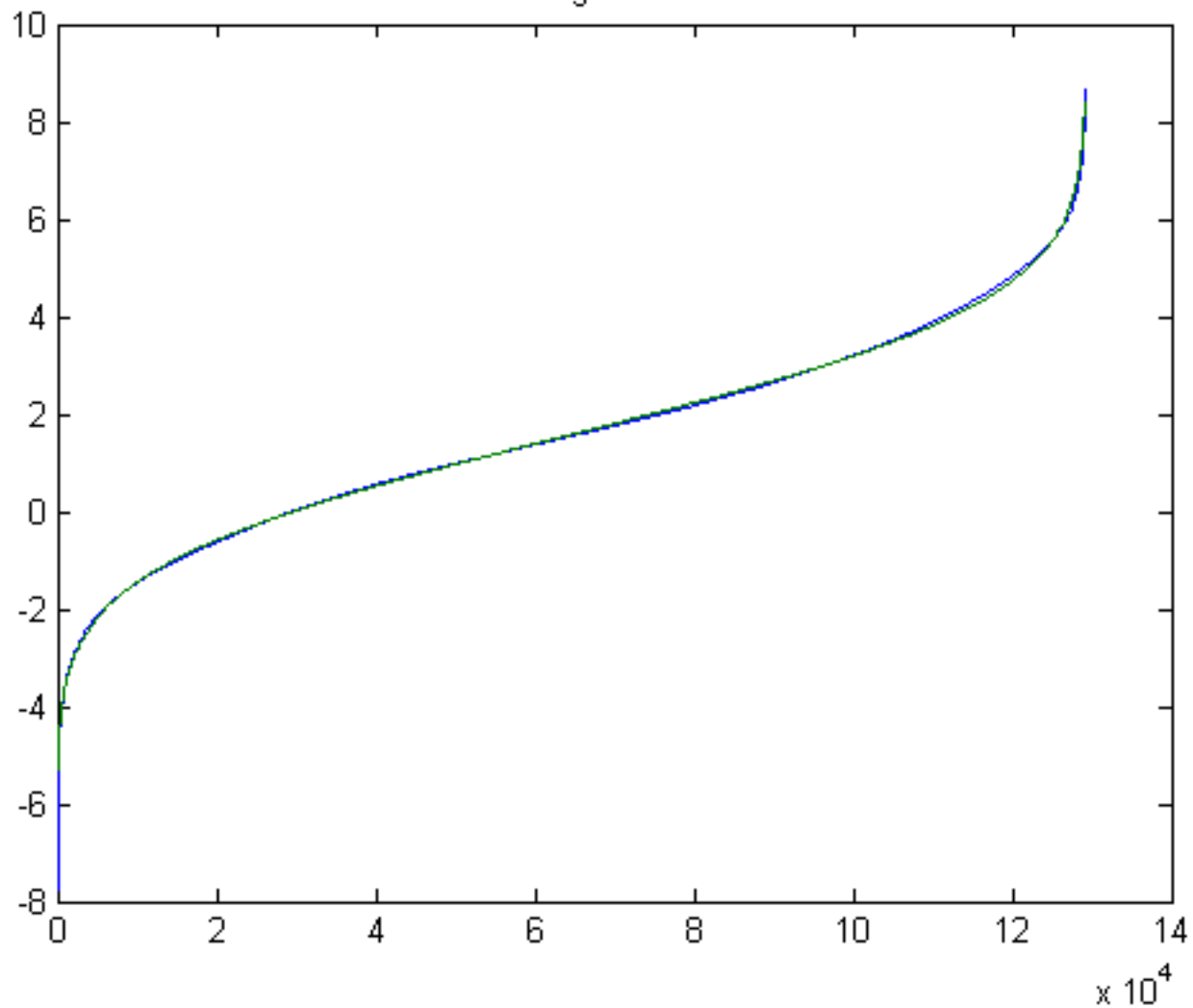


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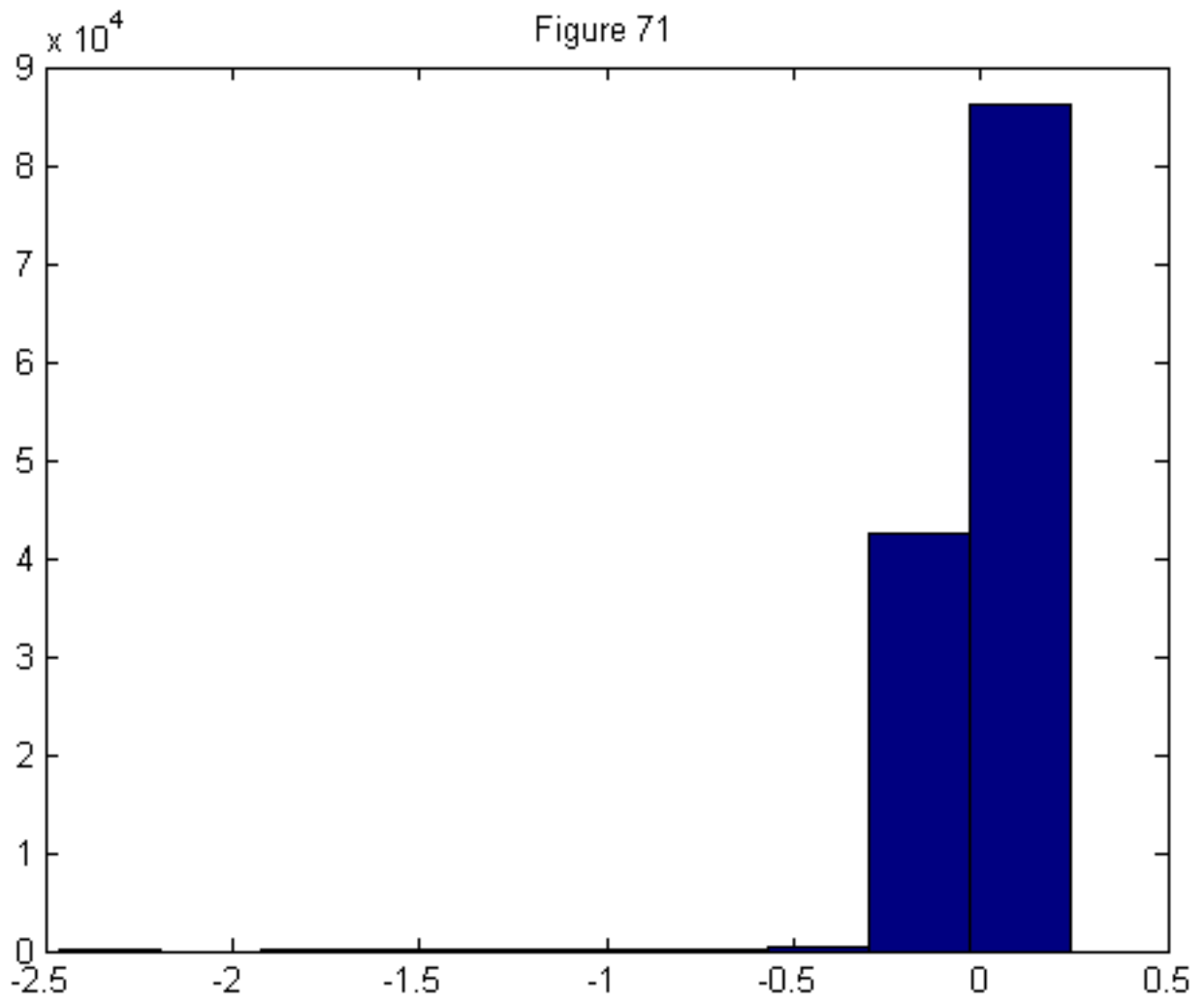


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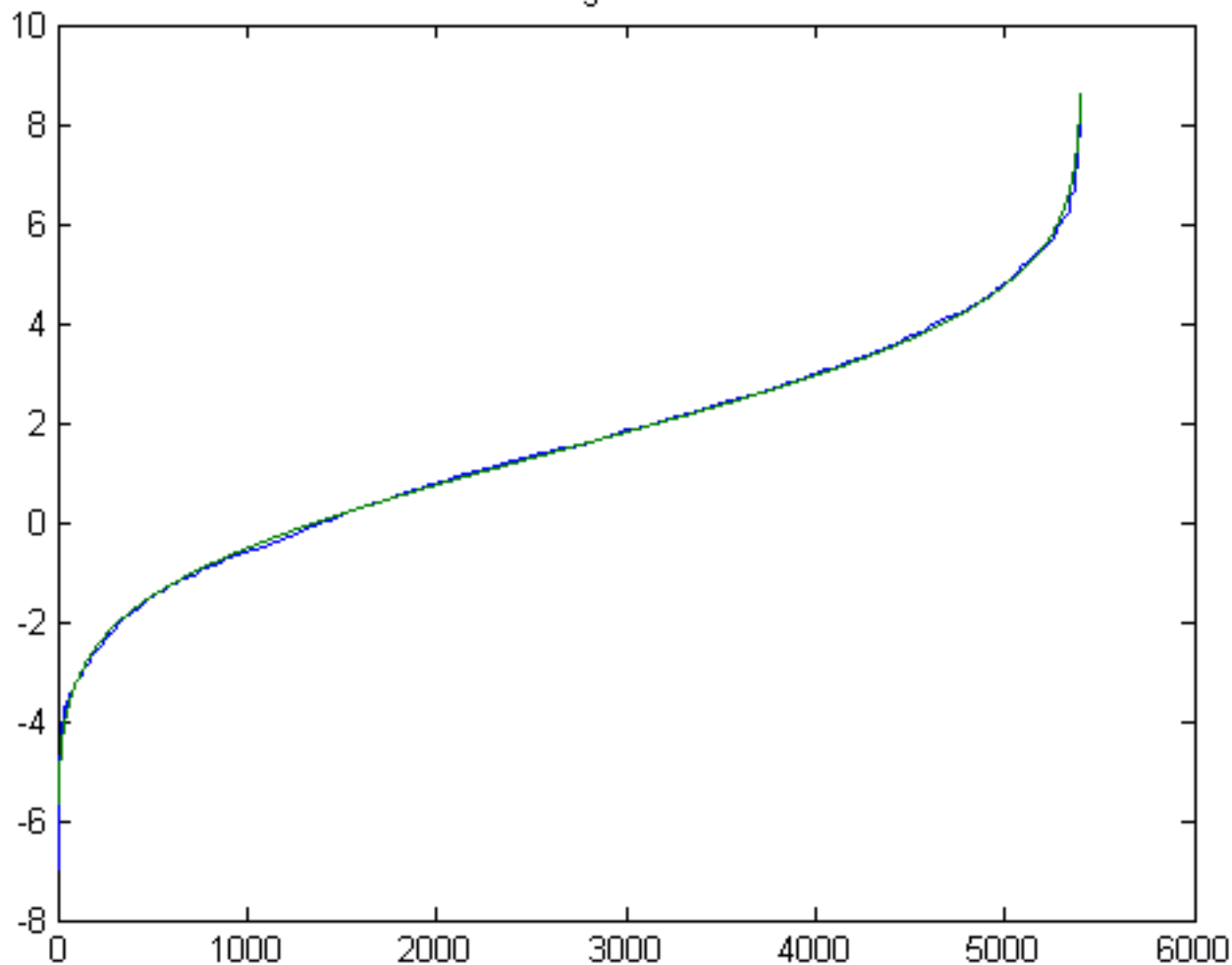


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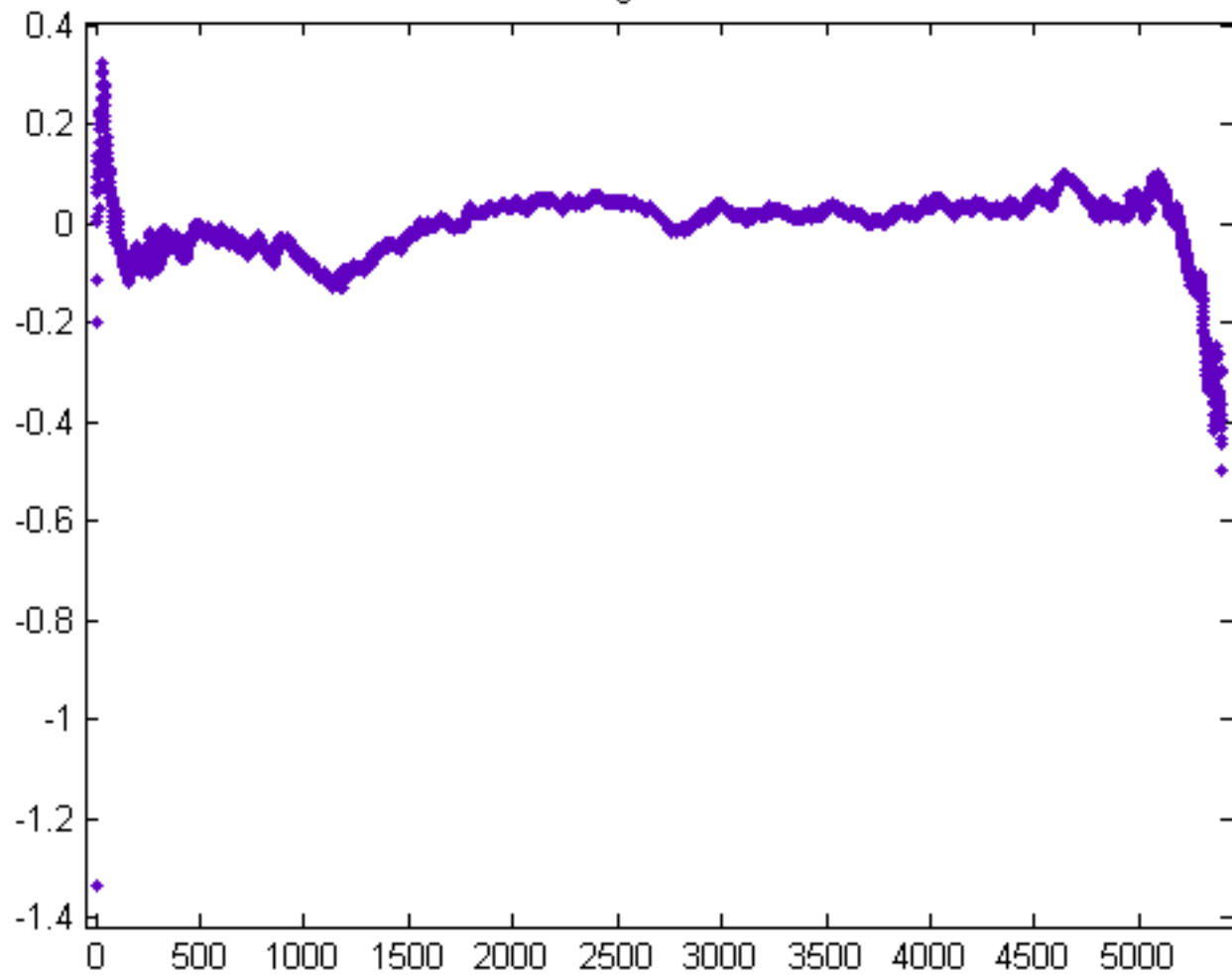


Figure 74

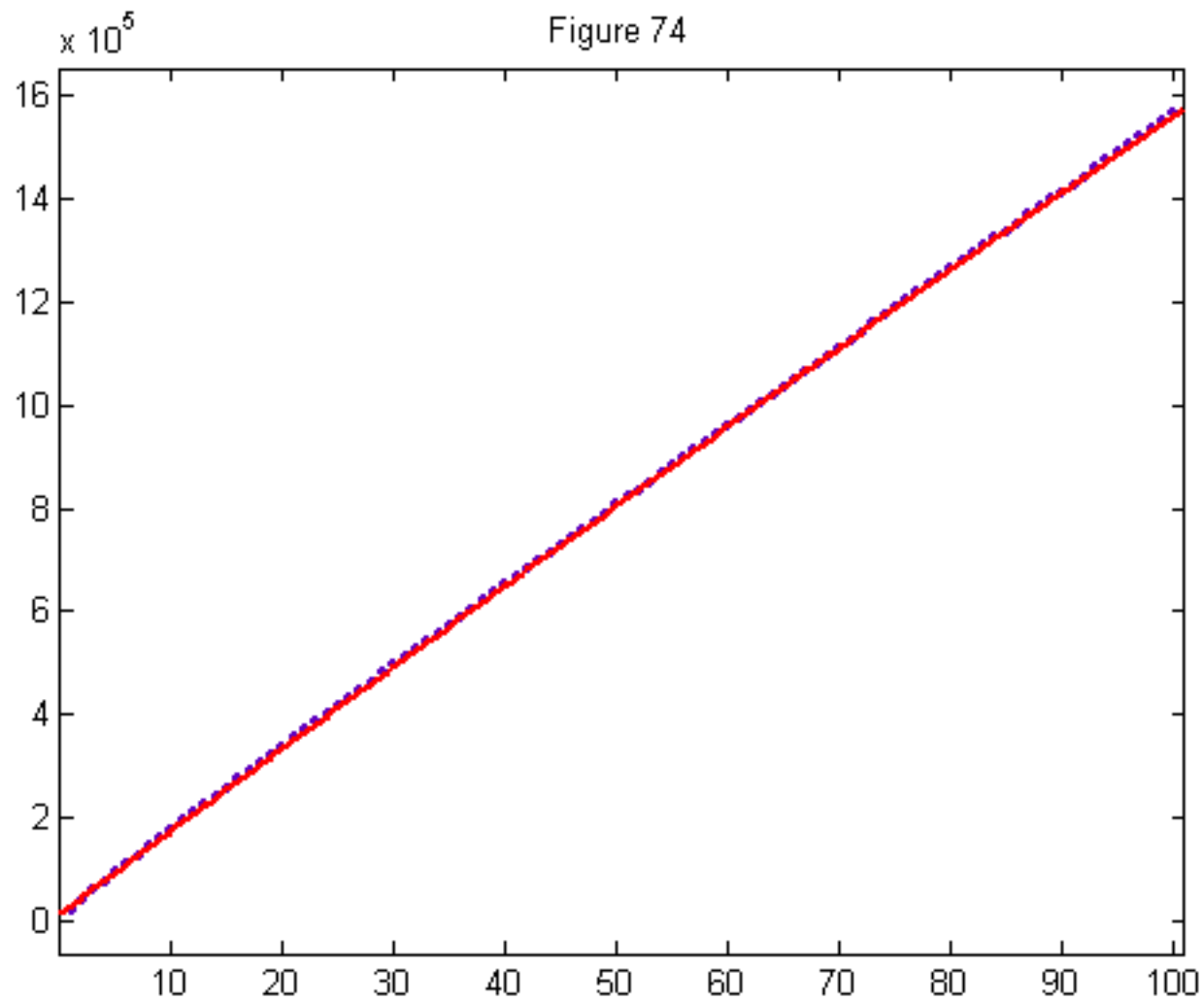


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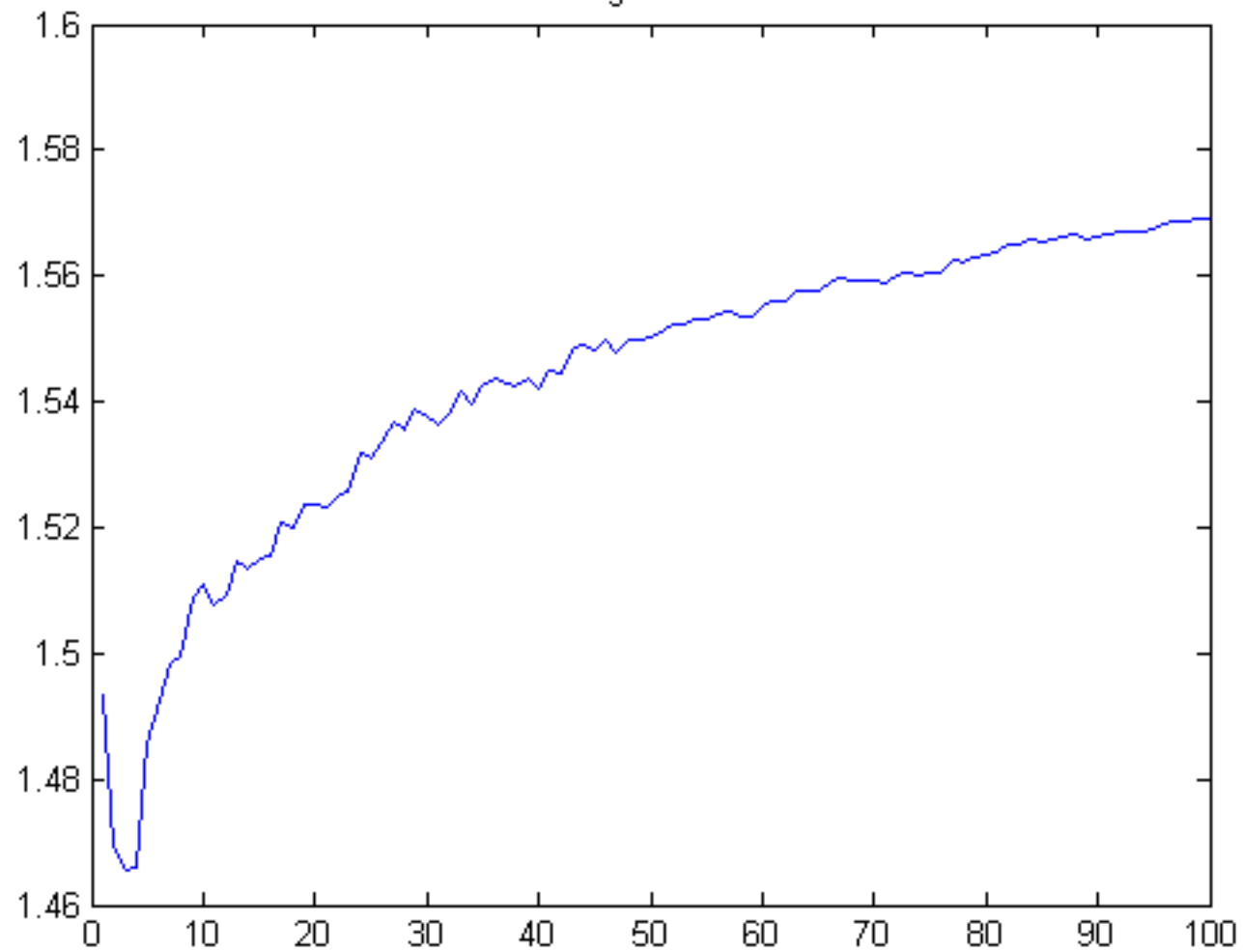


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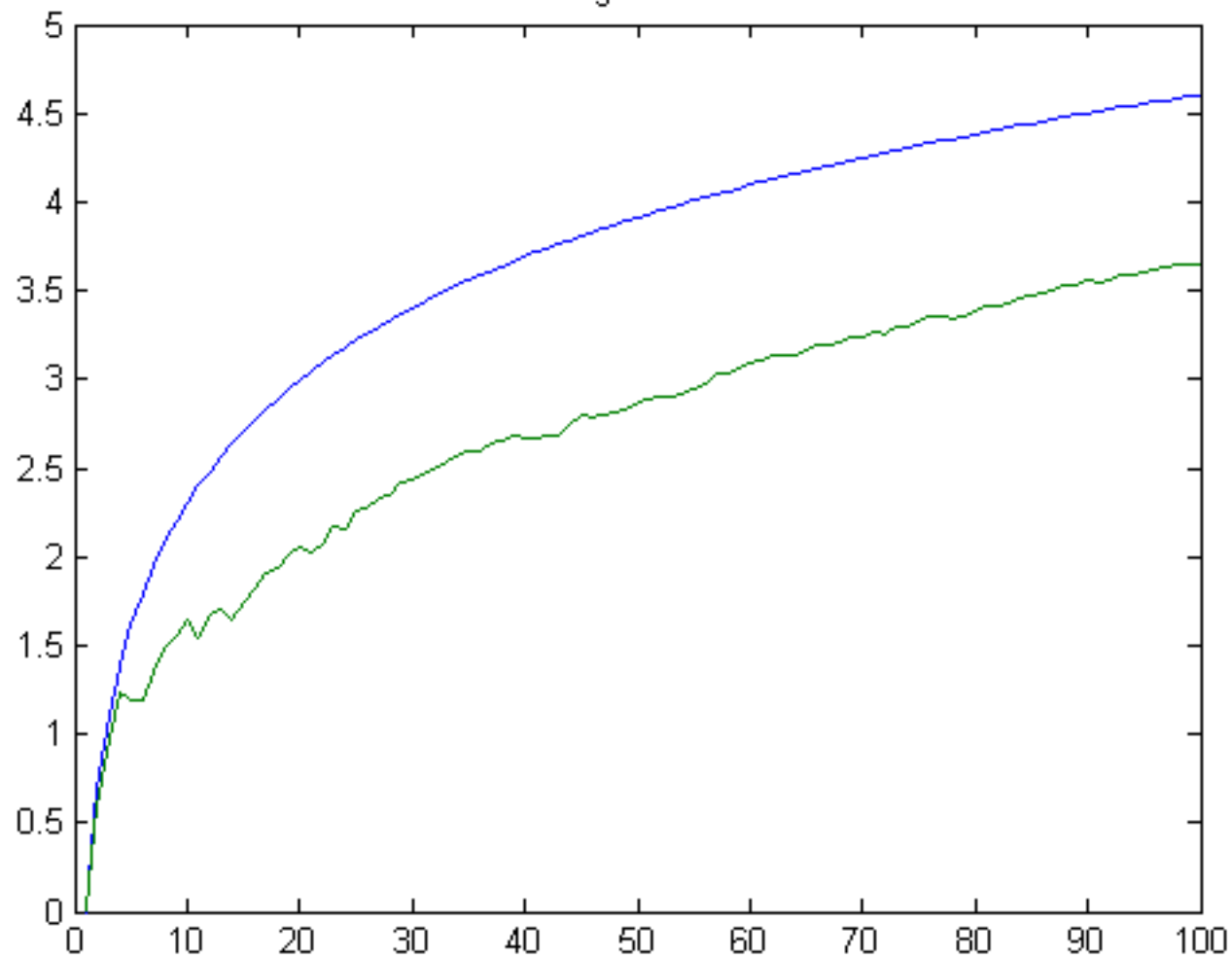


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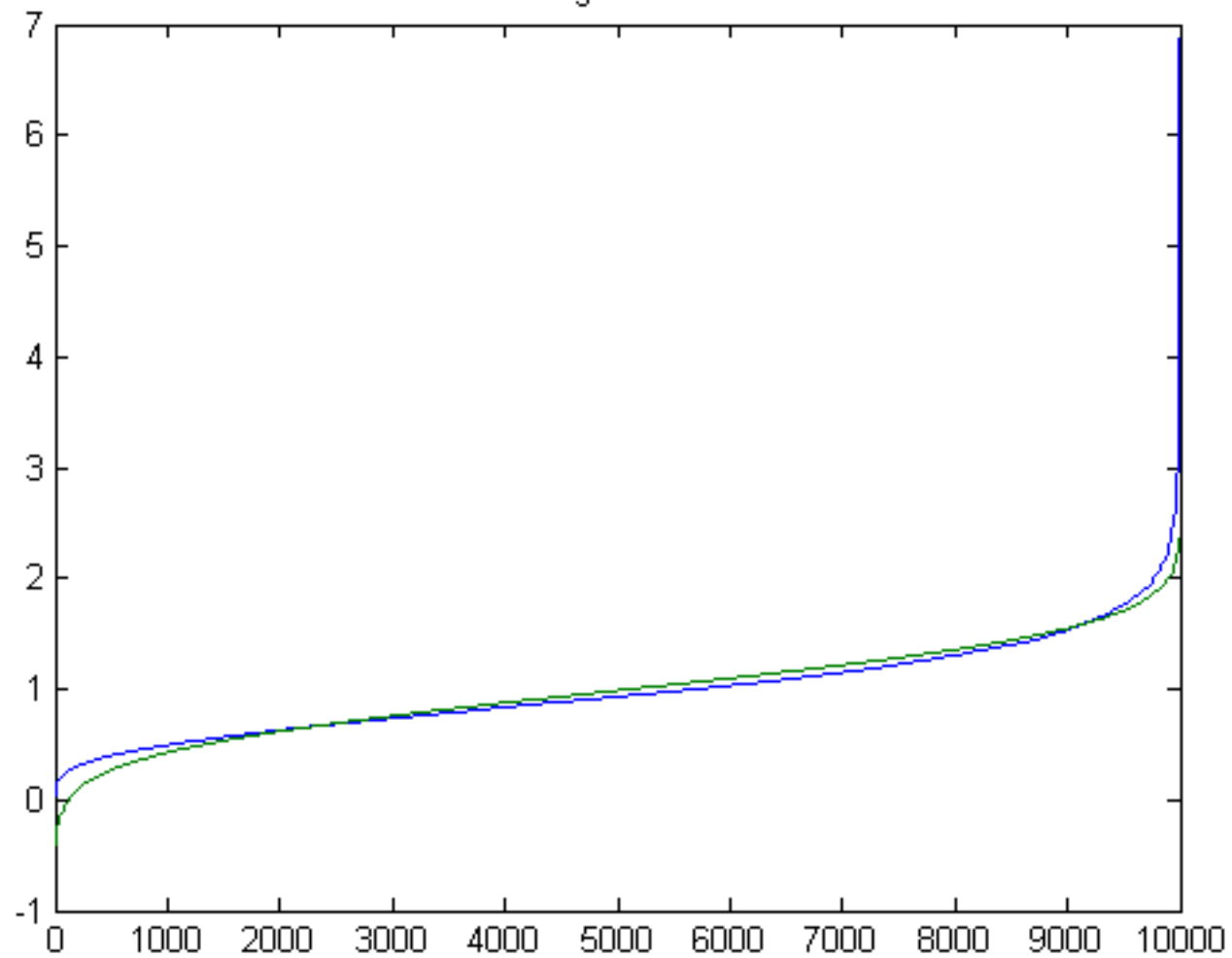


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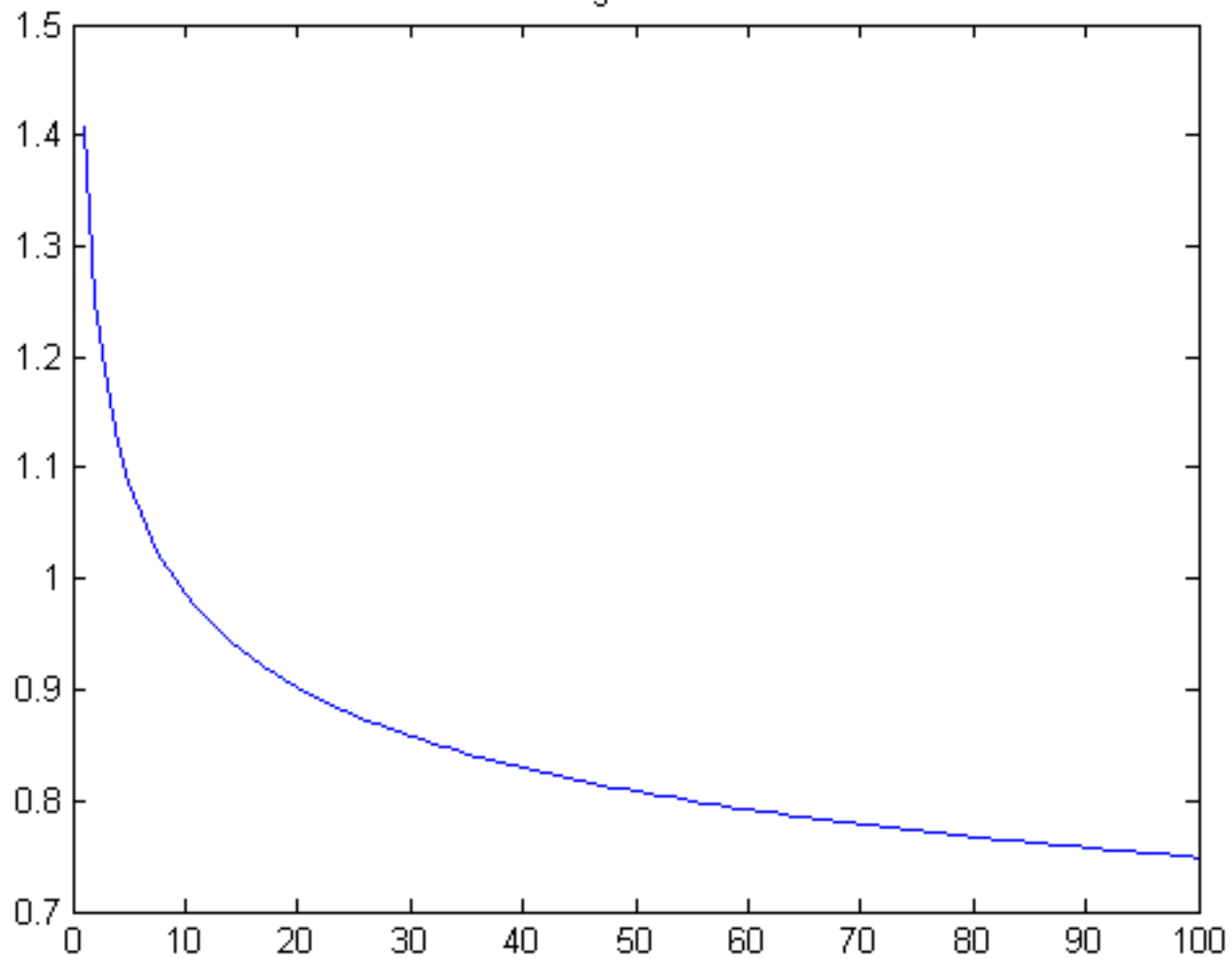


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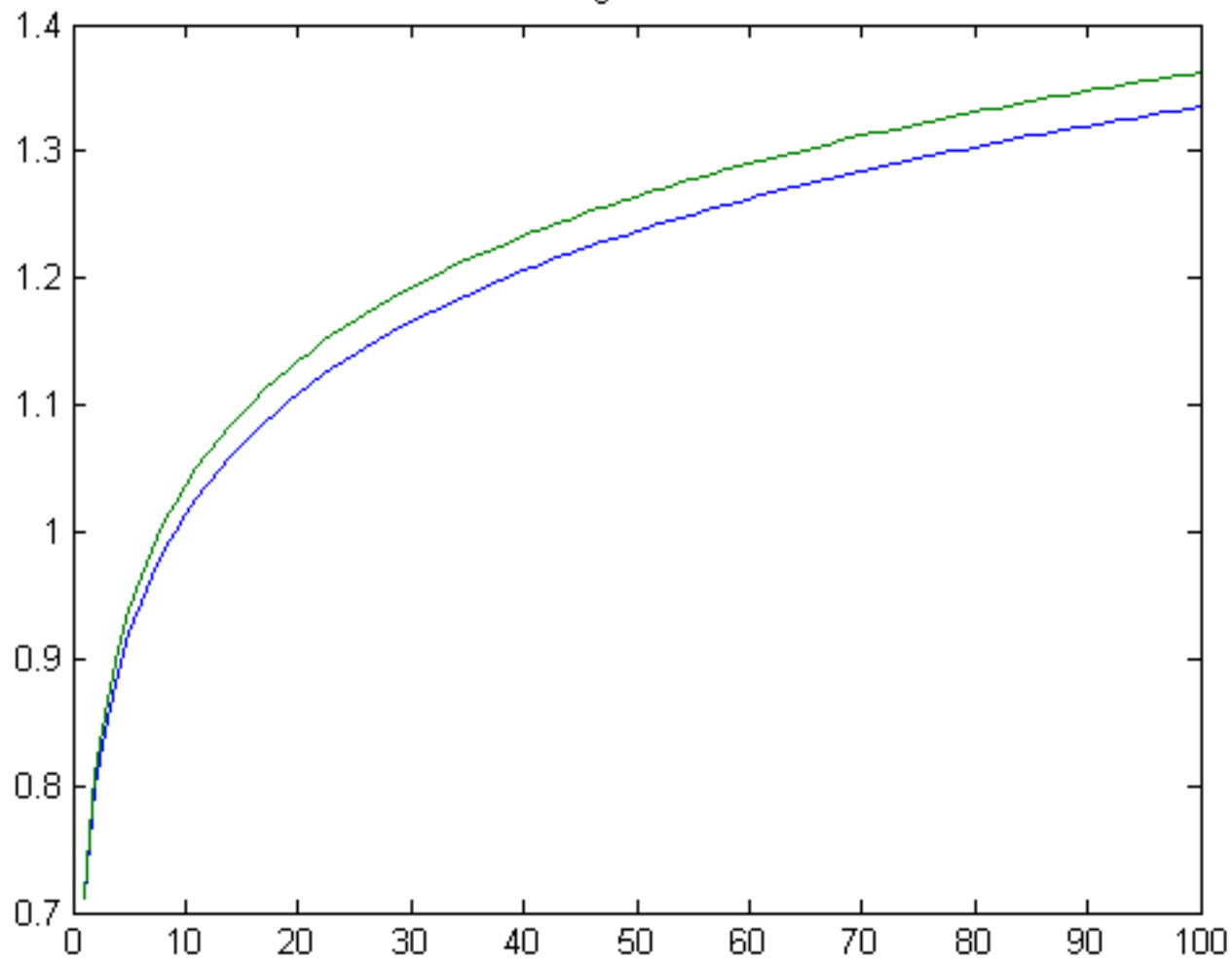


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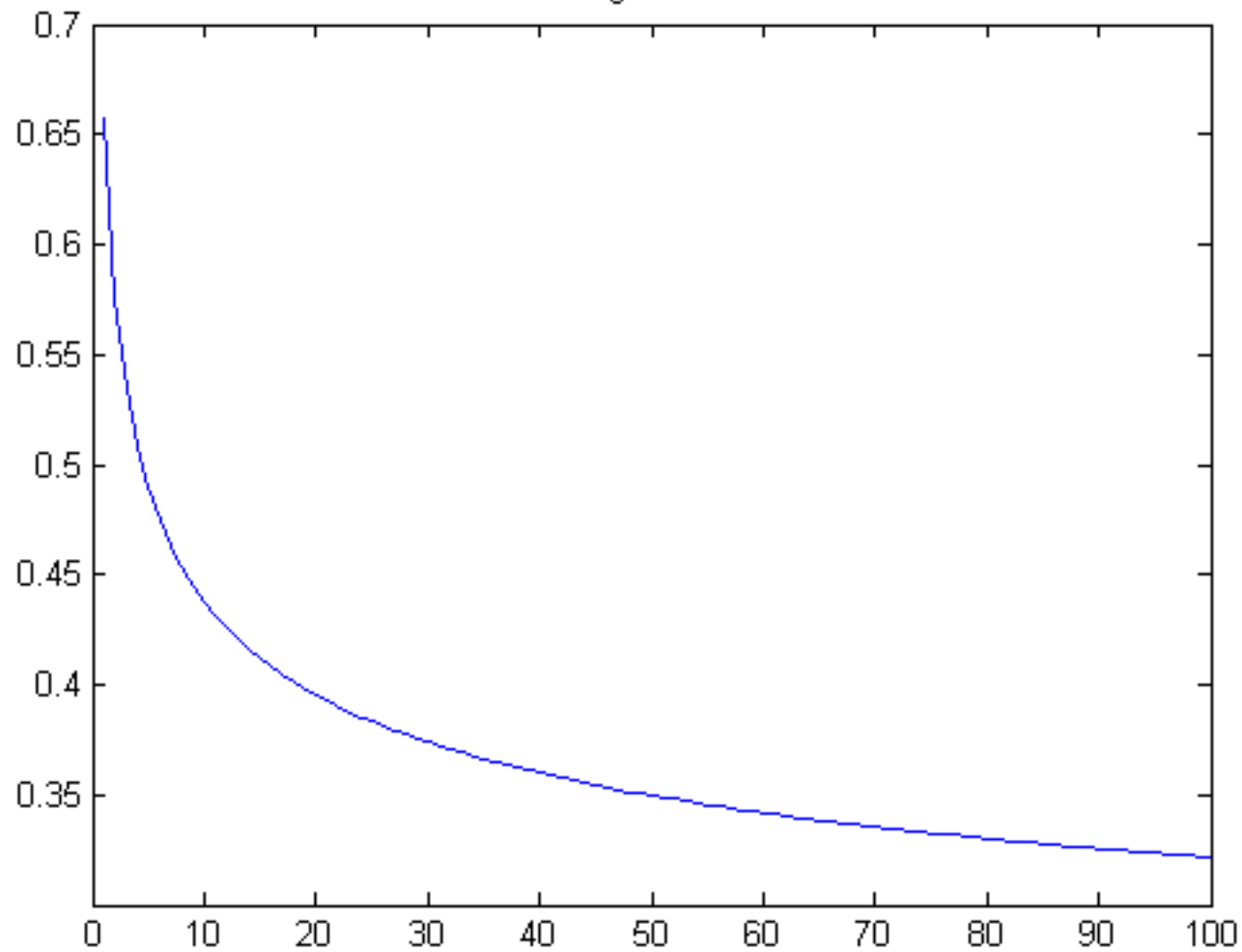


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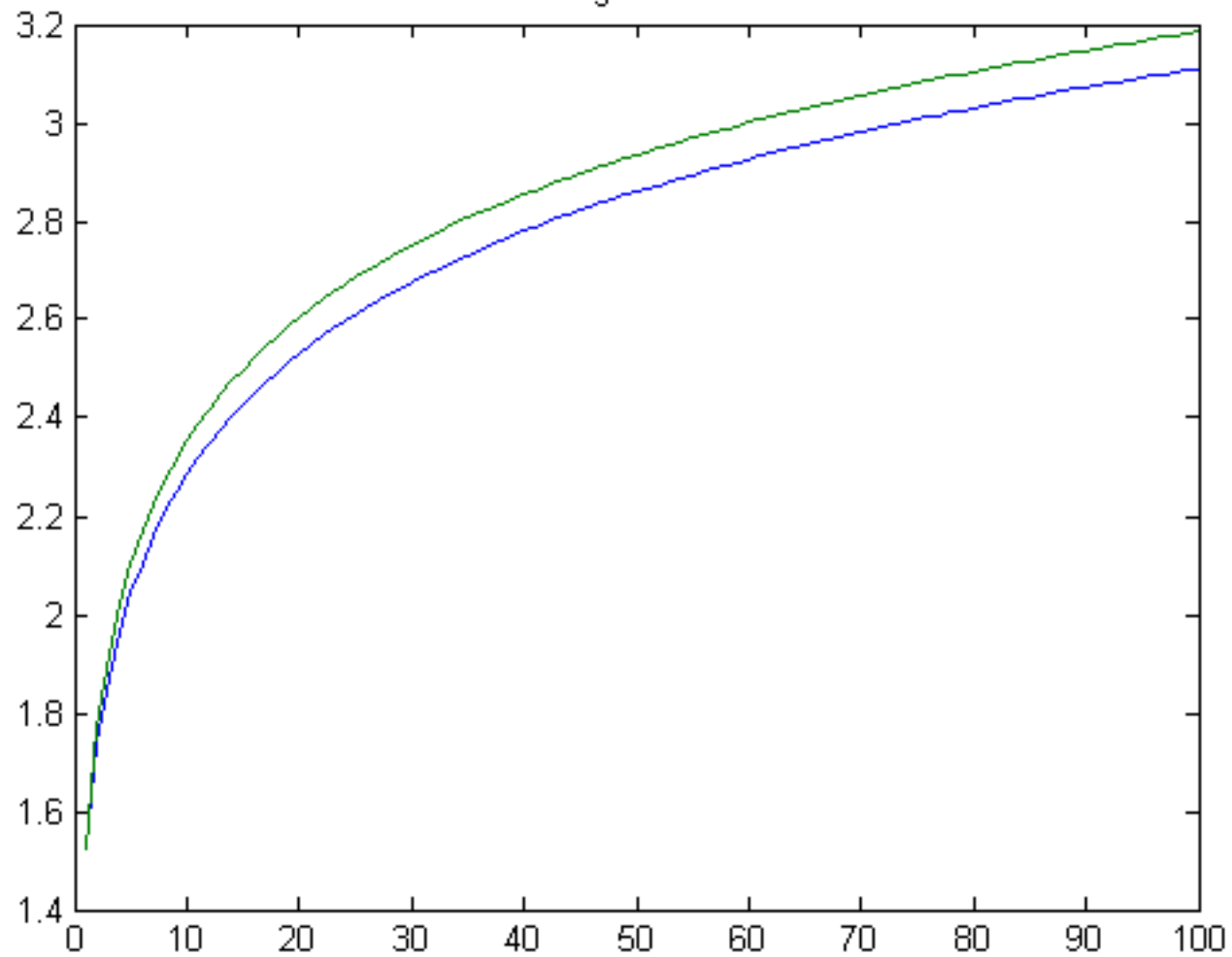


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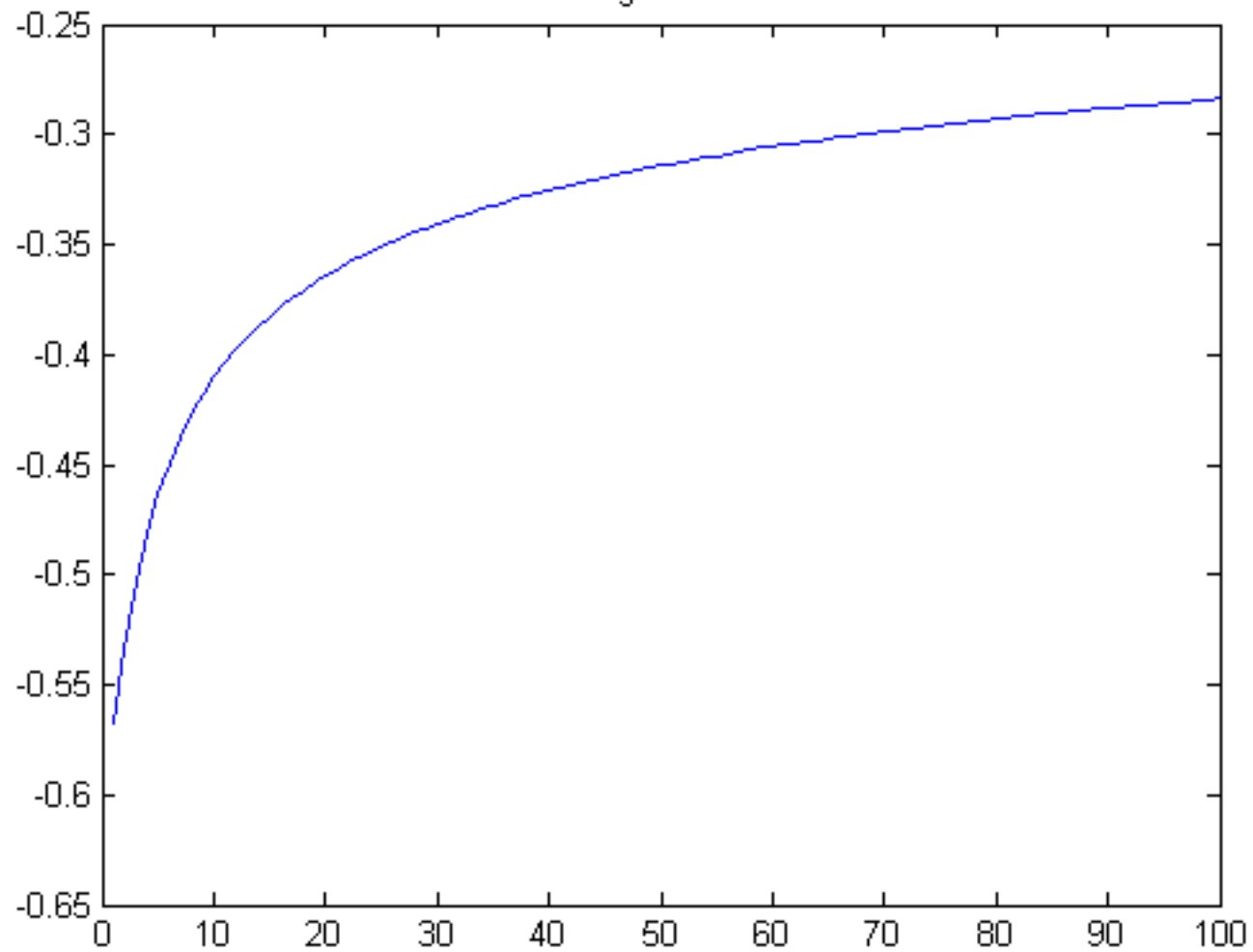


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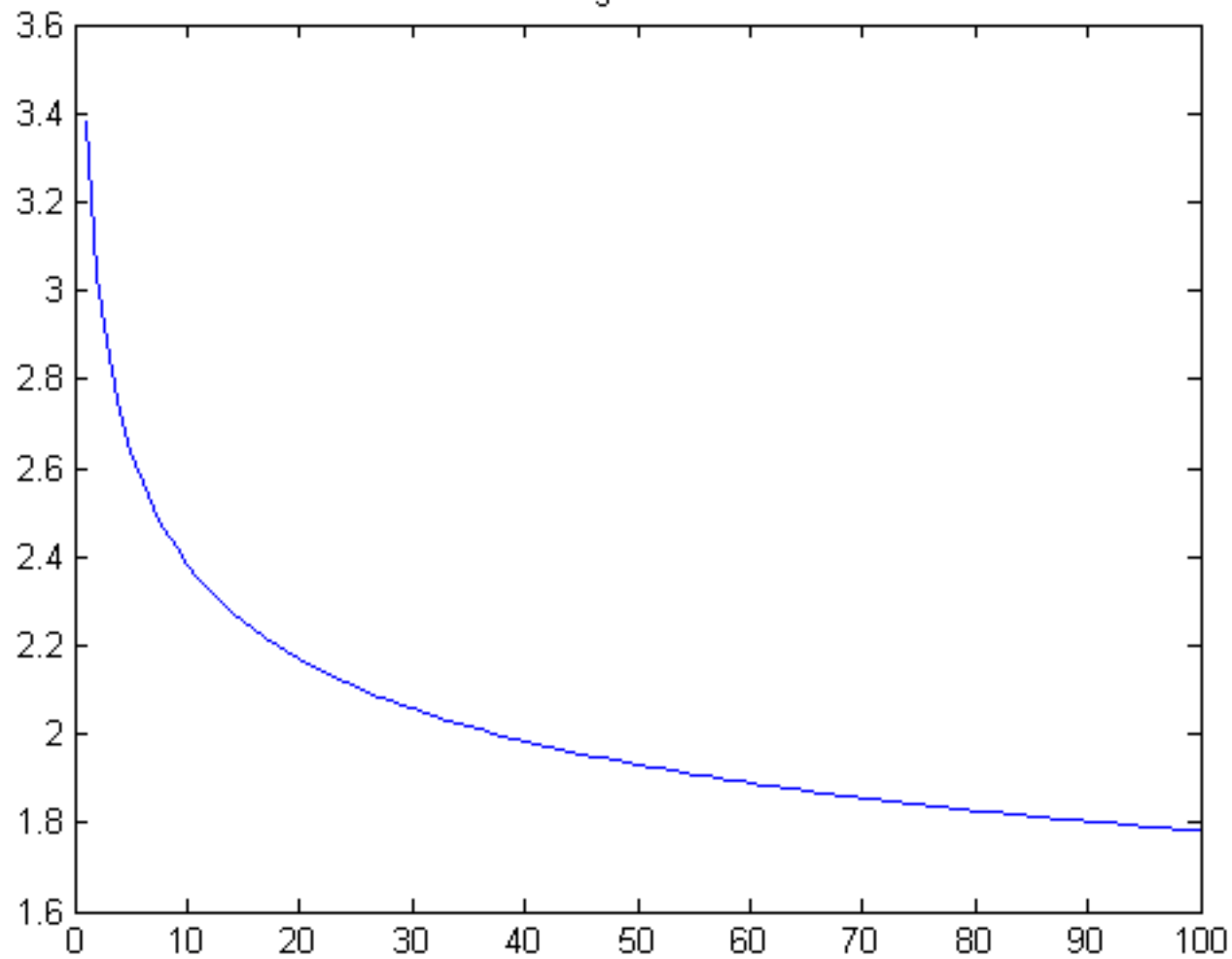


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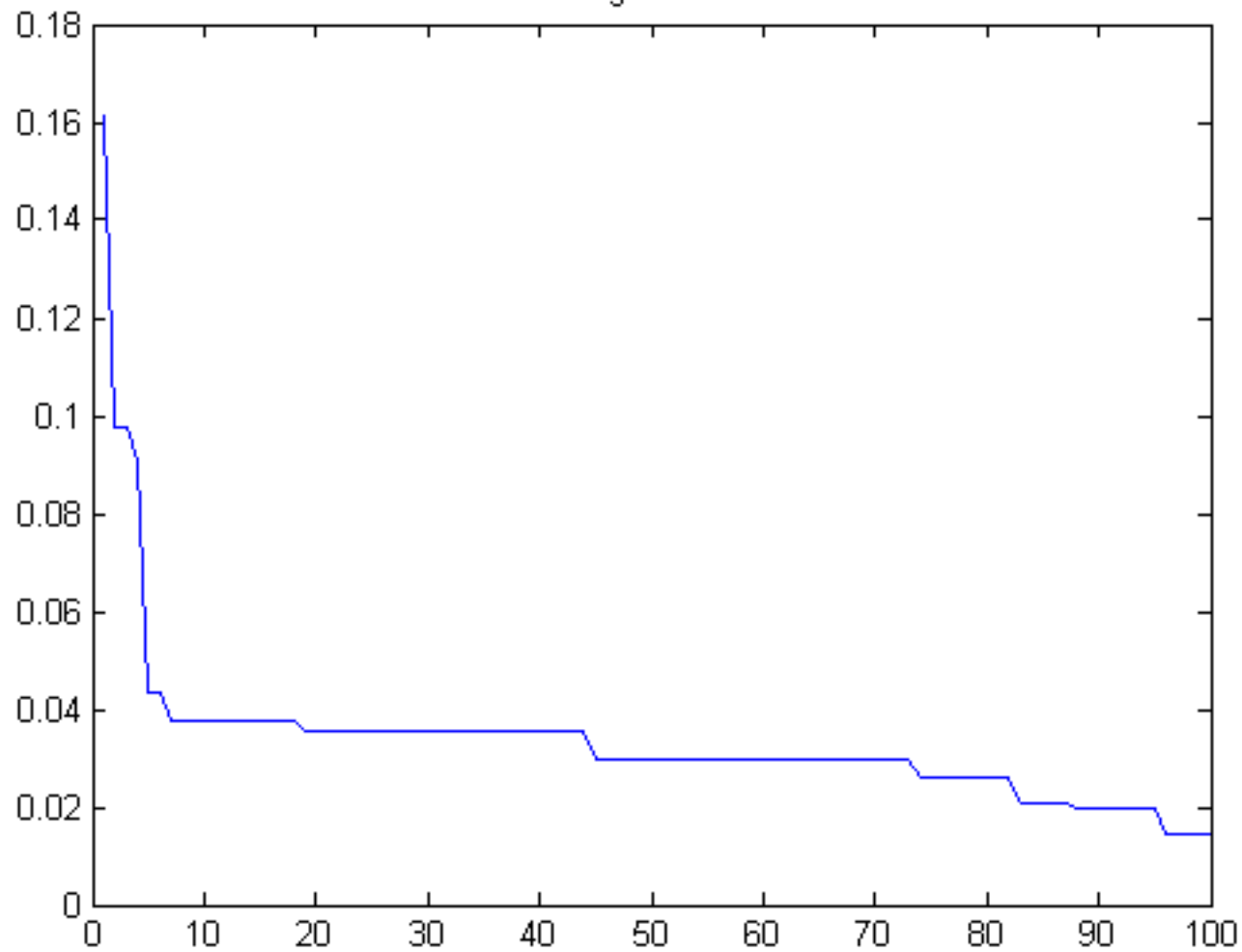


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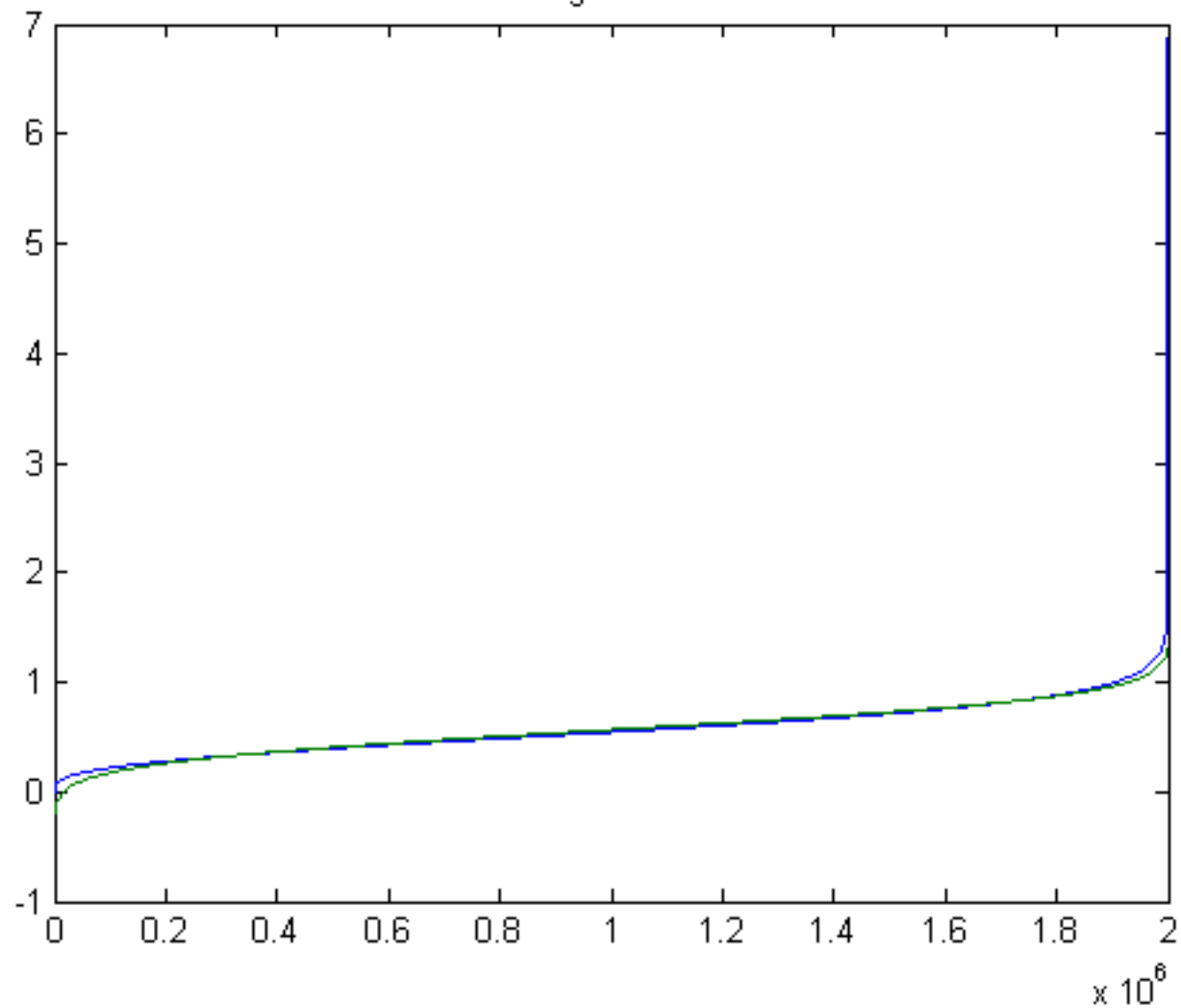


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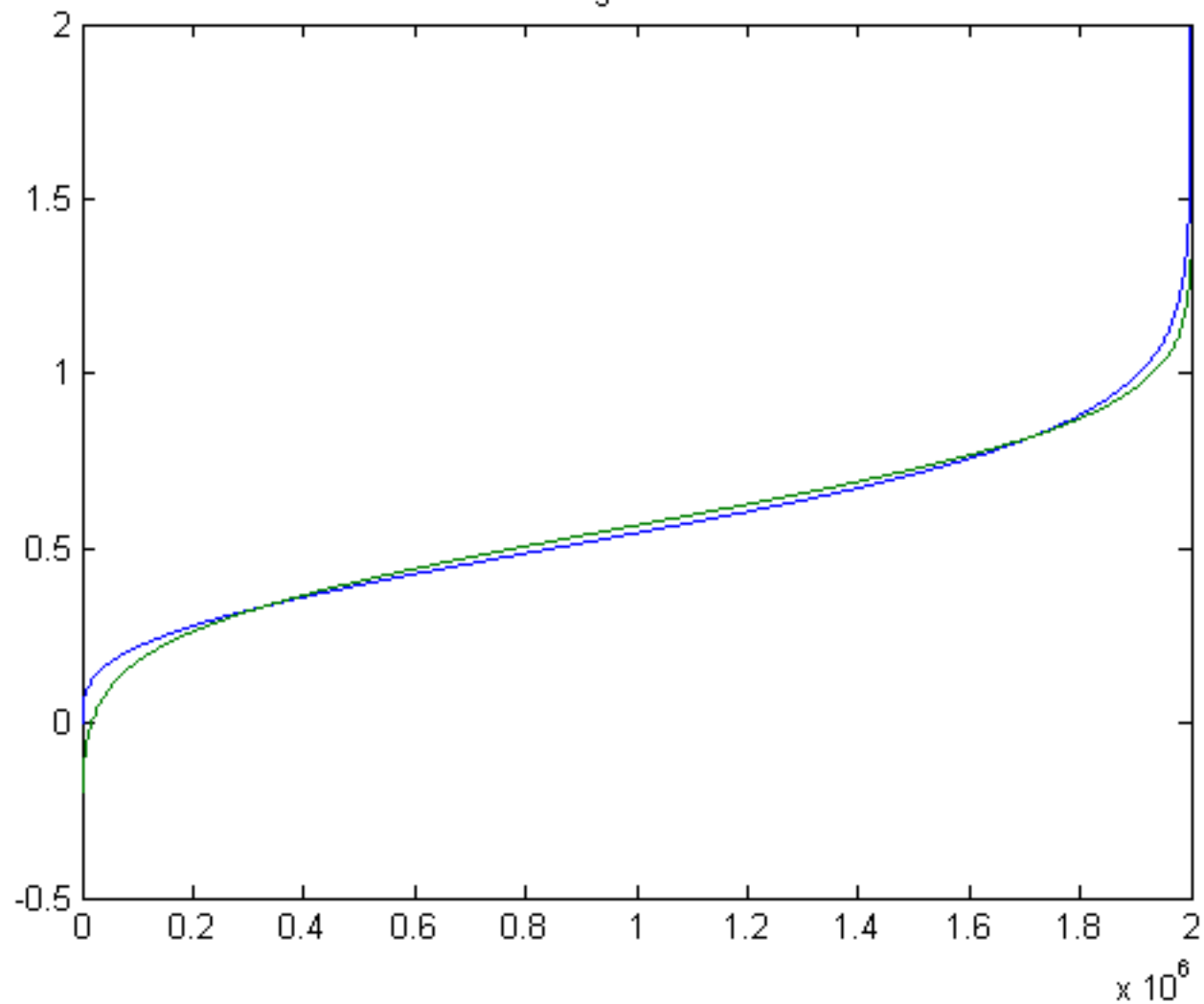


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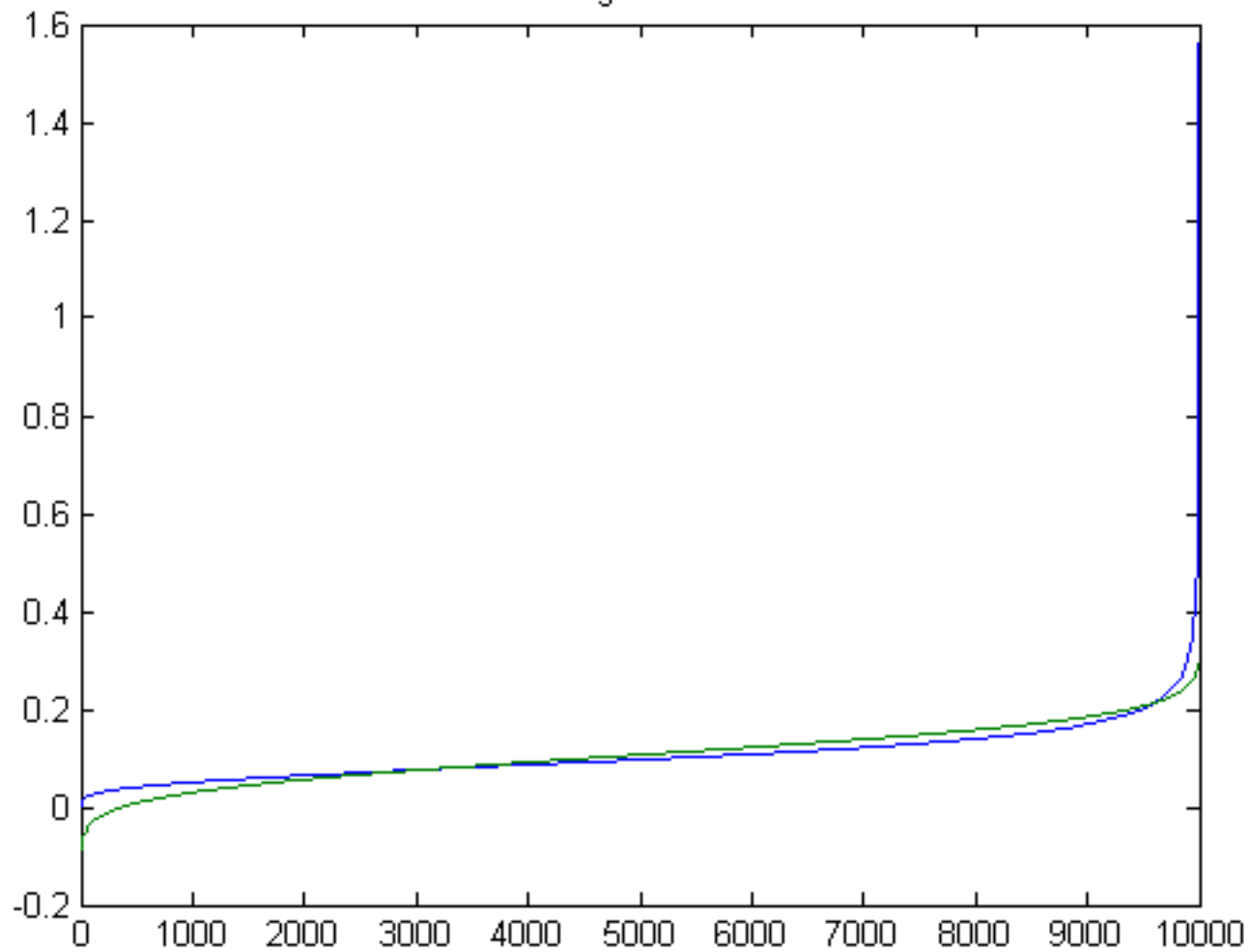


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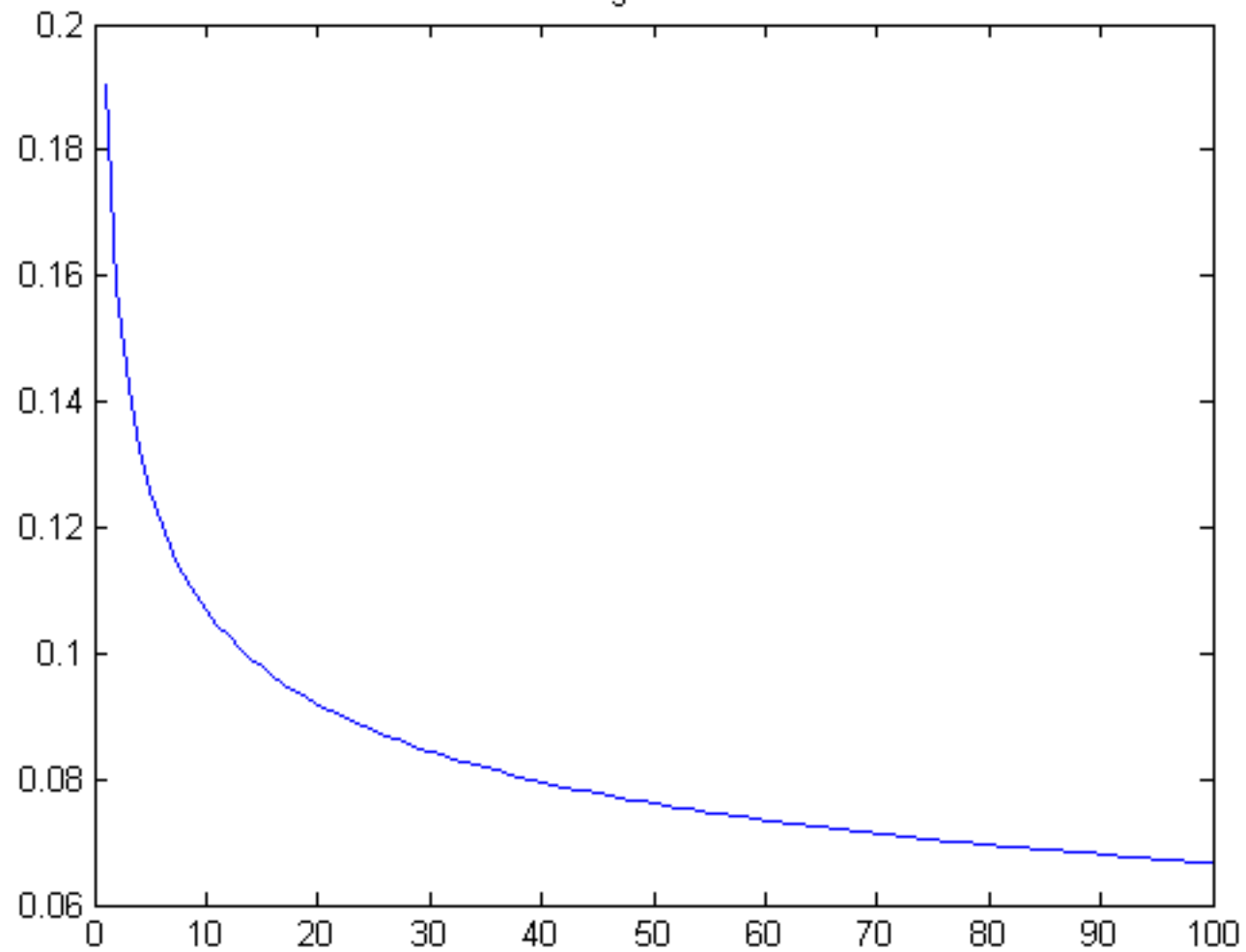


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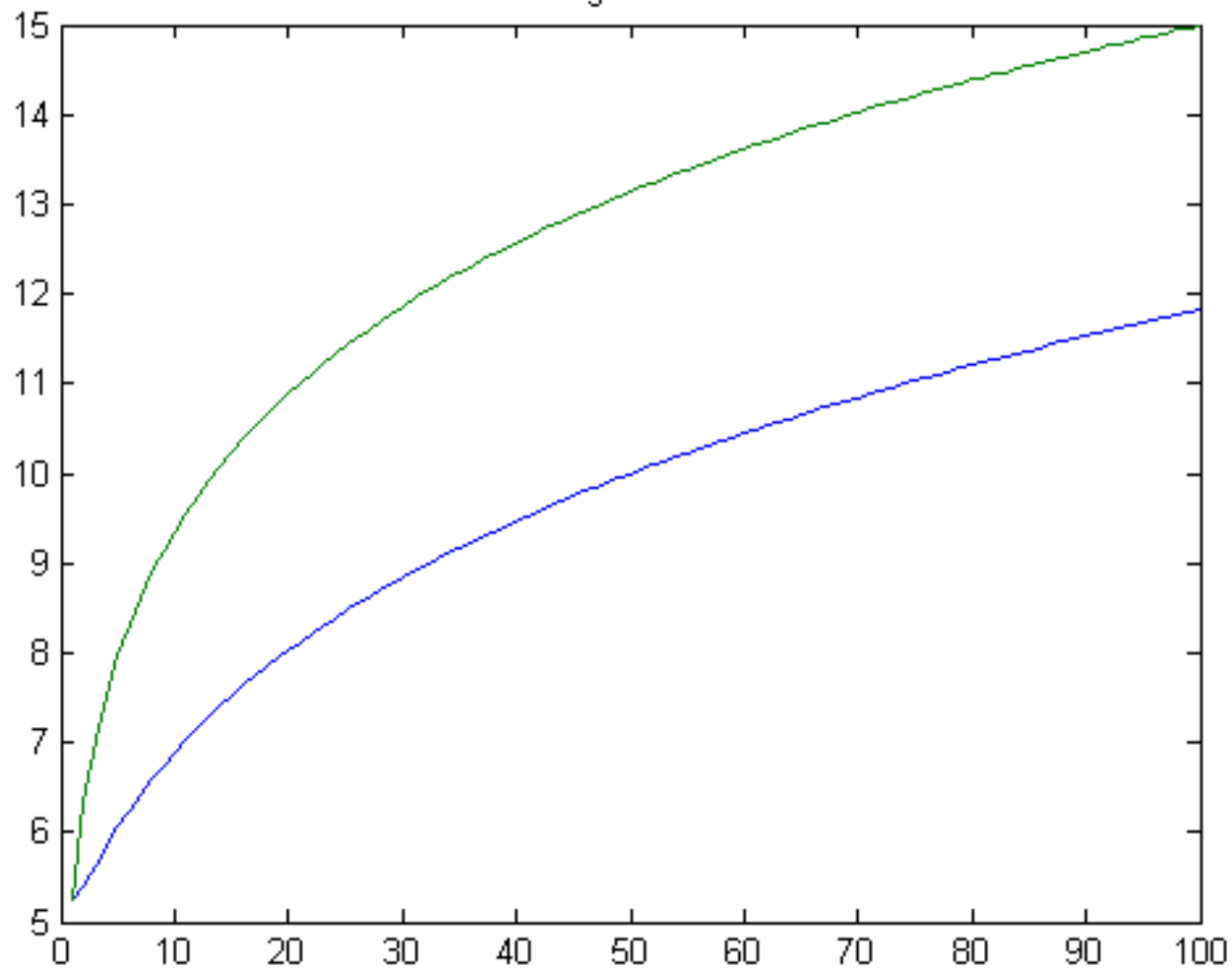


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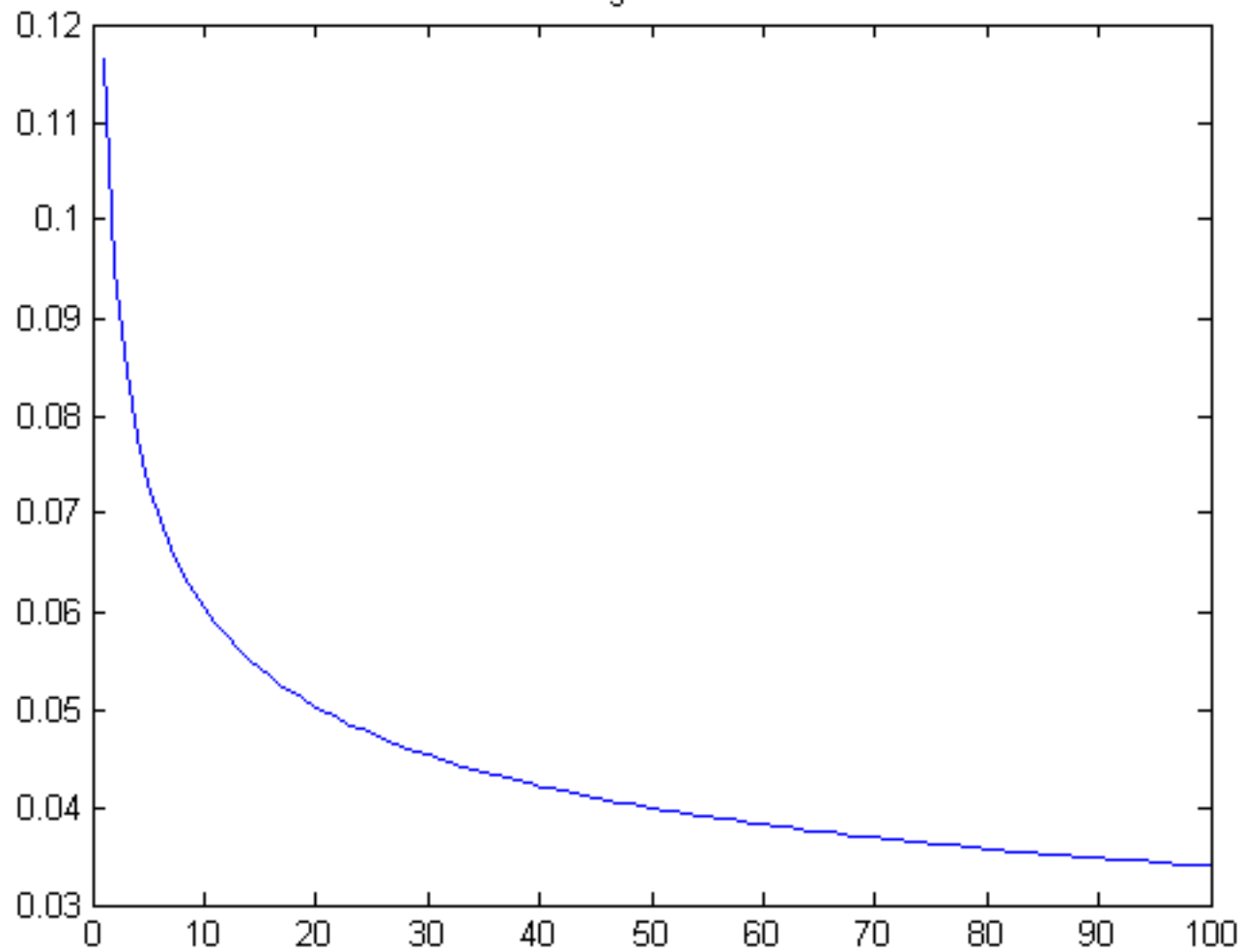


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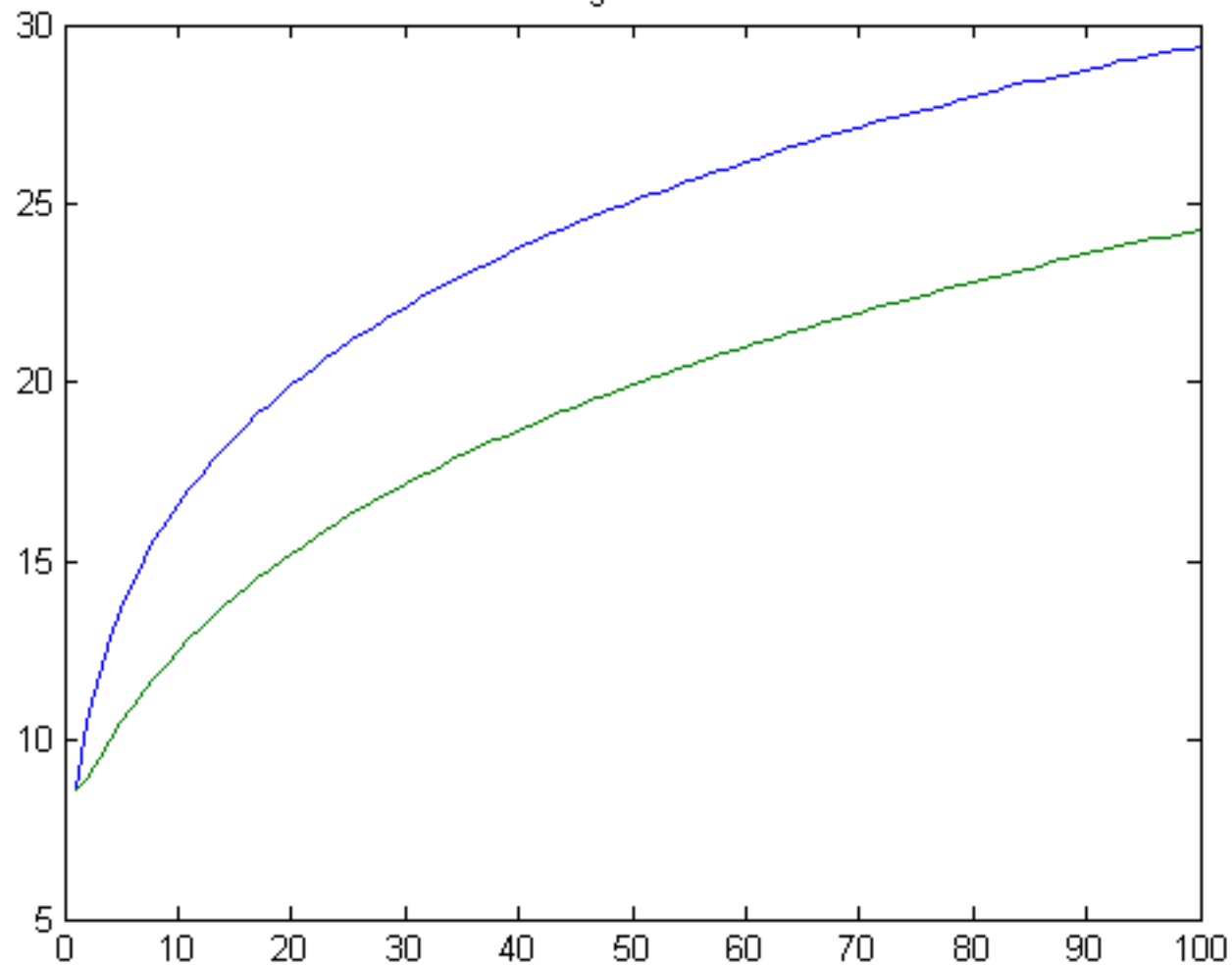


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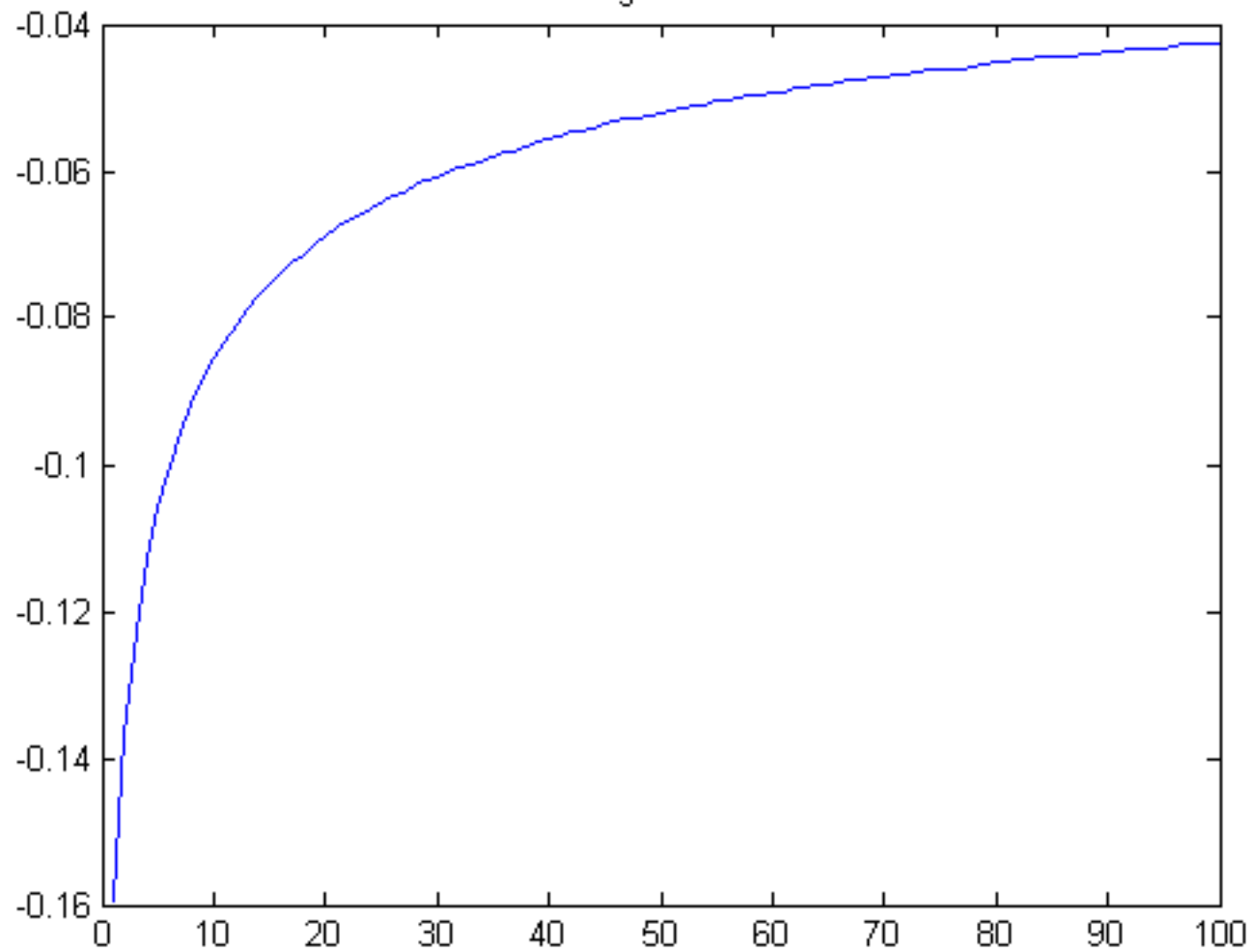


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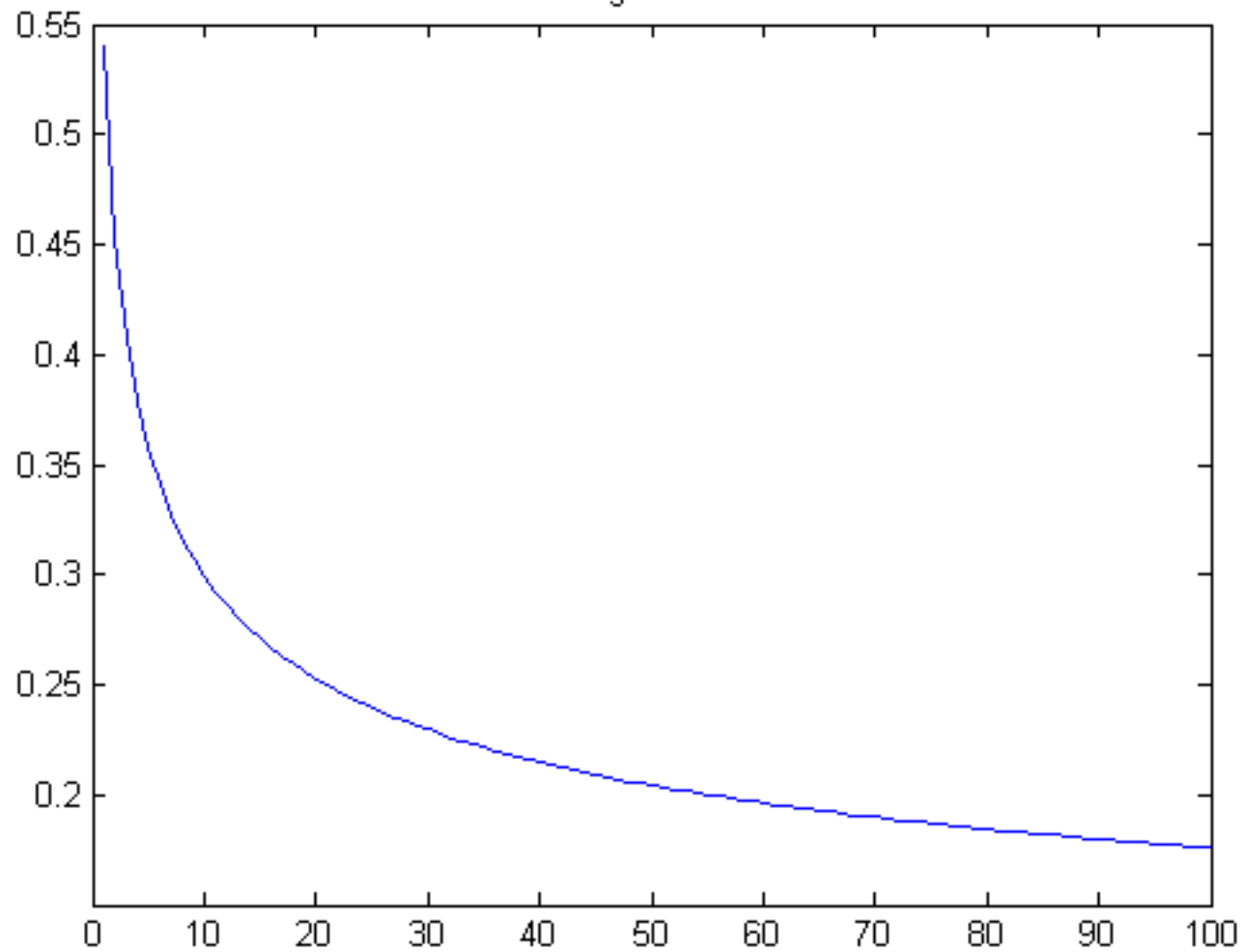


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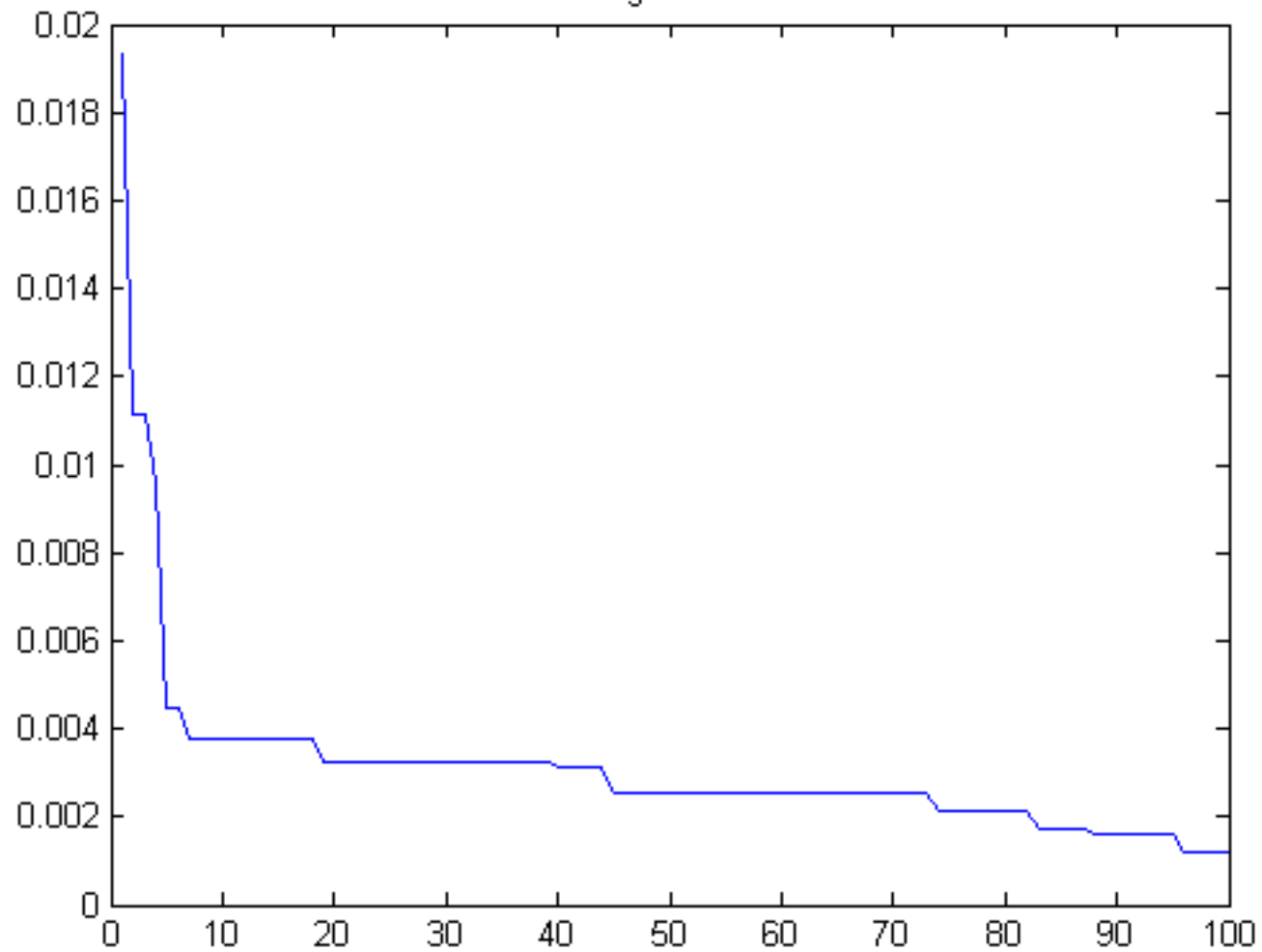


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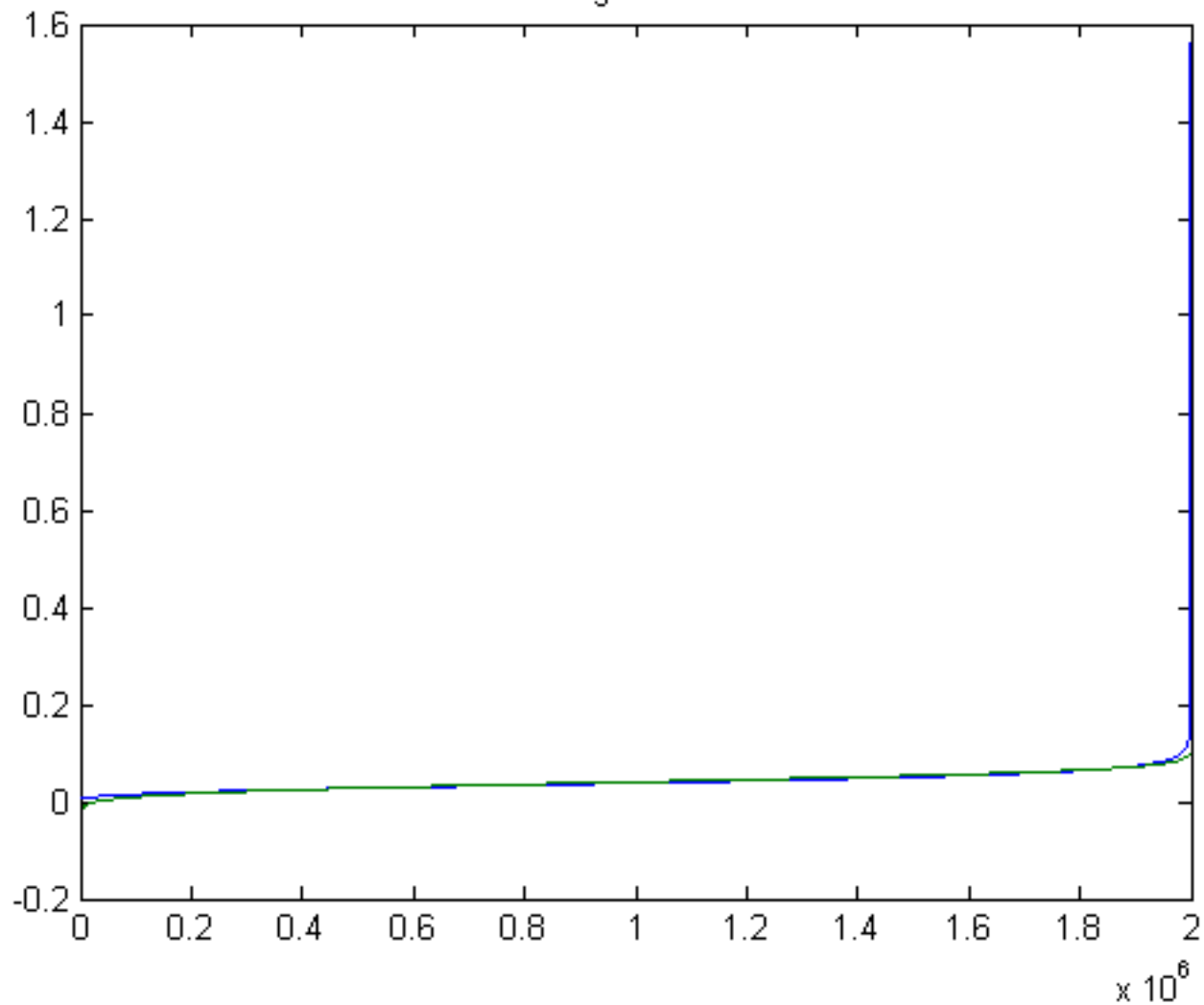


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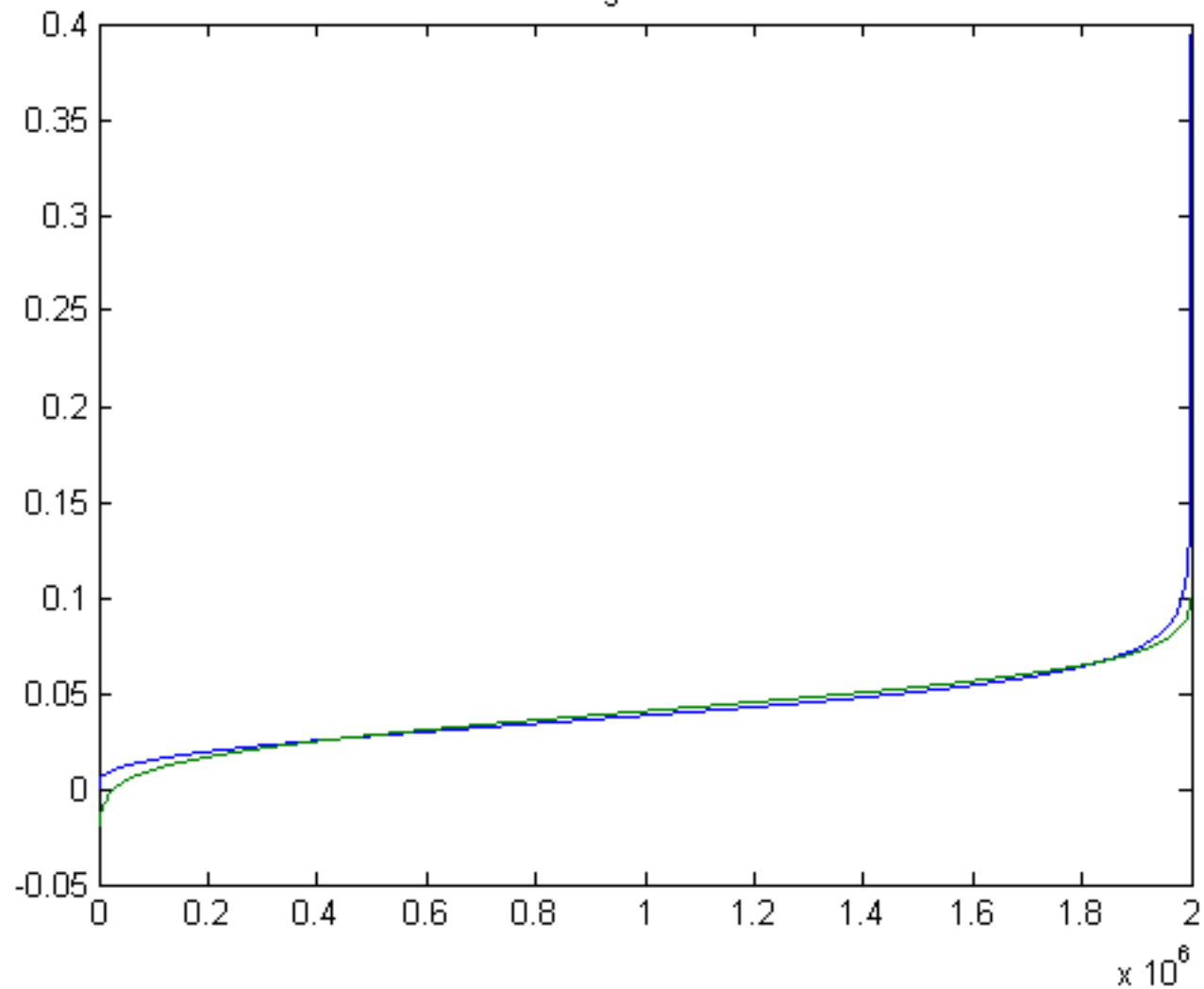


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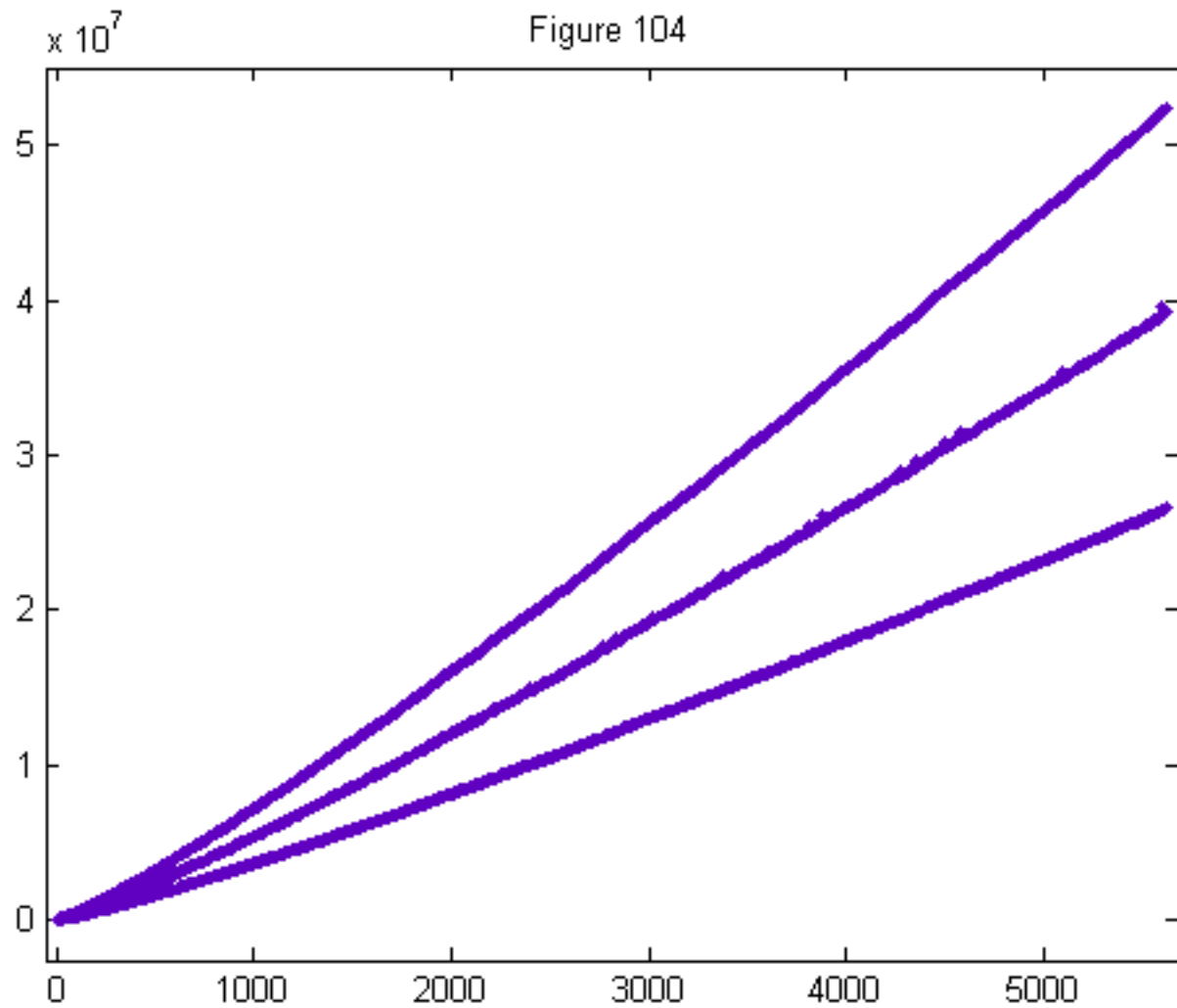


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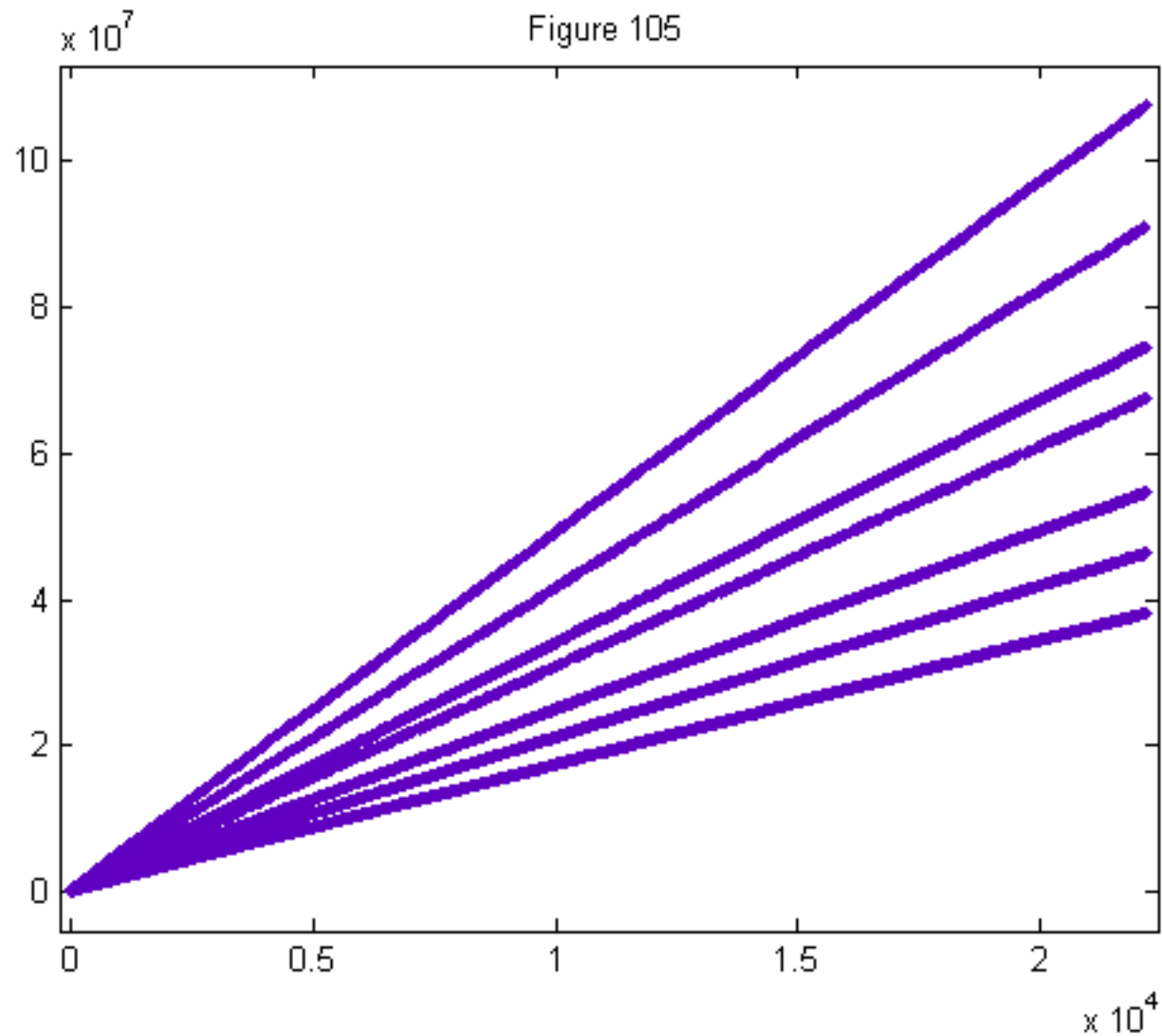


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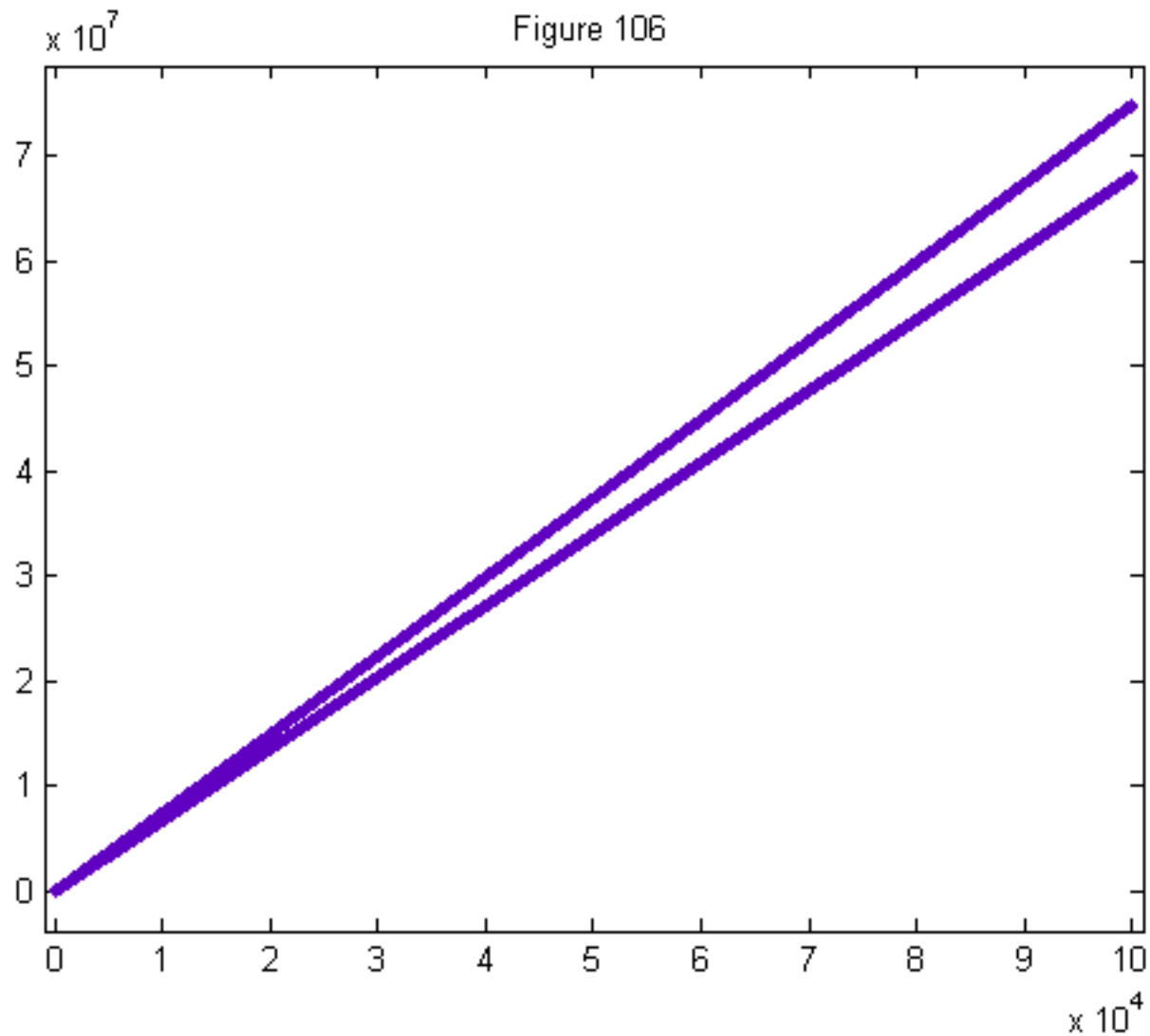


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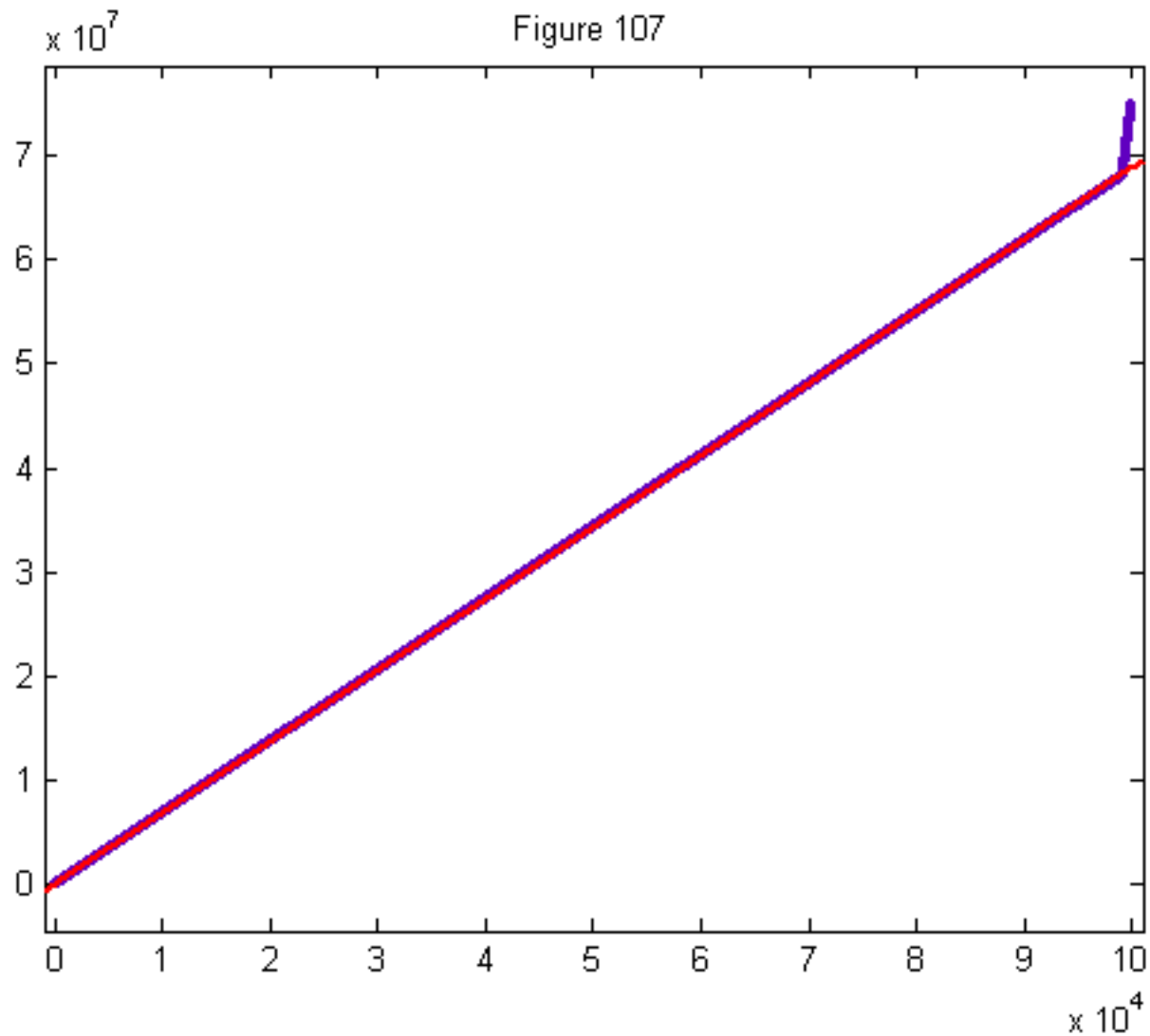


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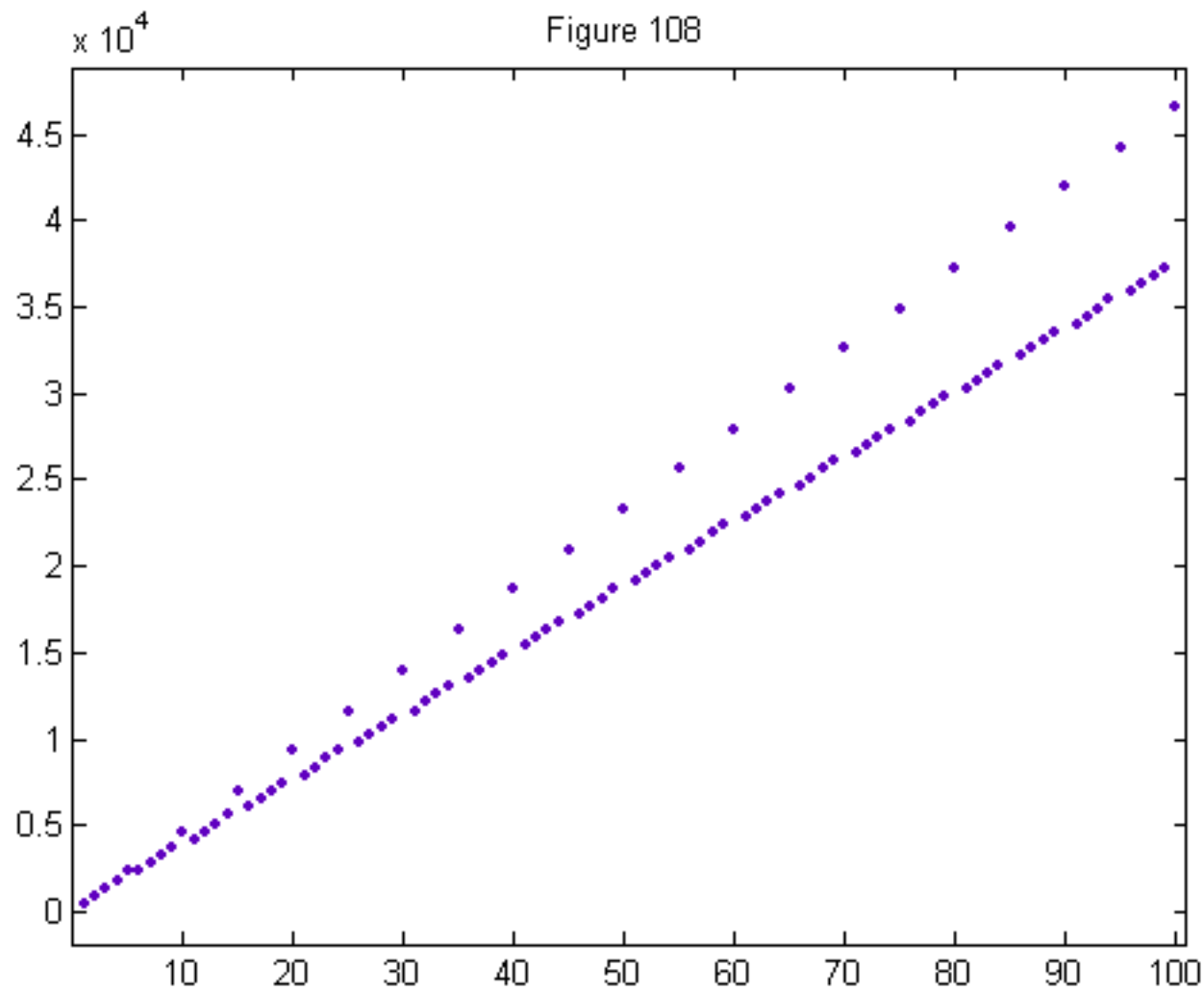


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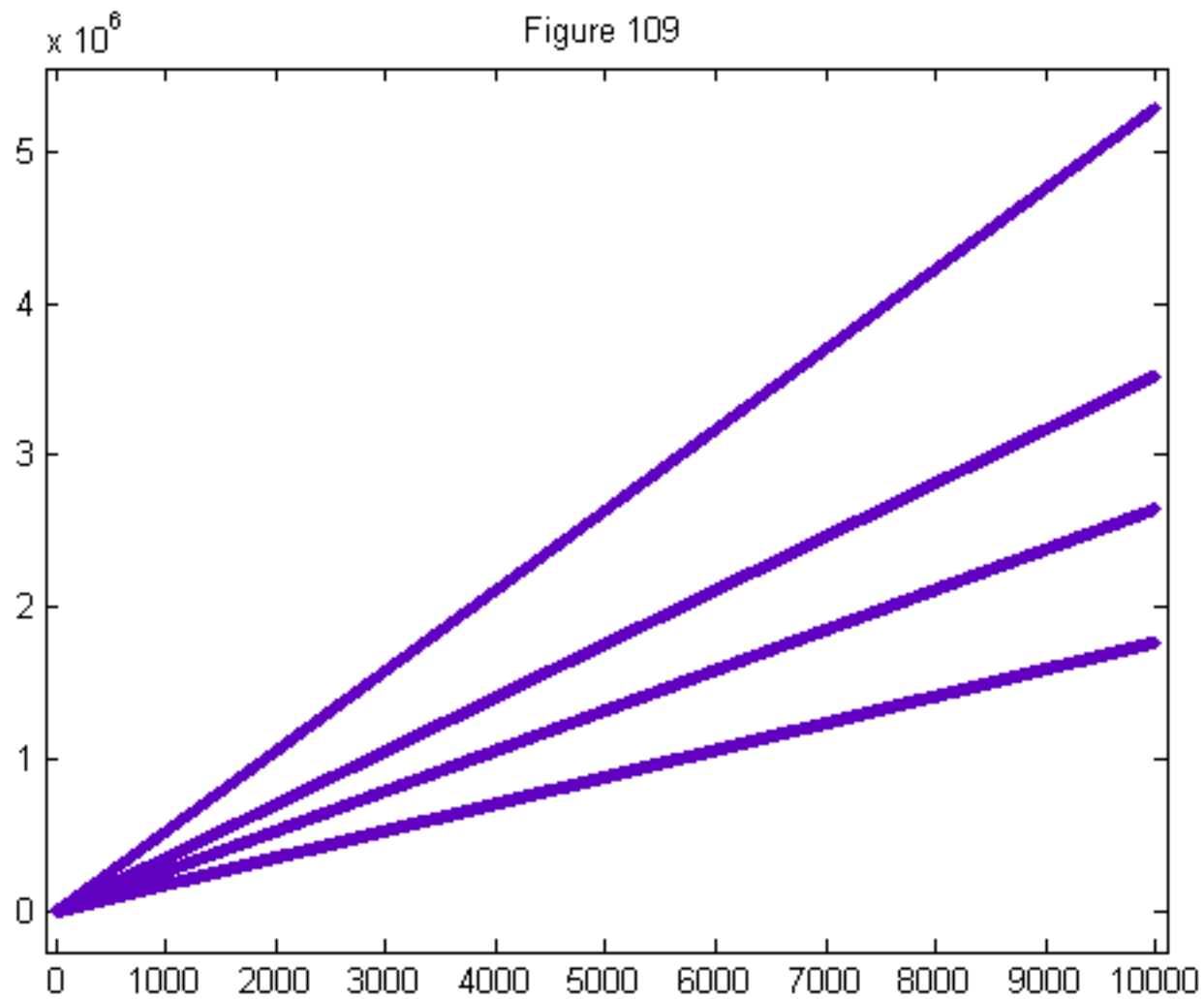


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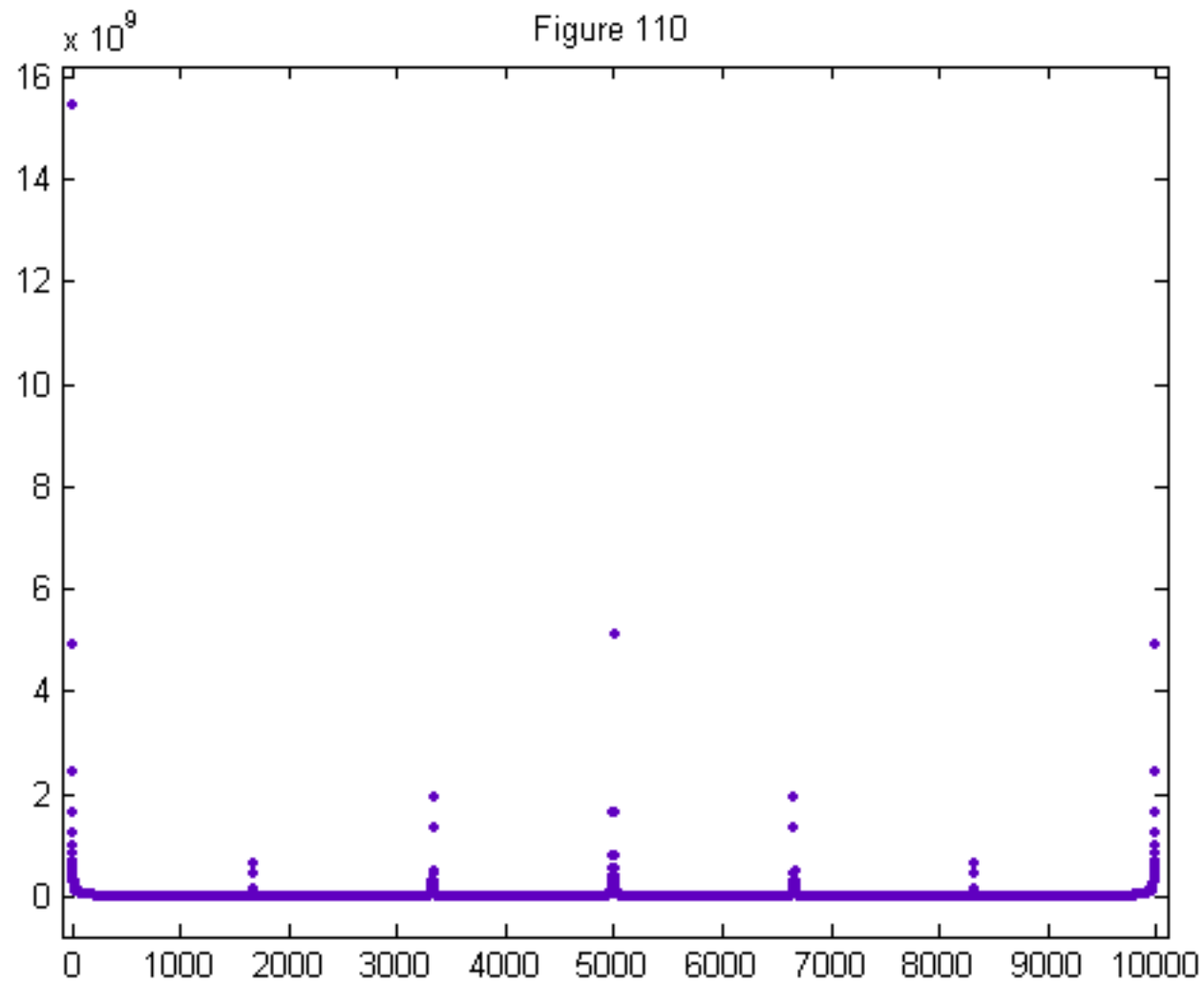


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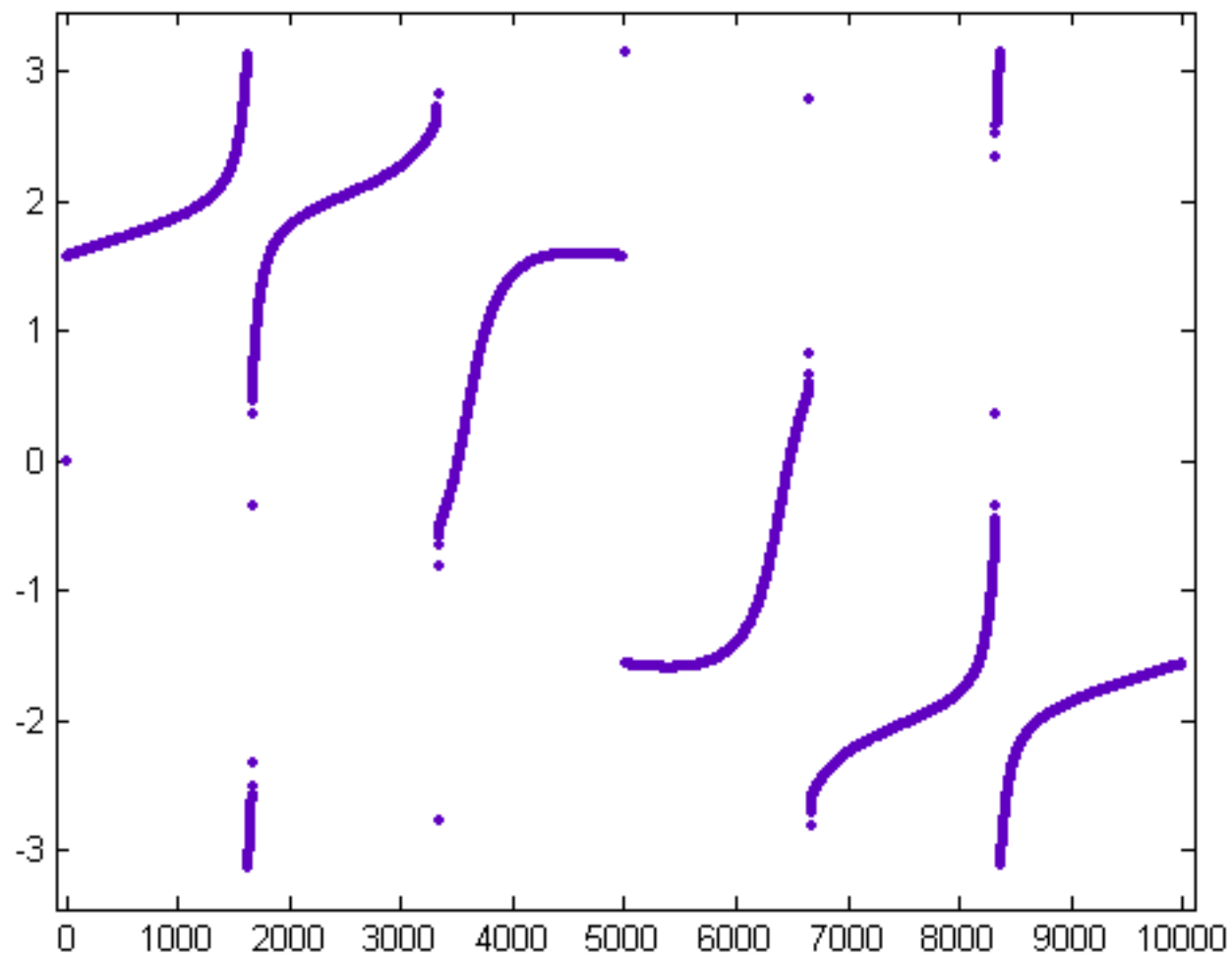


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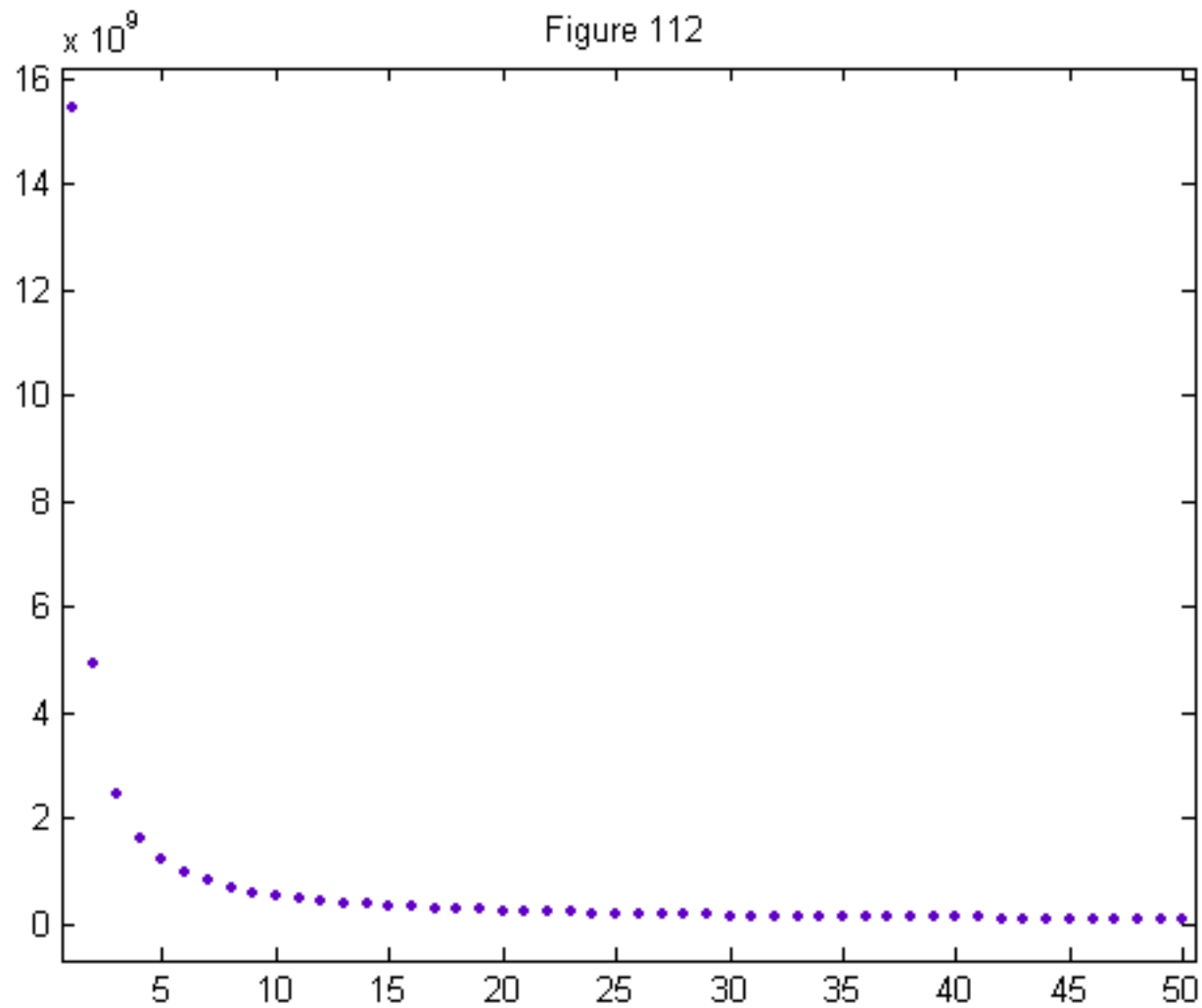


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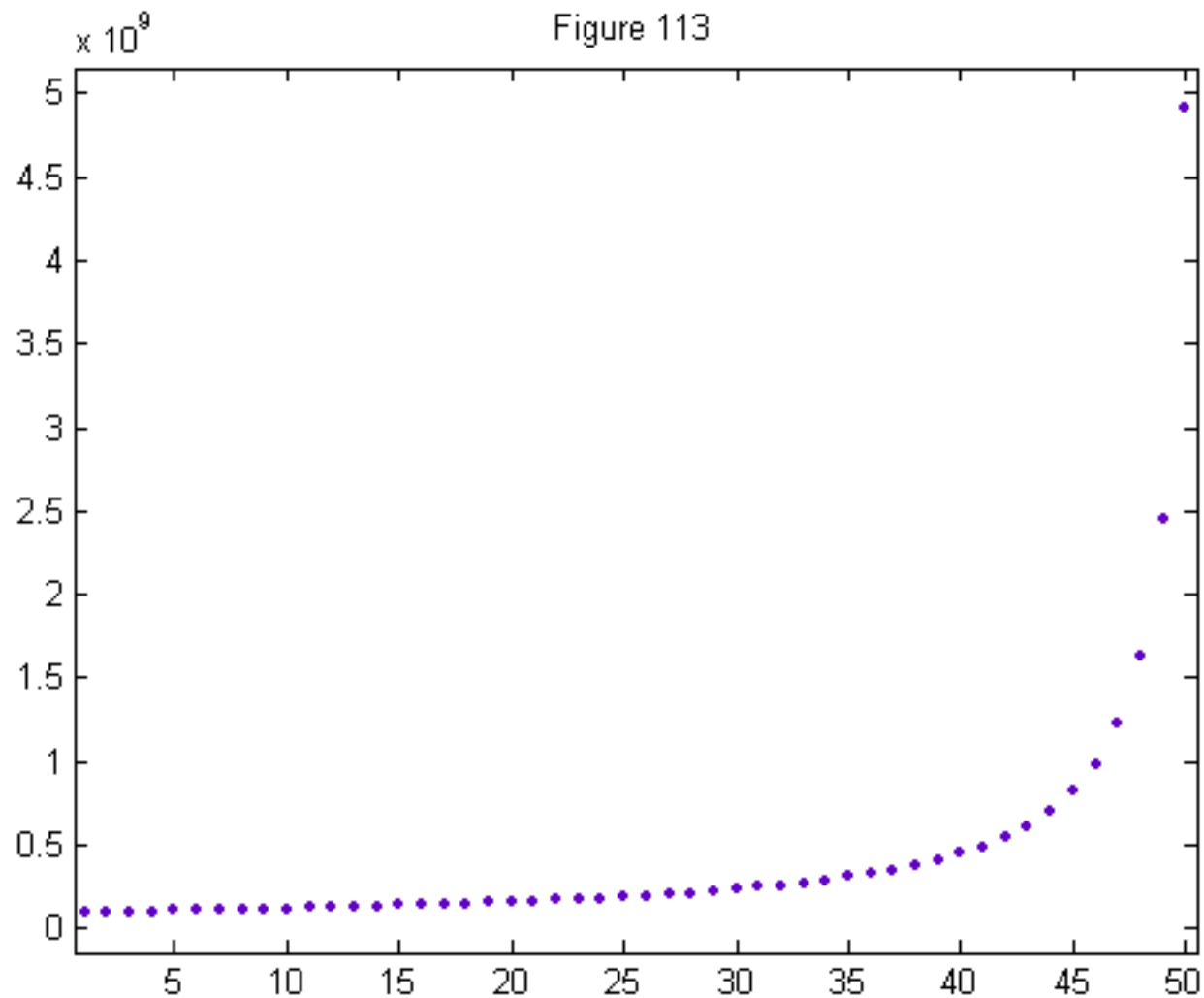


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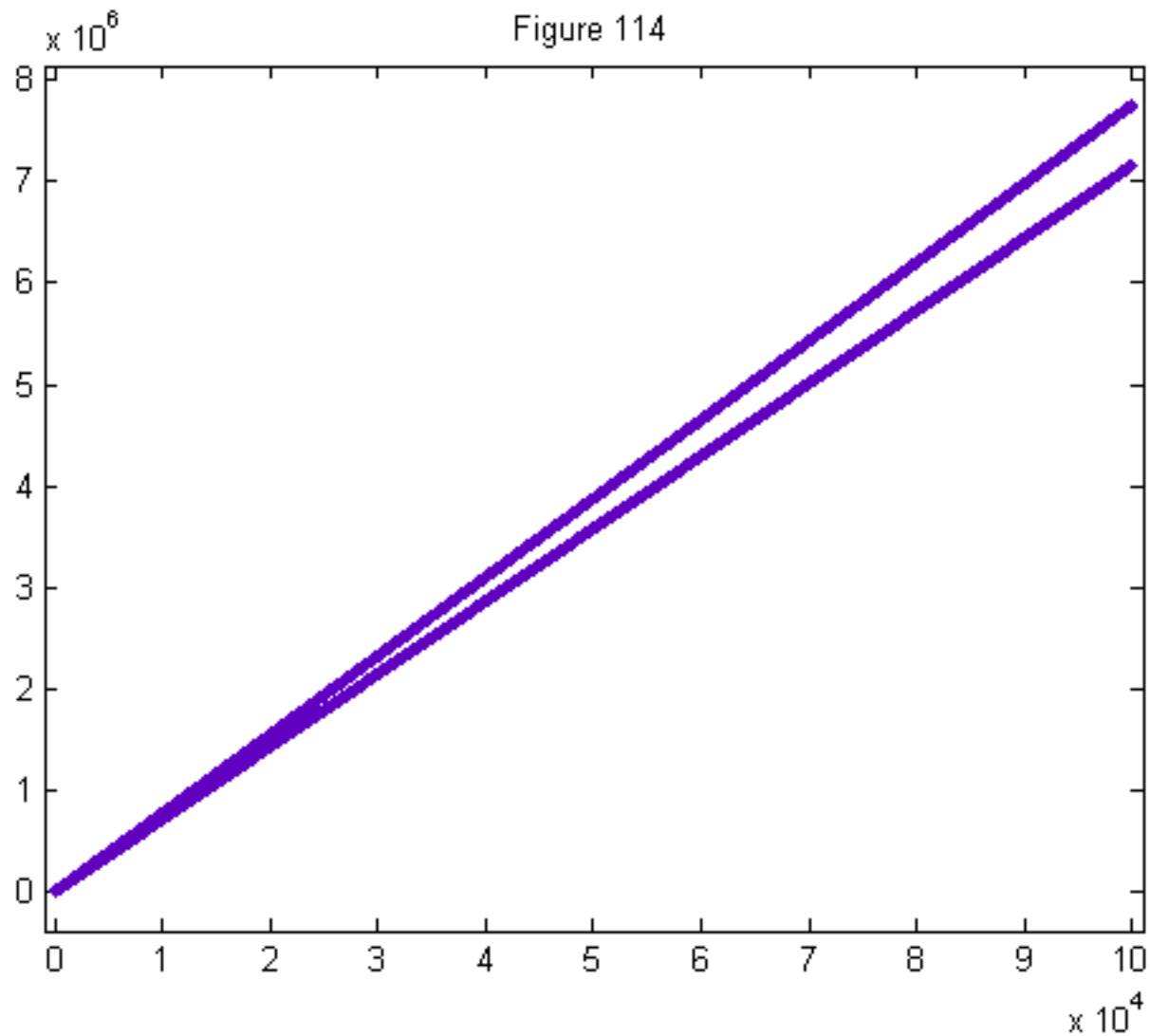


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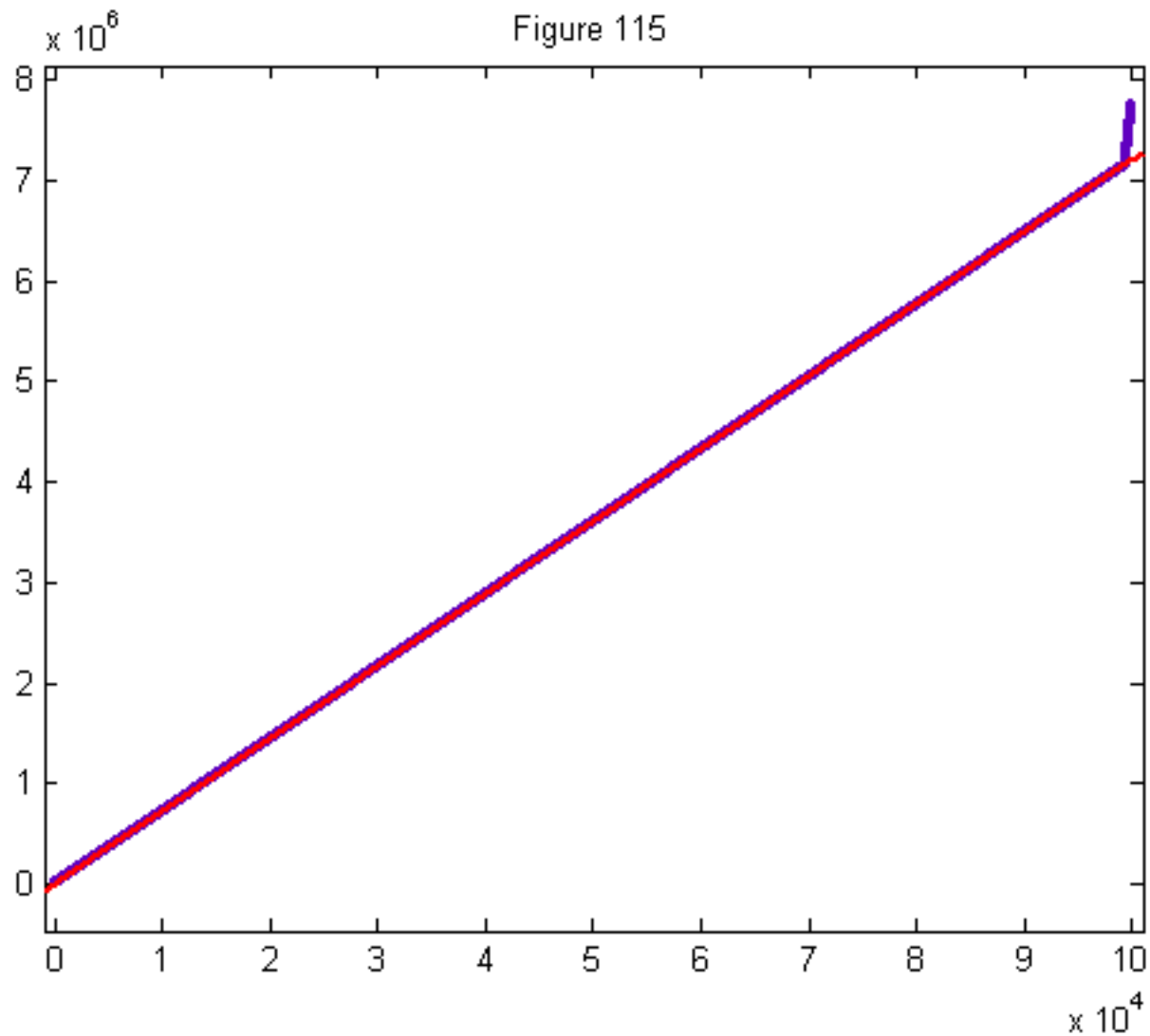


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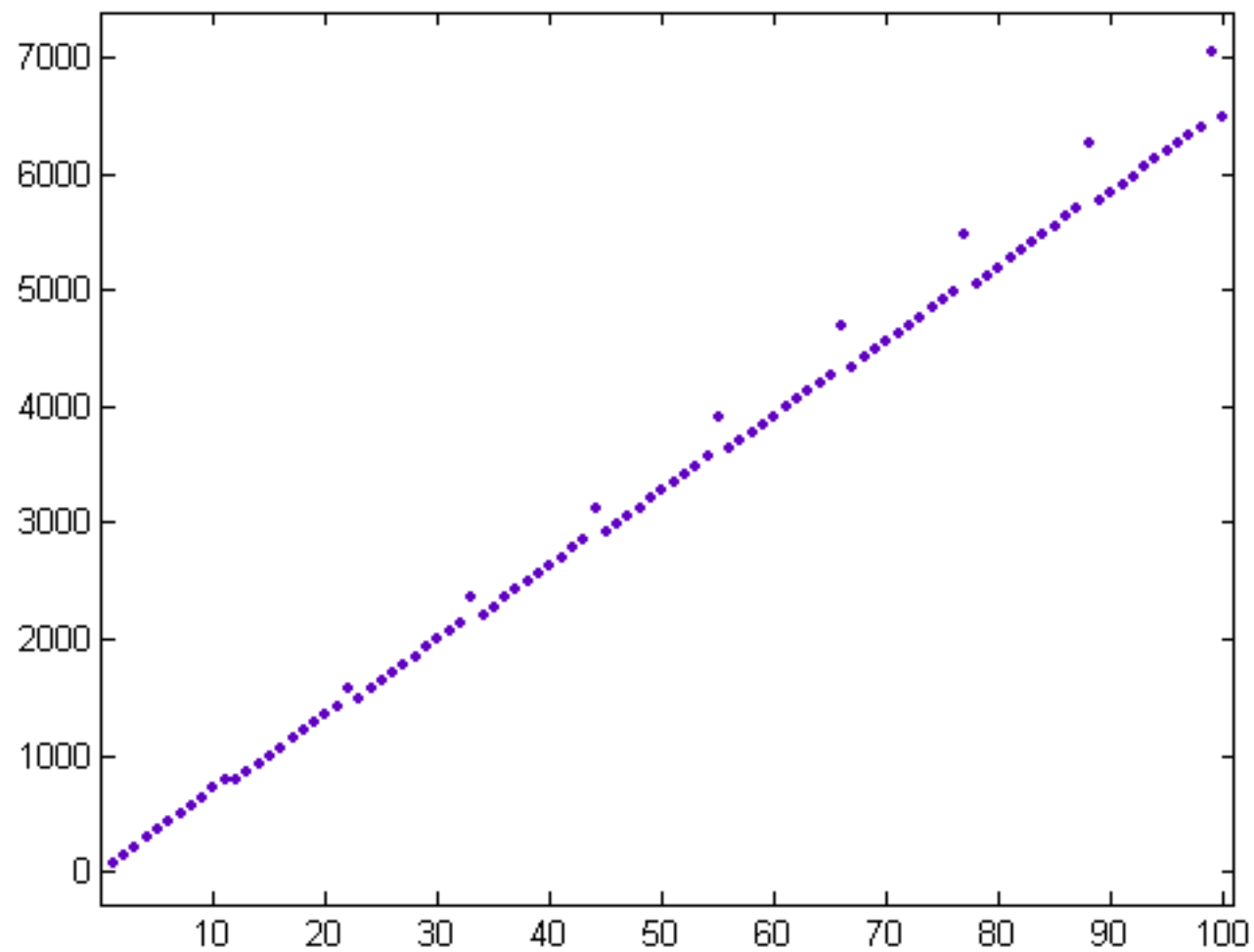


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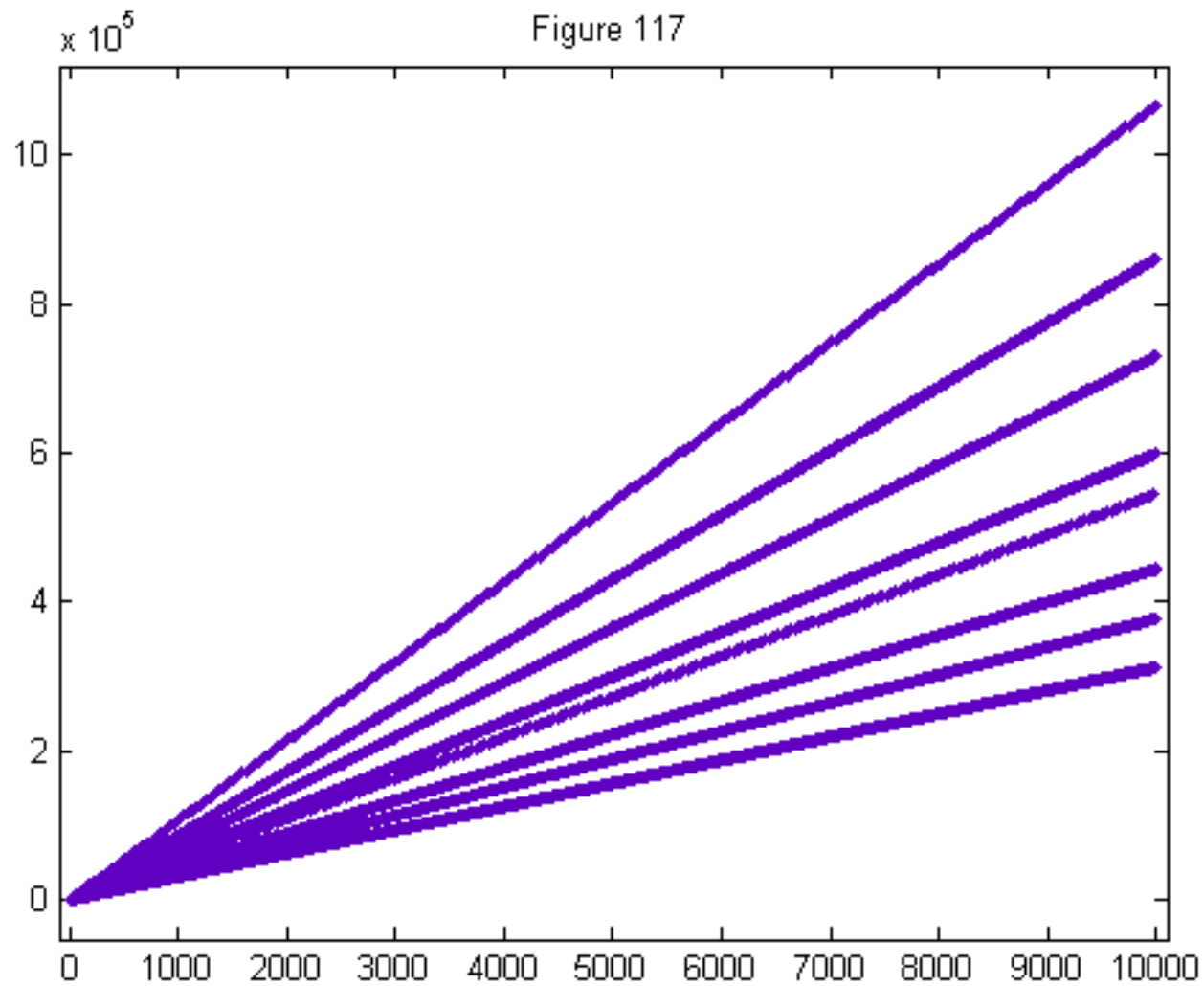


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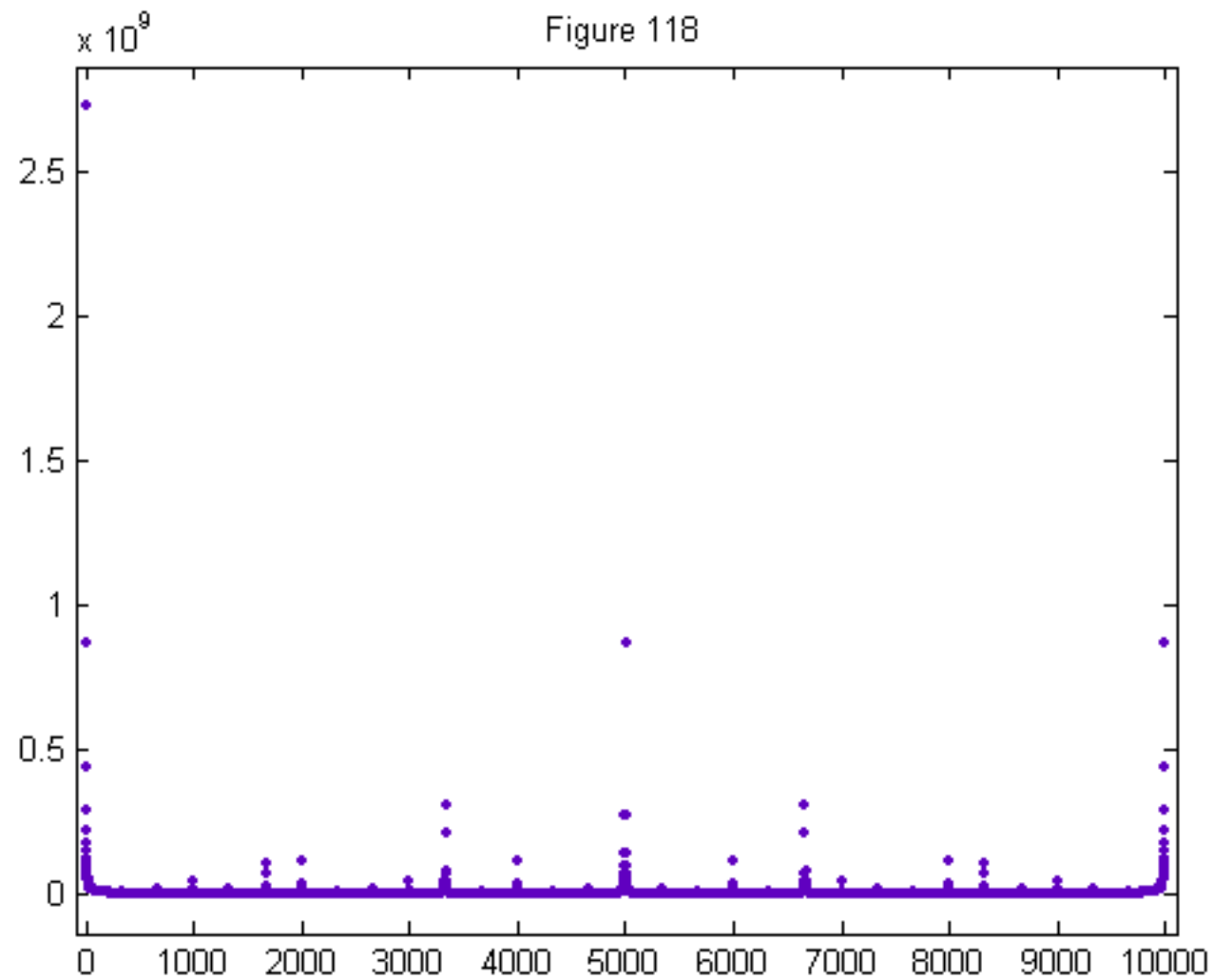


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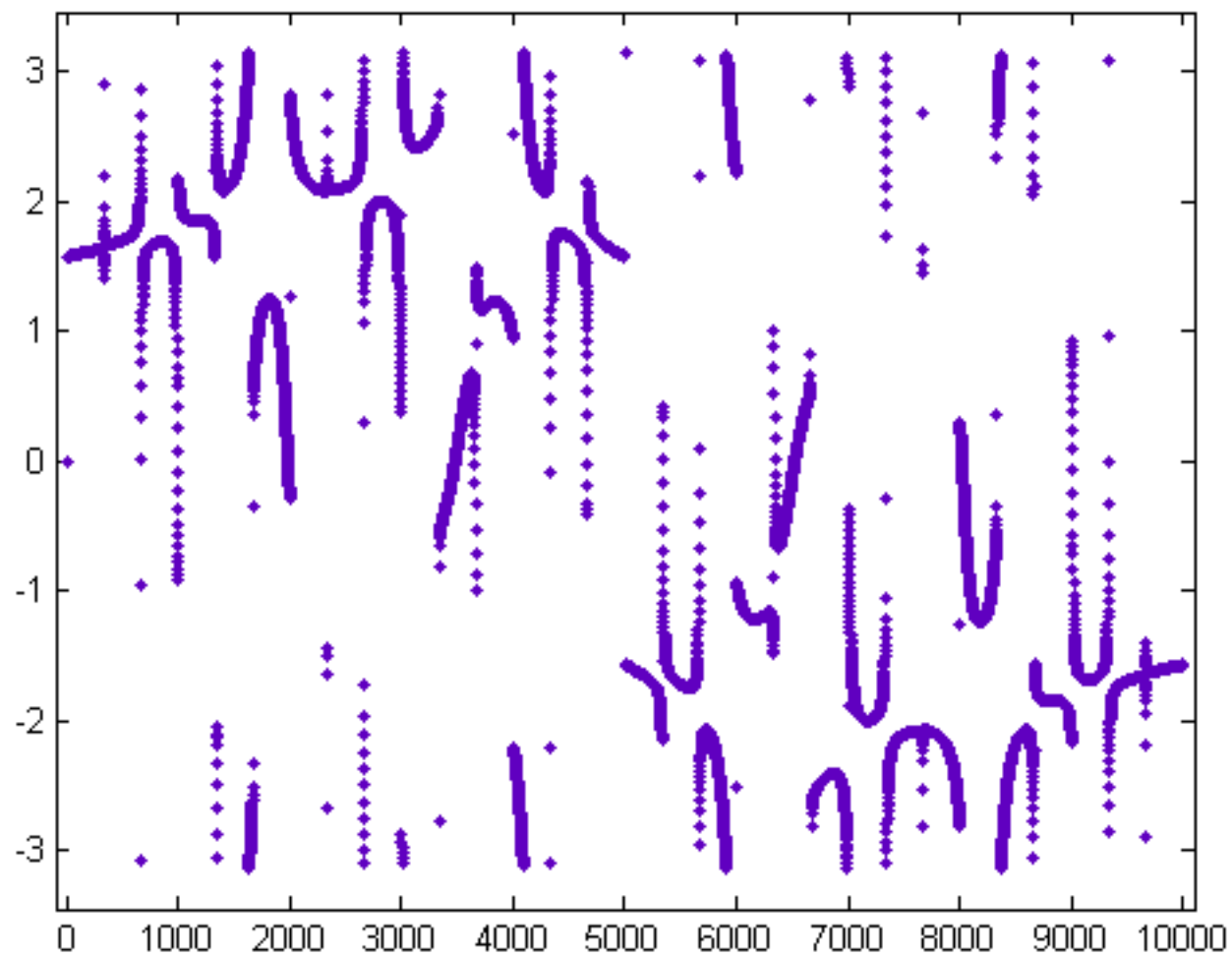


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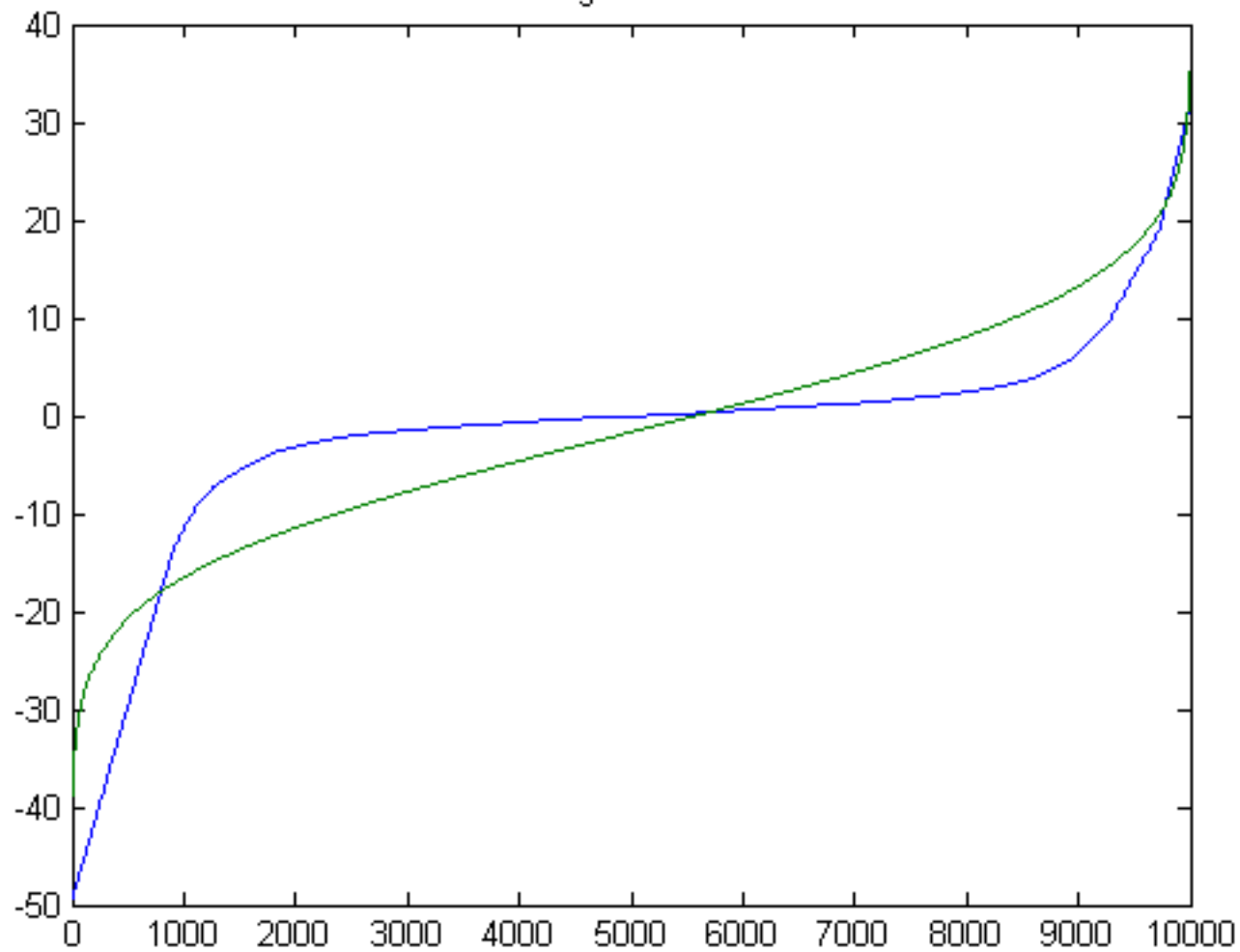


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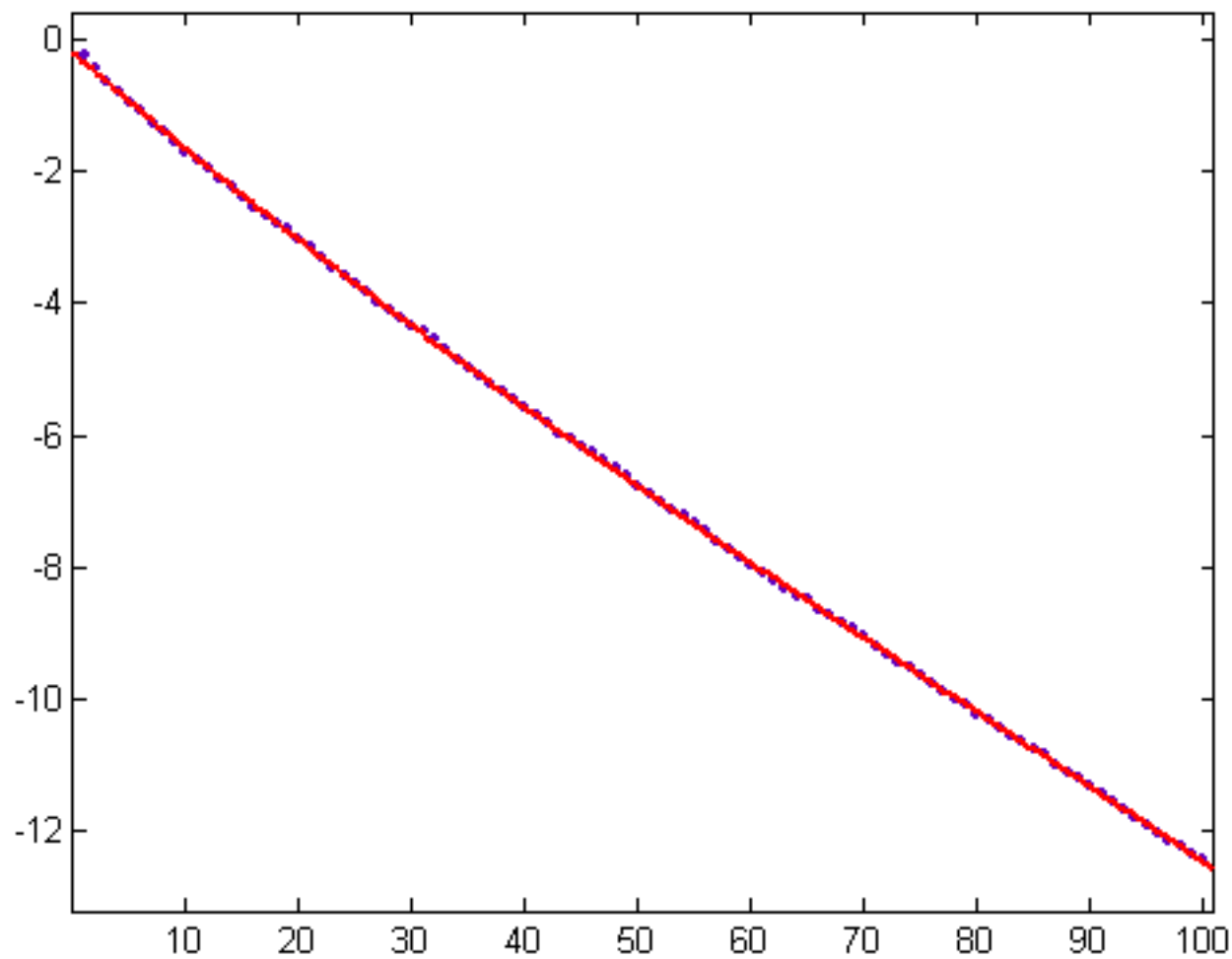


Figure 122

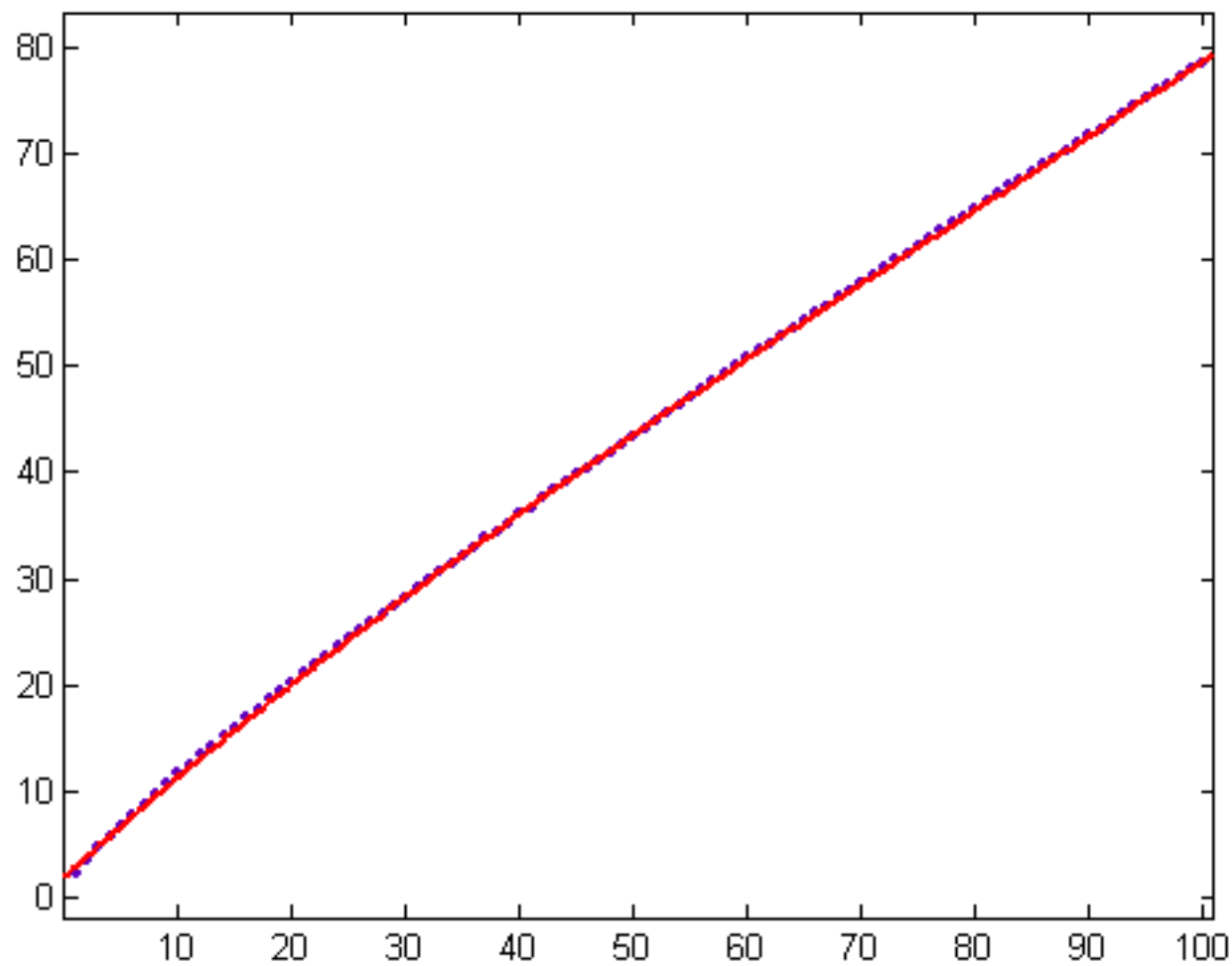


Figure 123

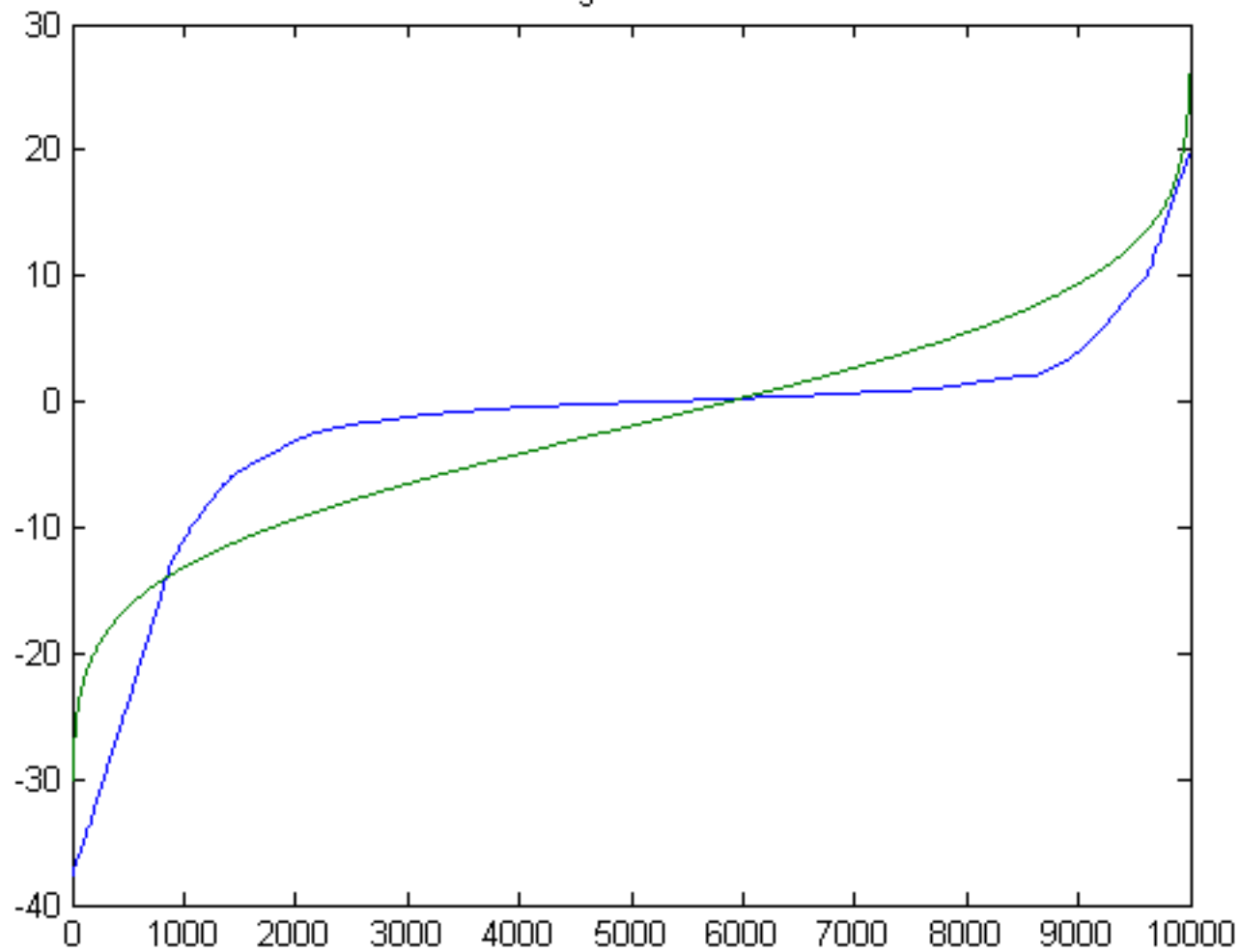


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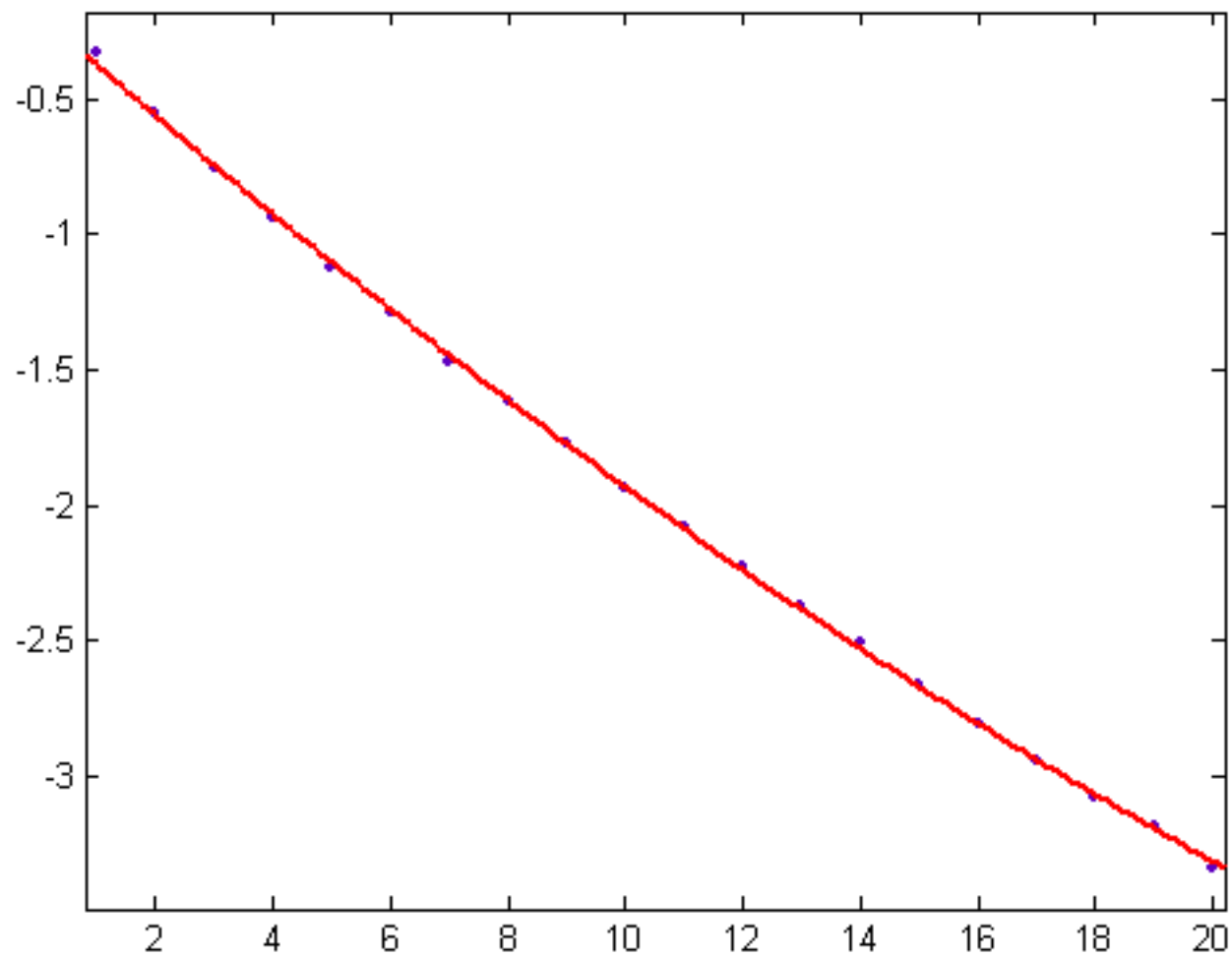


Figure 125

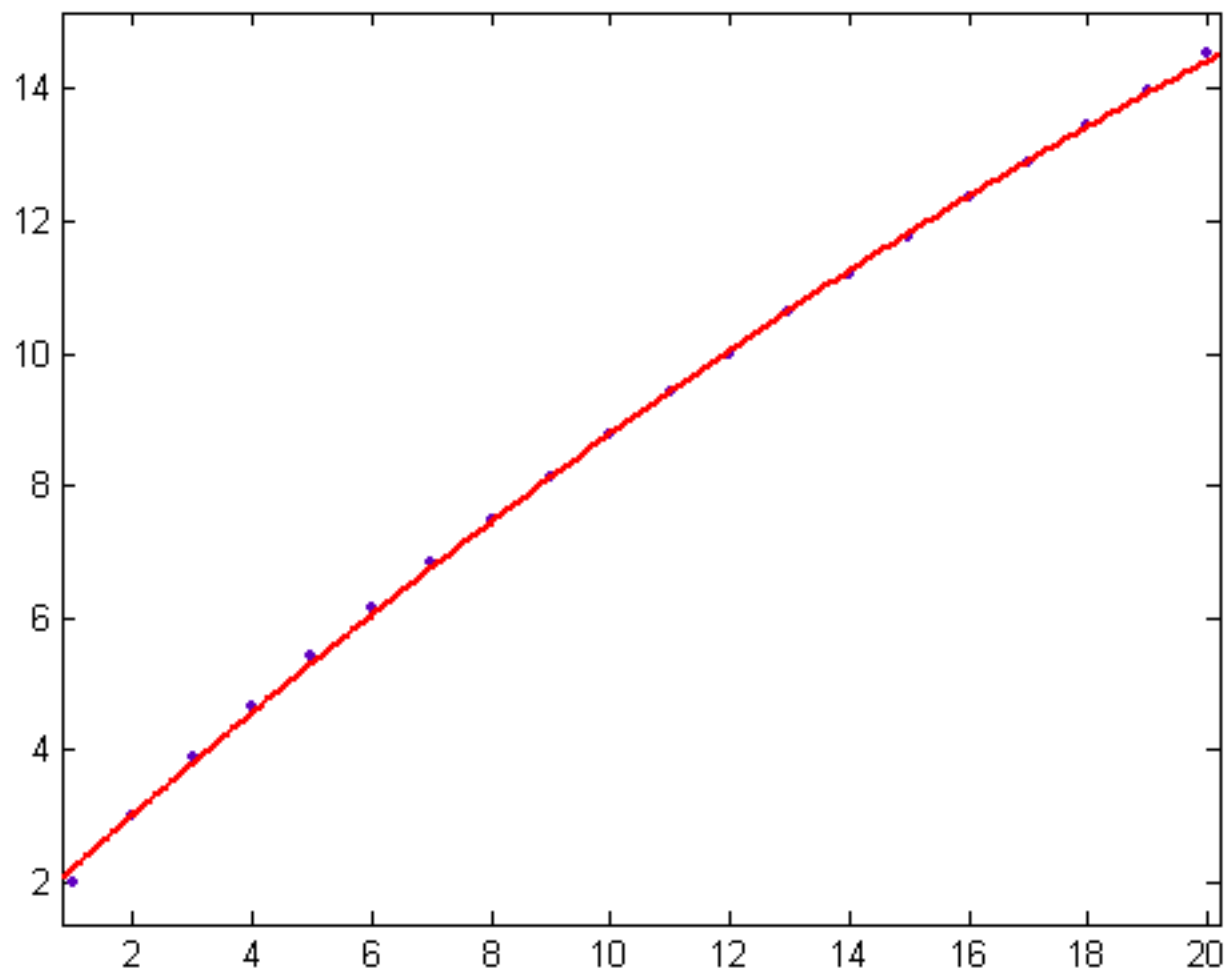


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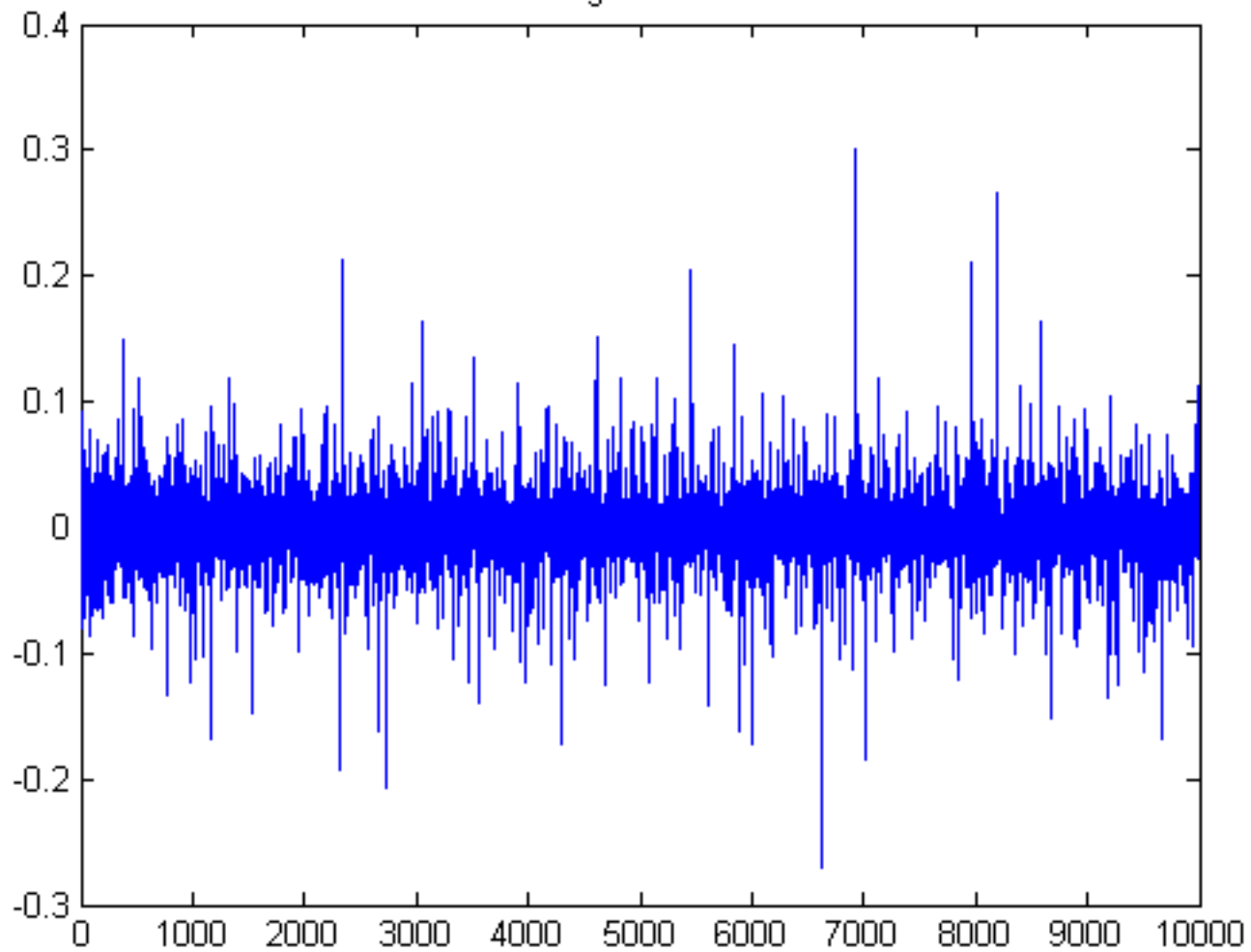


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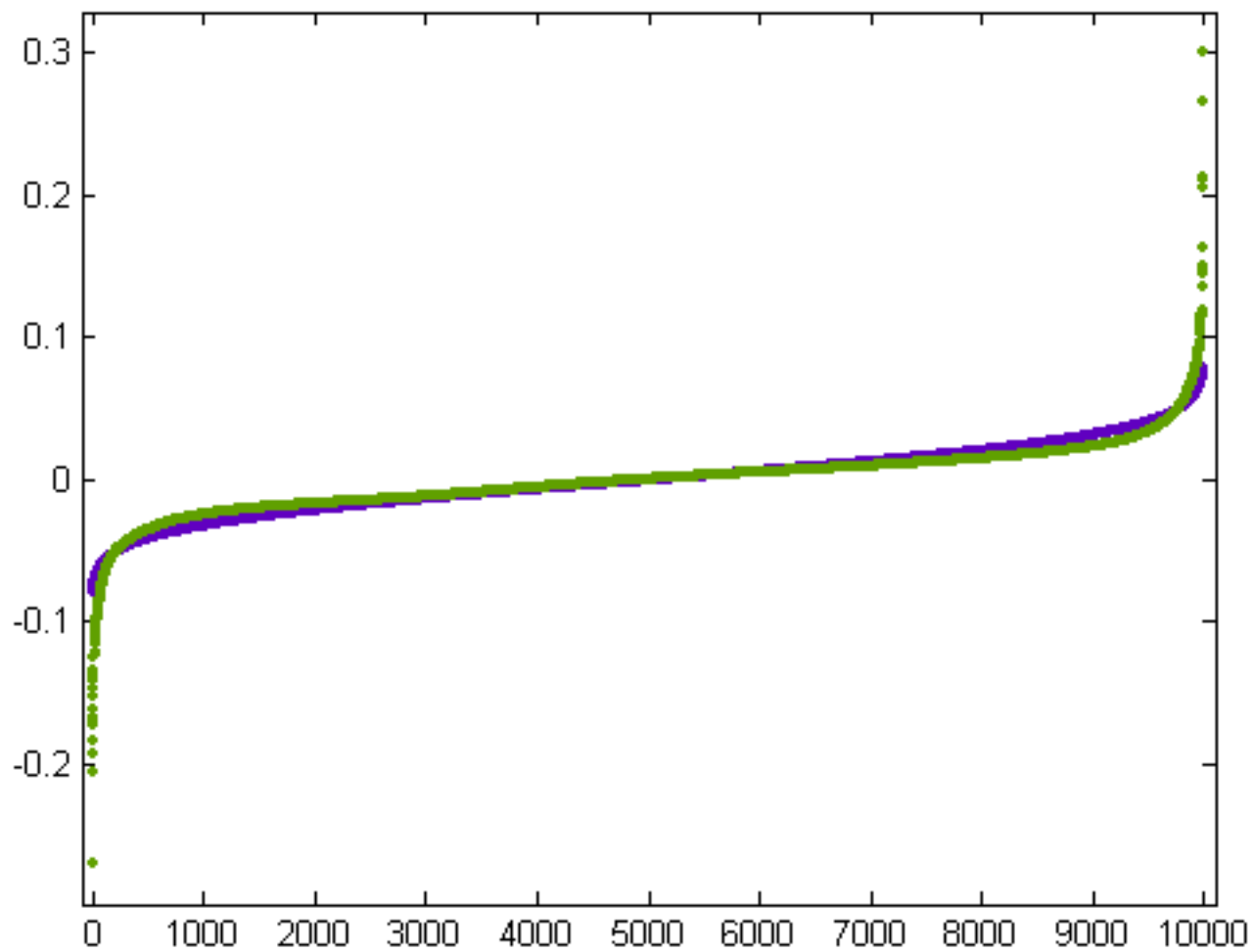


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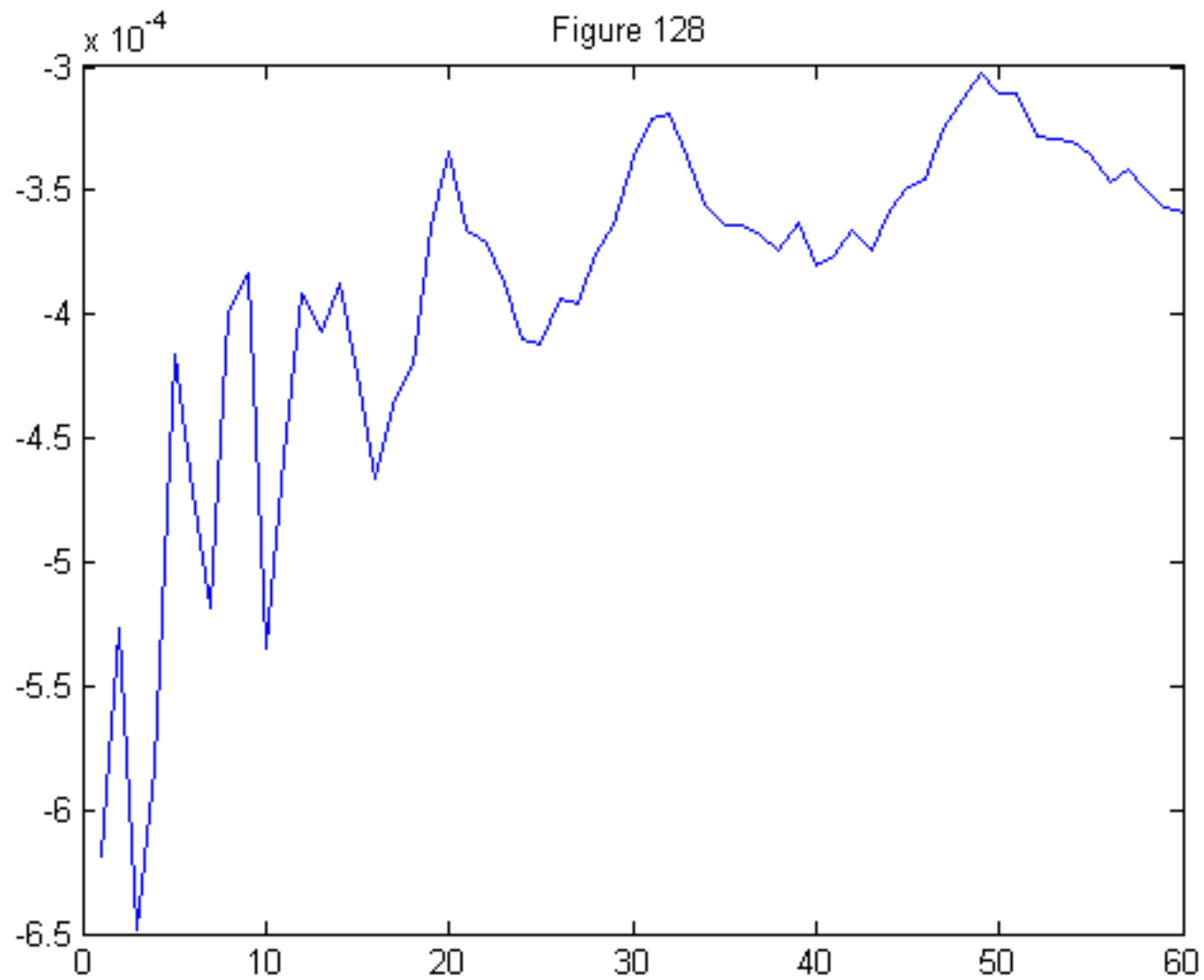


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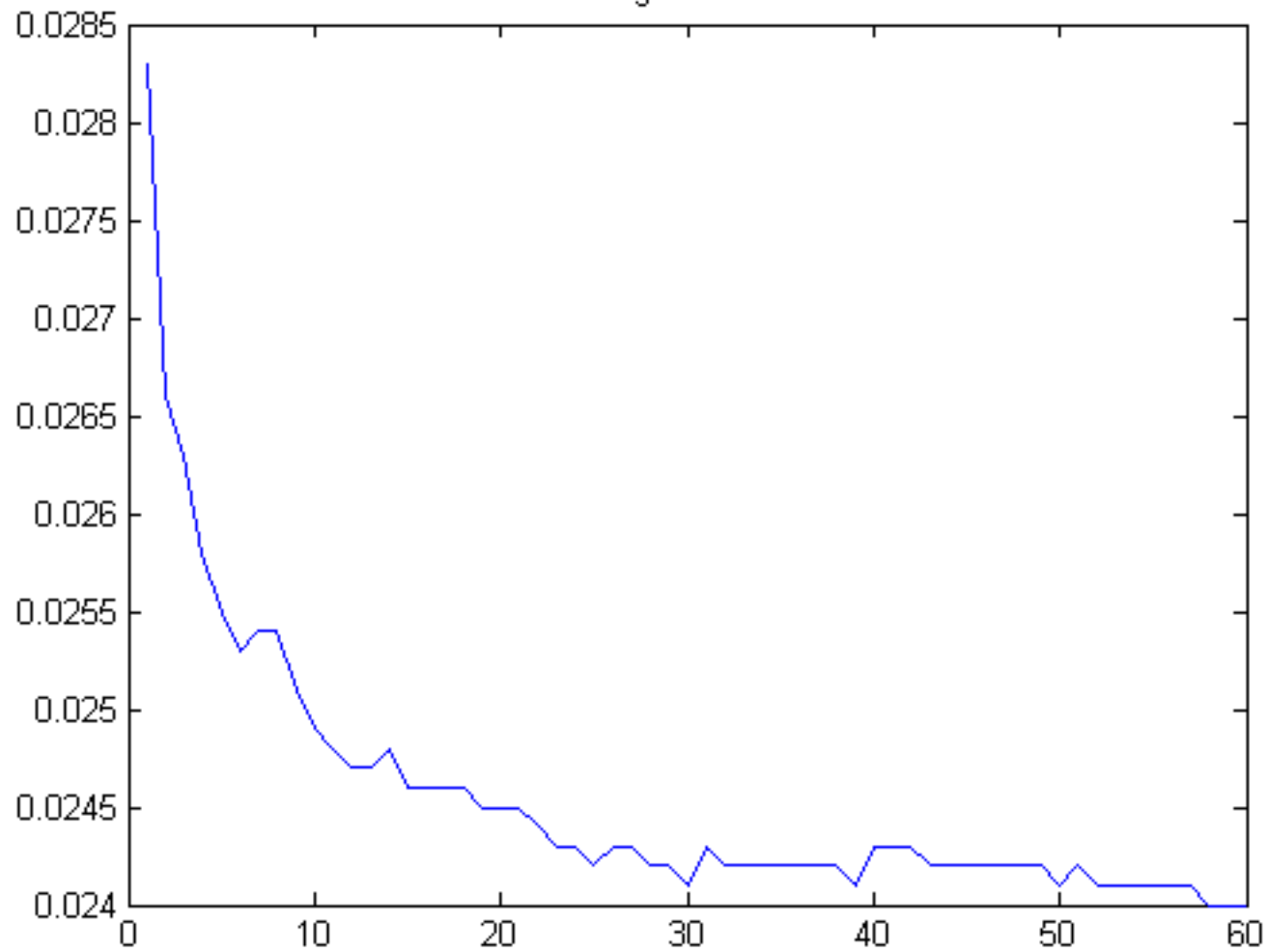


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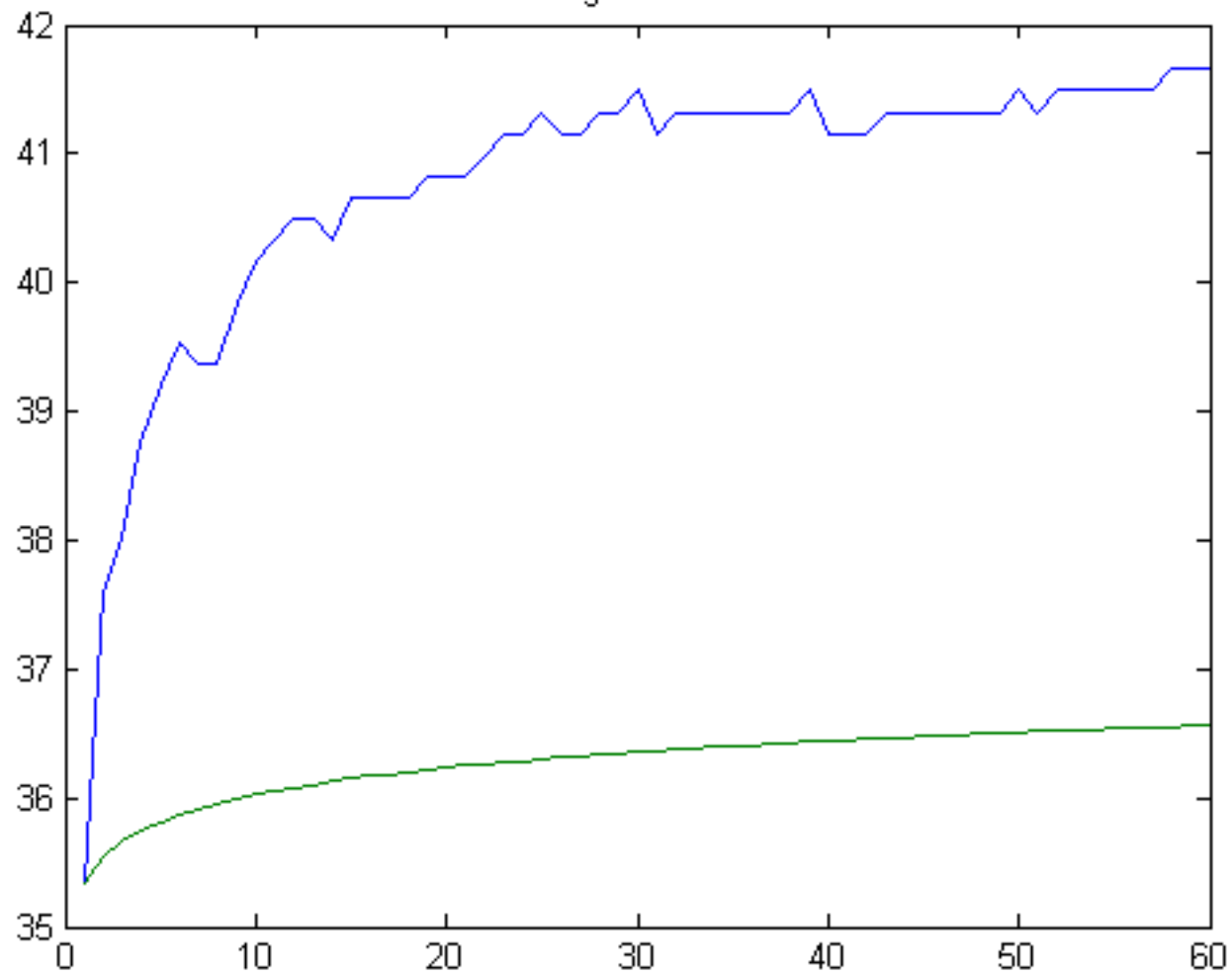


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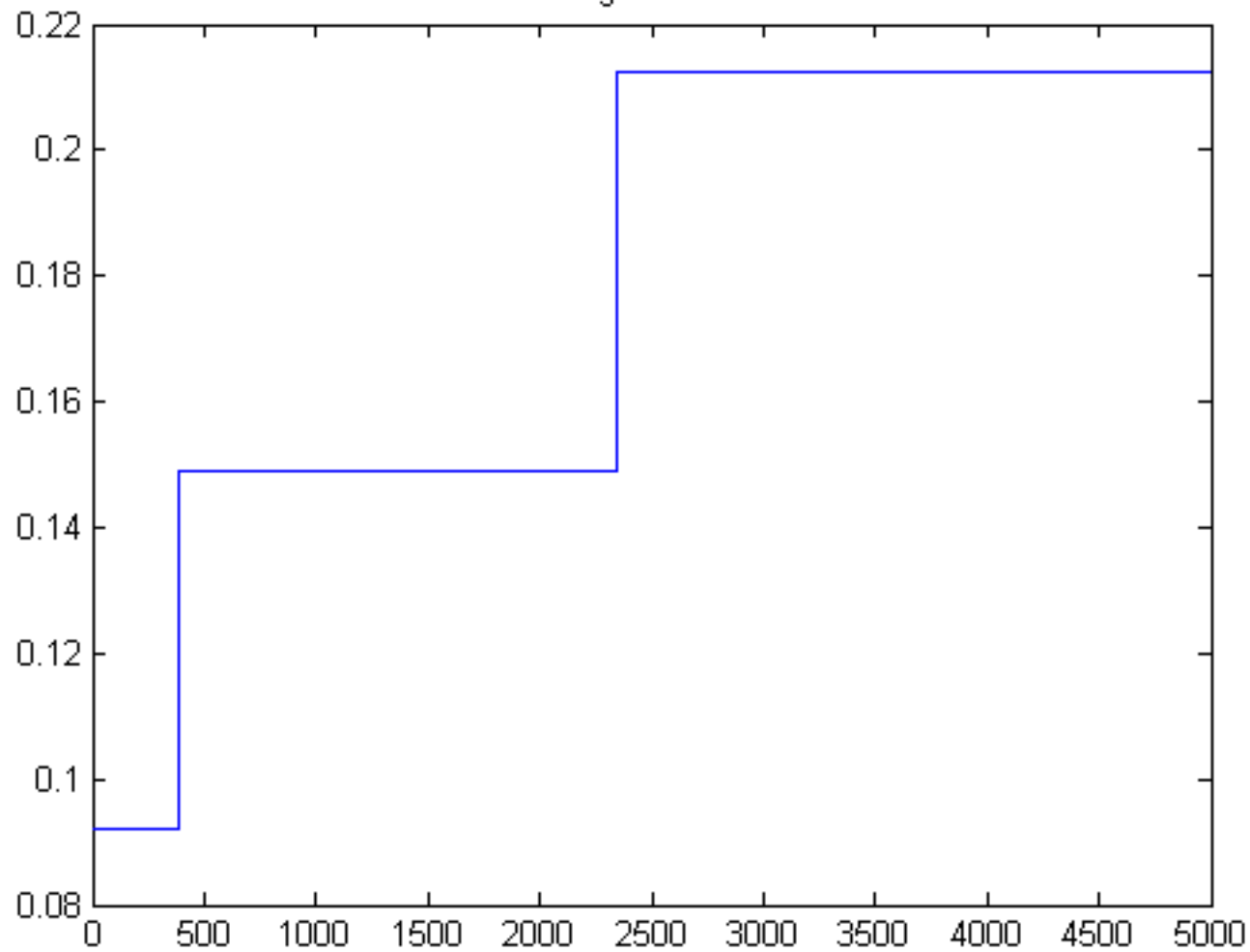


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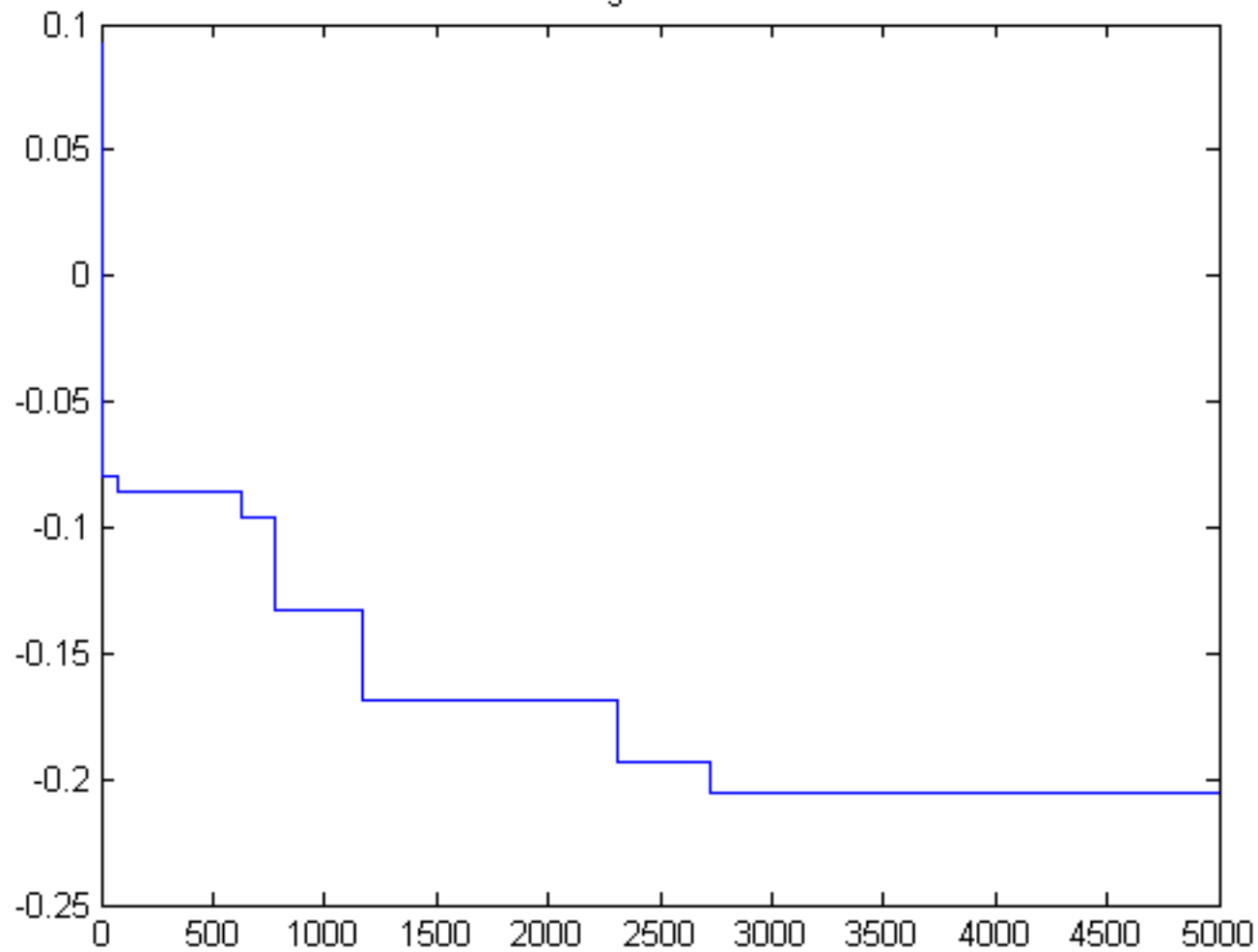


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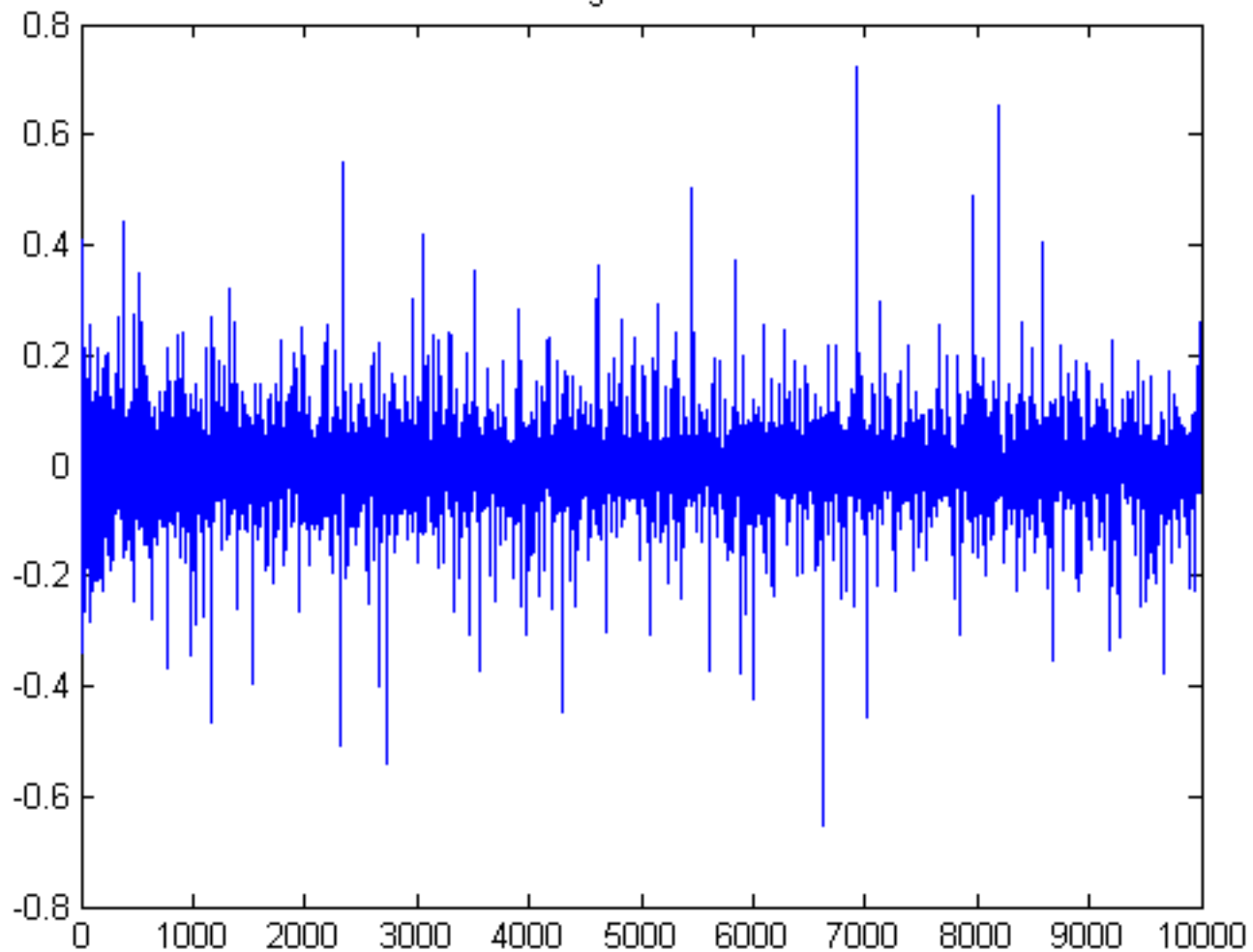


Figure 134

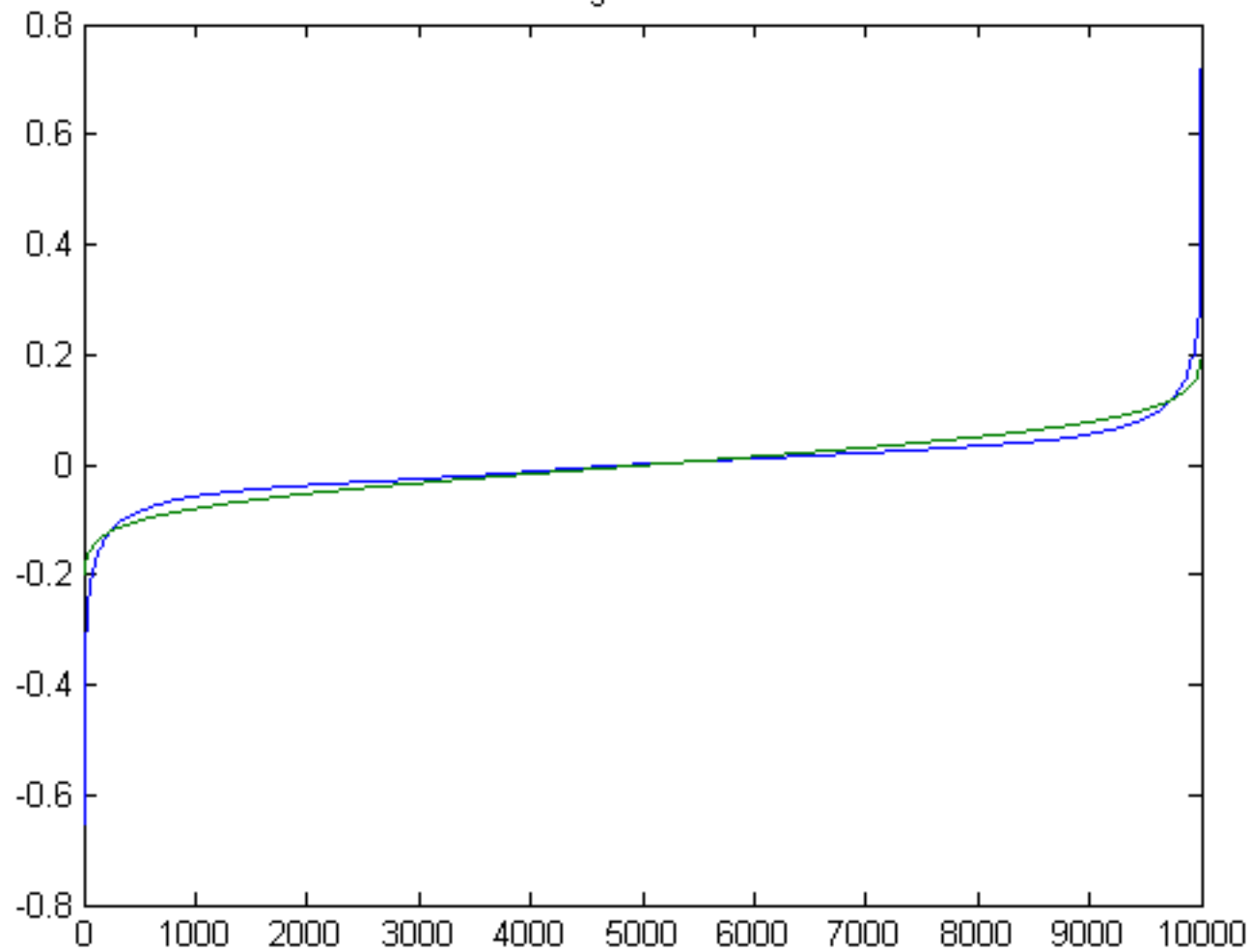


Figure 135

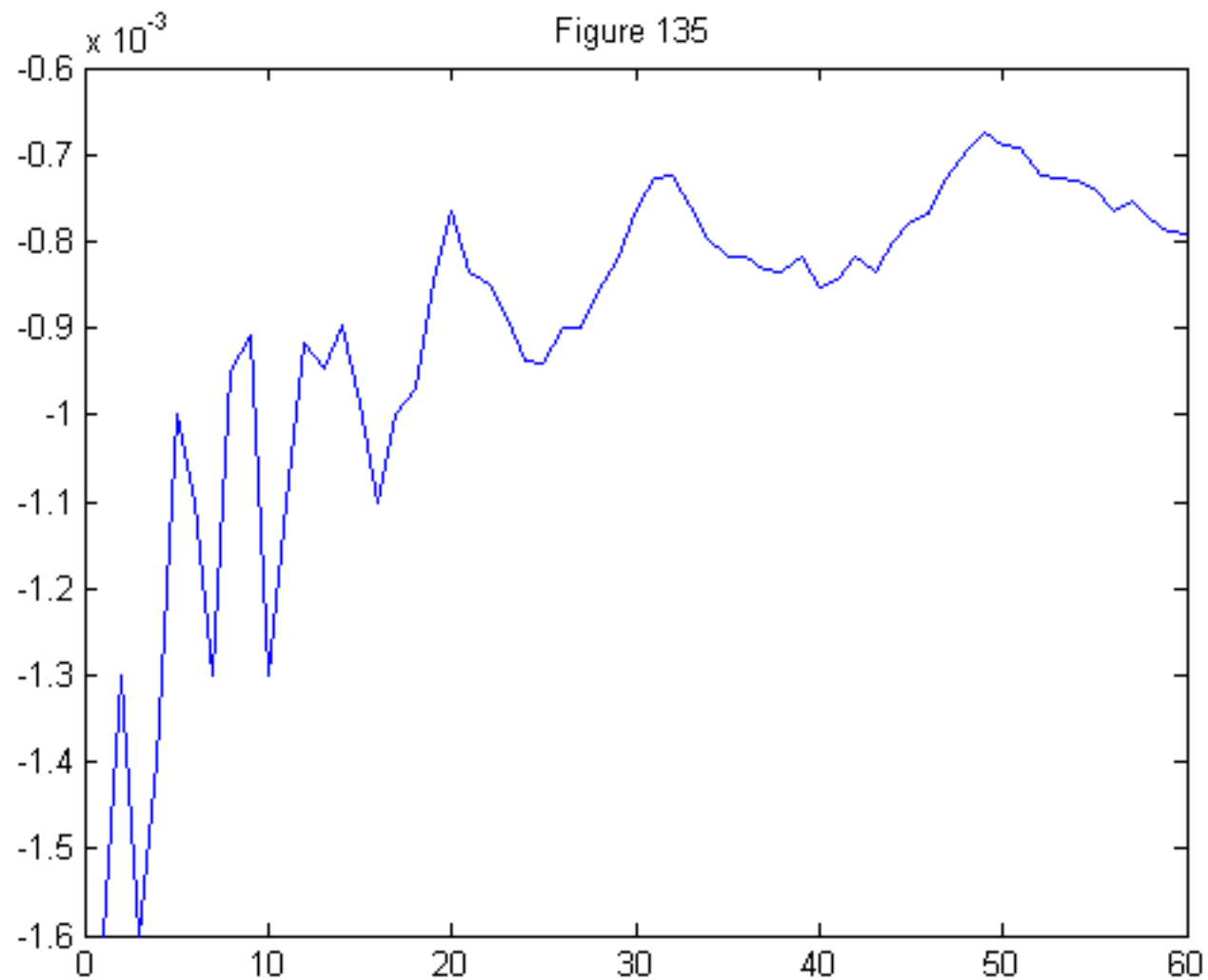


Figure 136

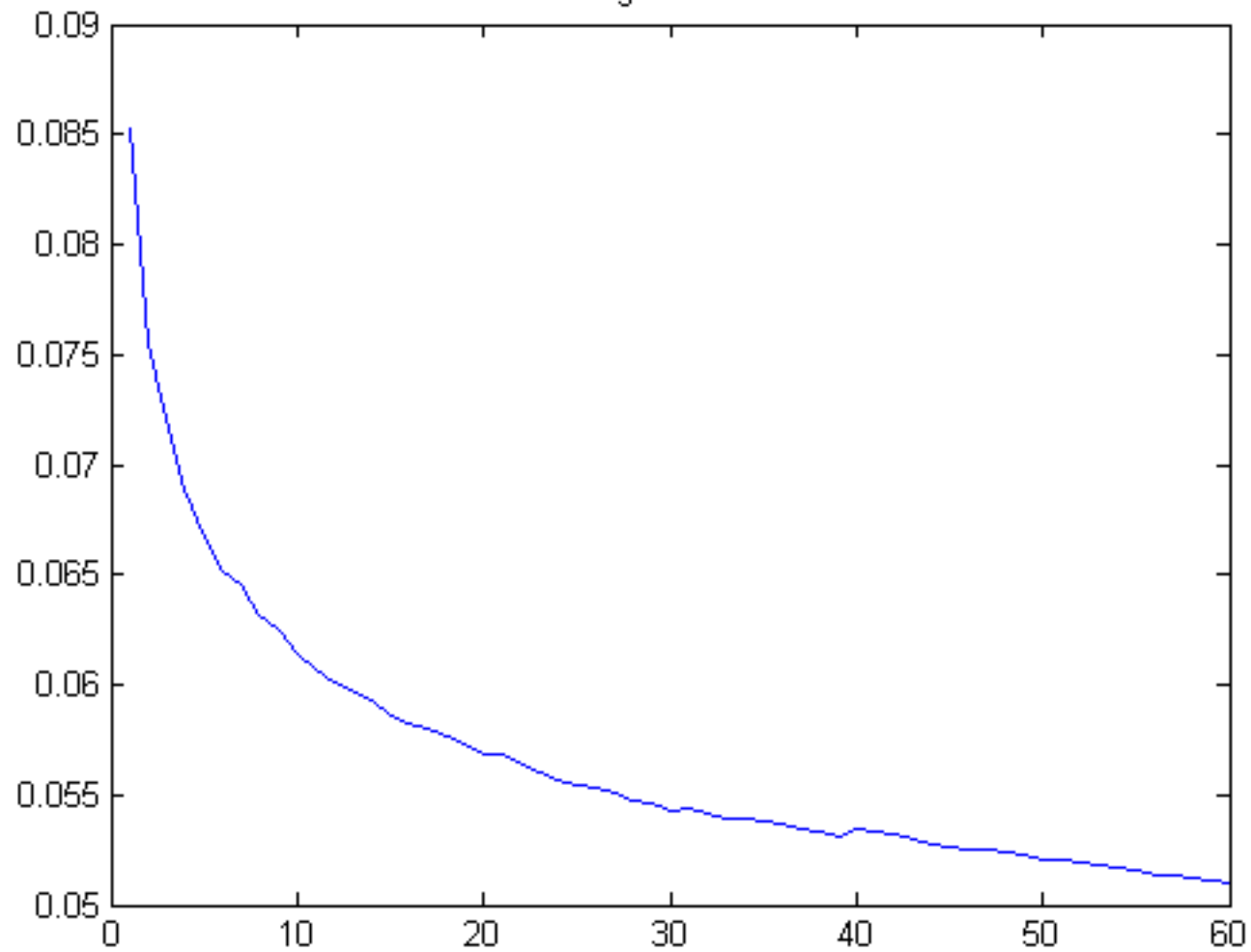


Figure 137

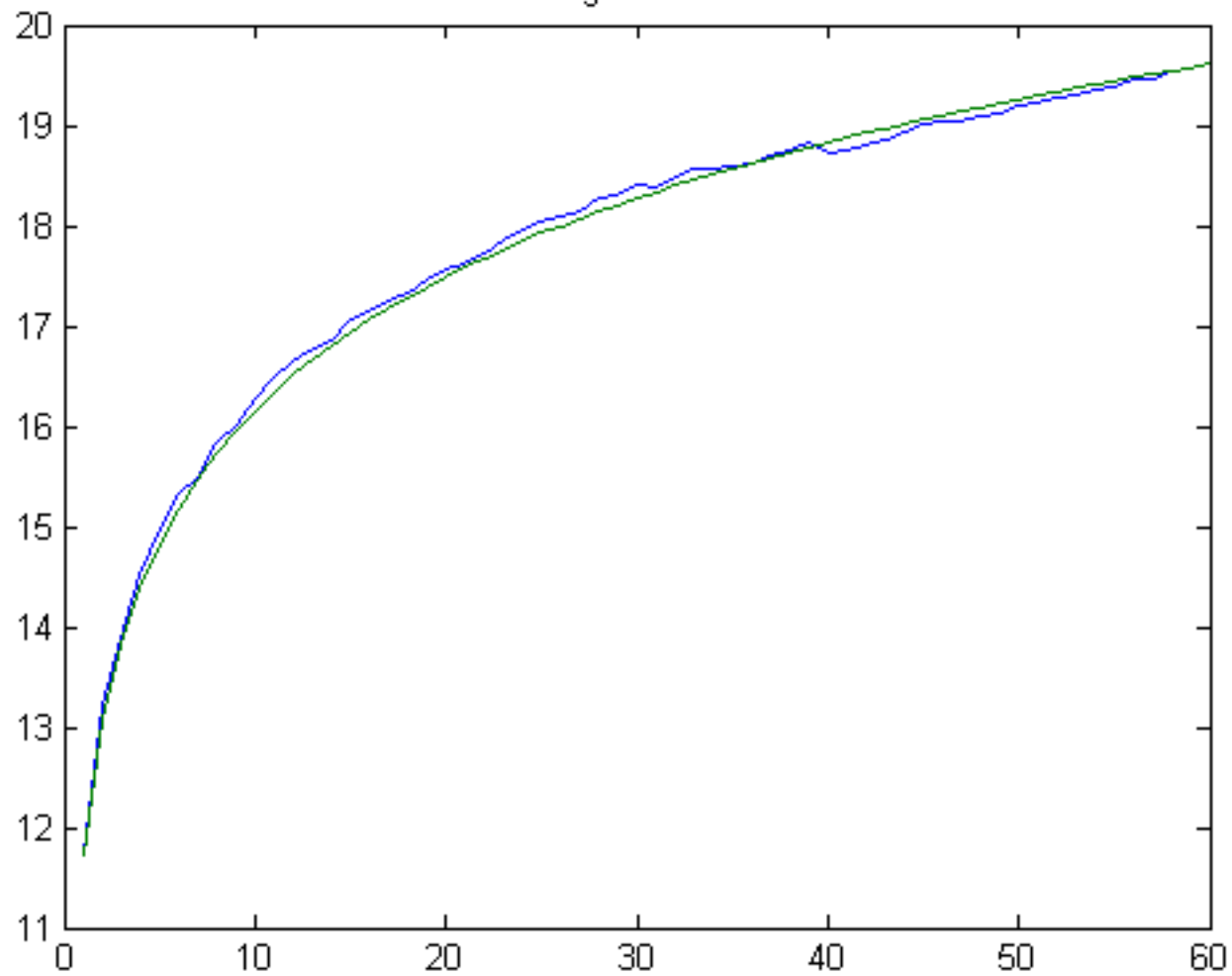


Figure 138

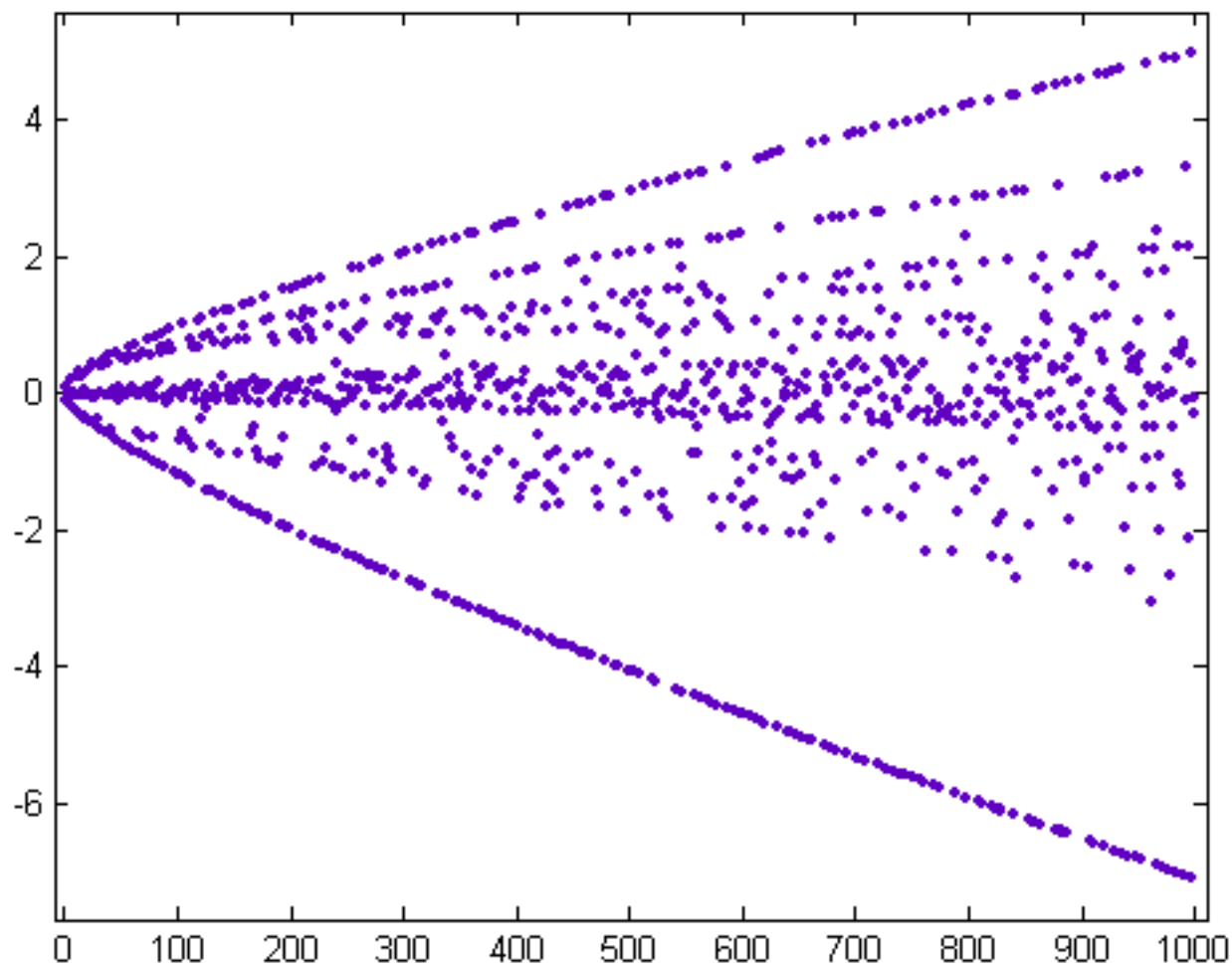


Figure 139

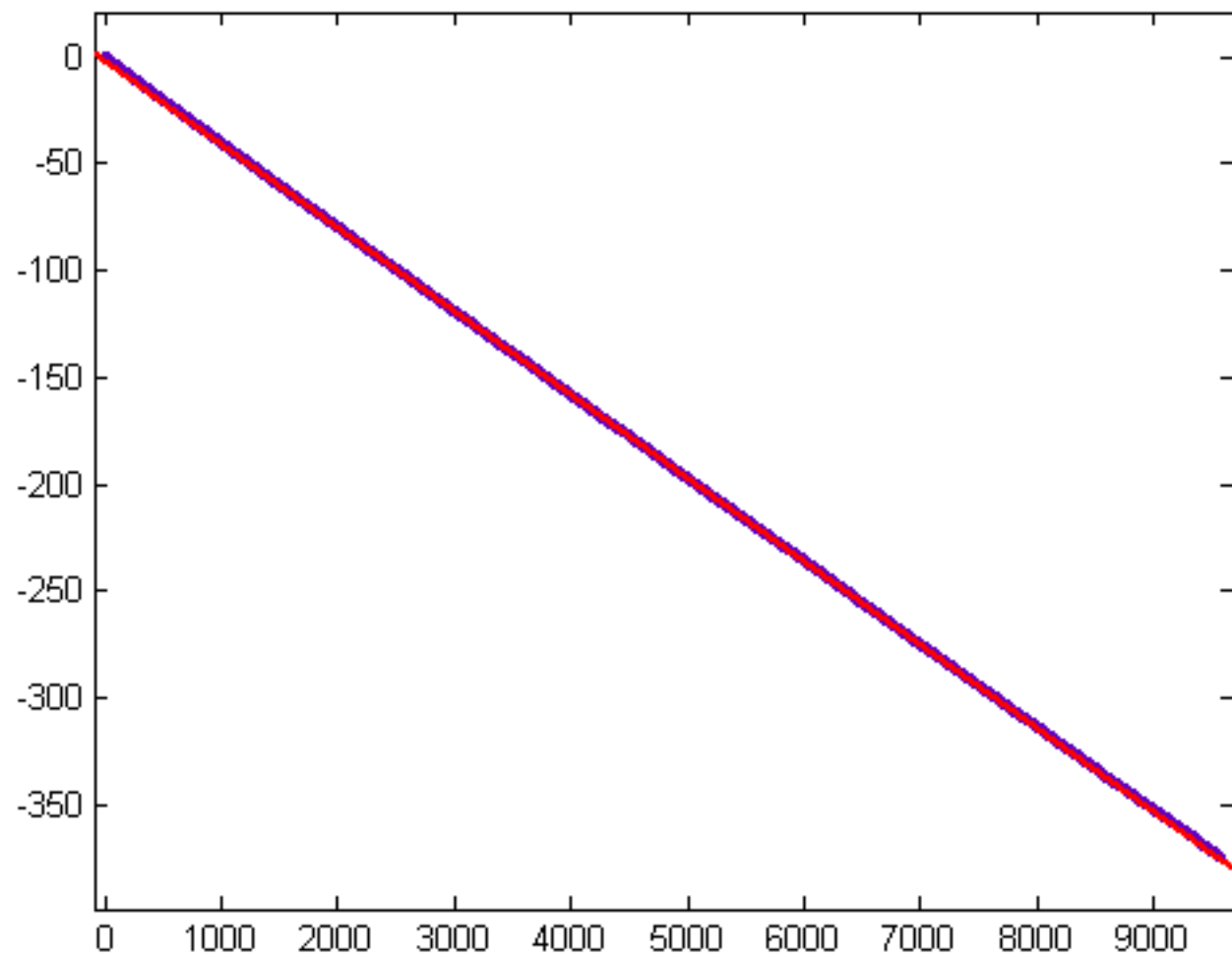


Figure 140

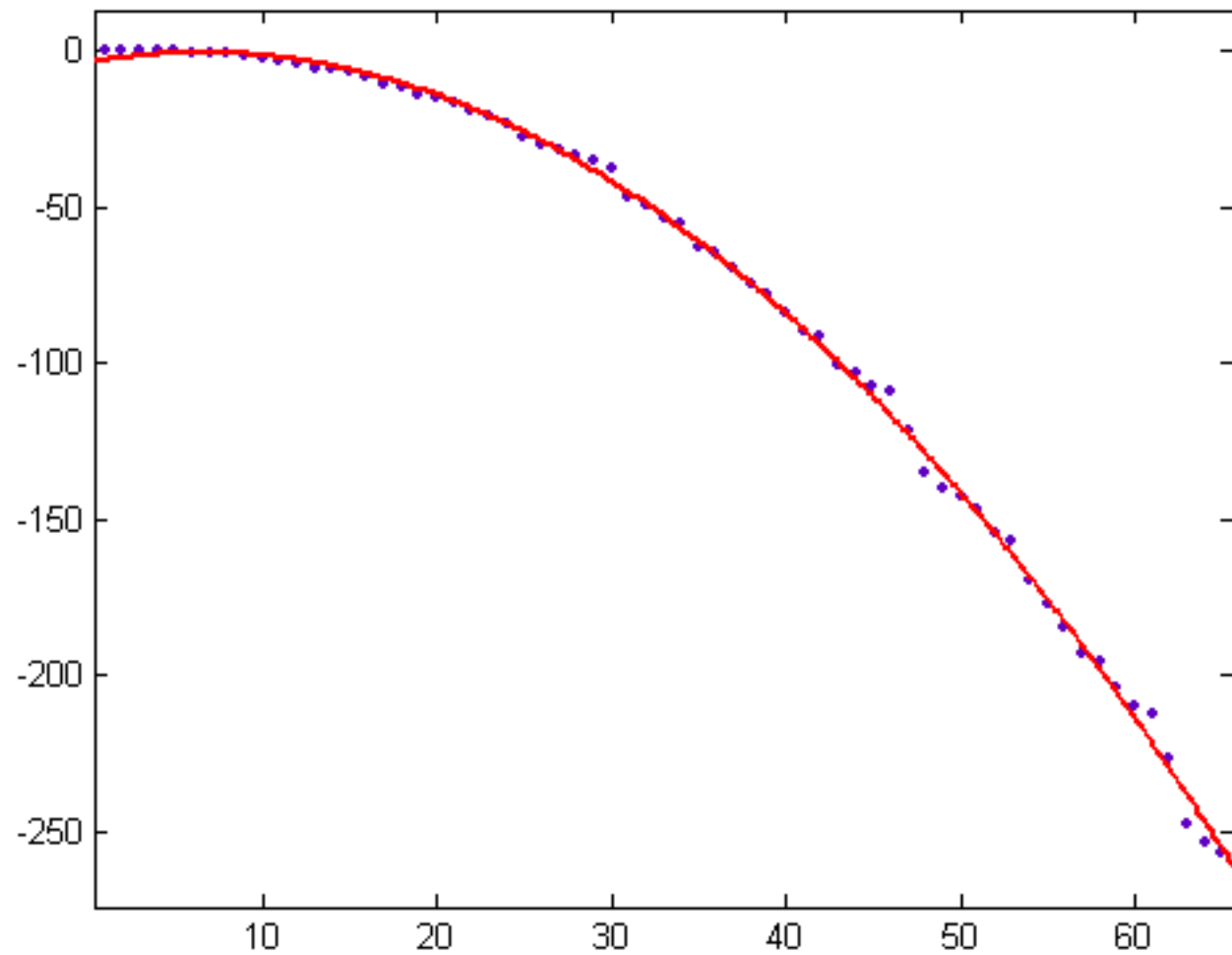


Figure 141

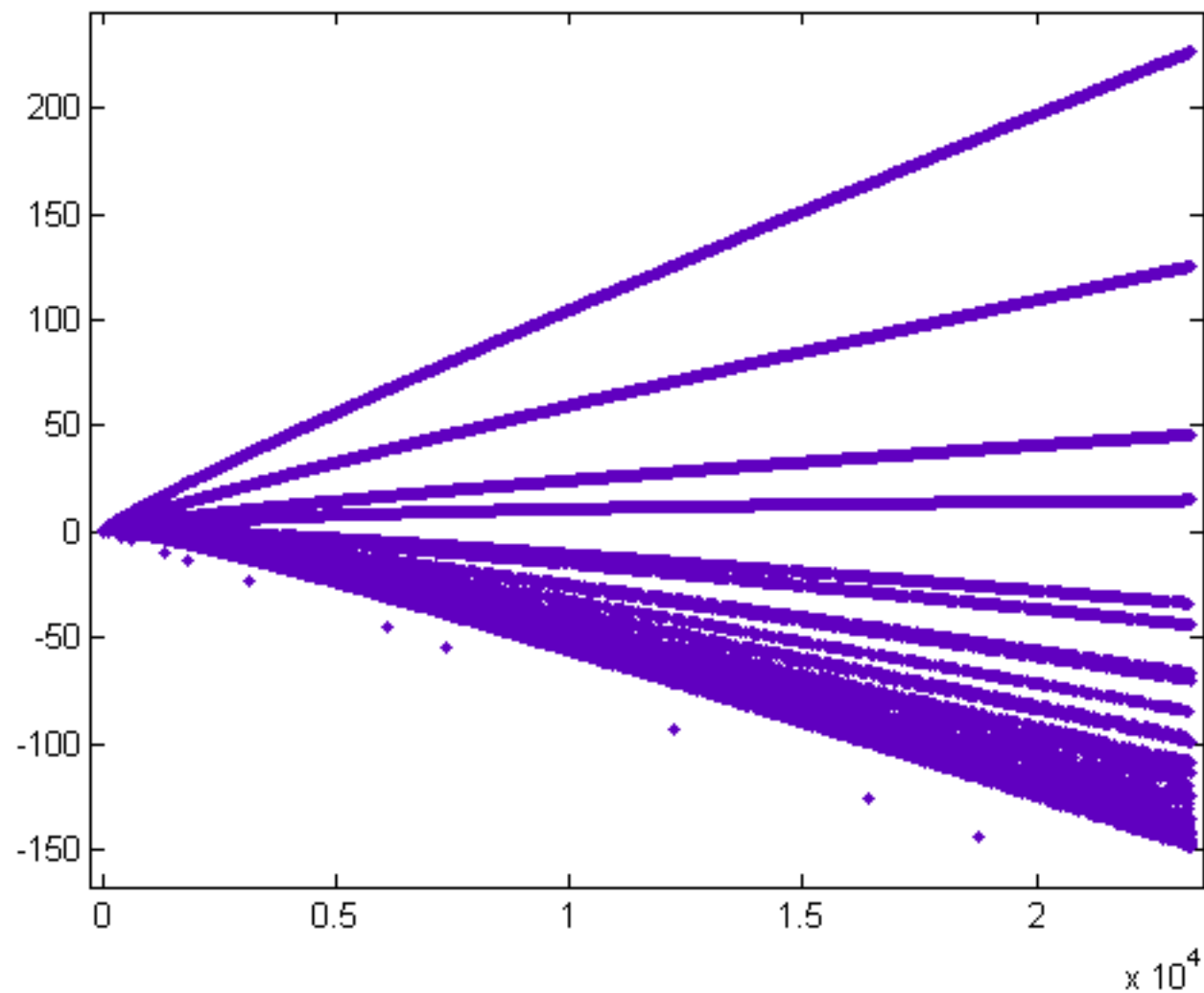


Figure 142

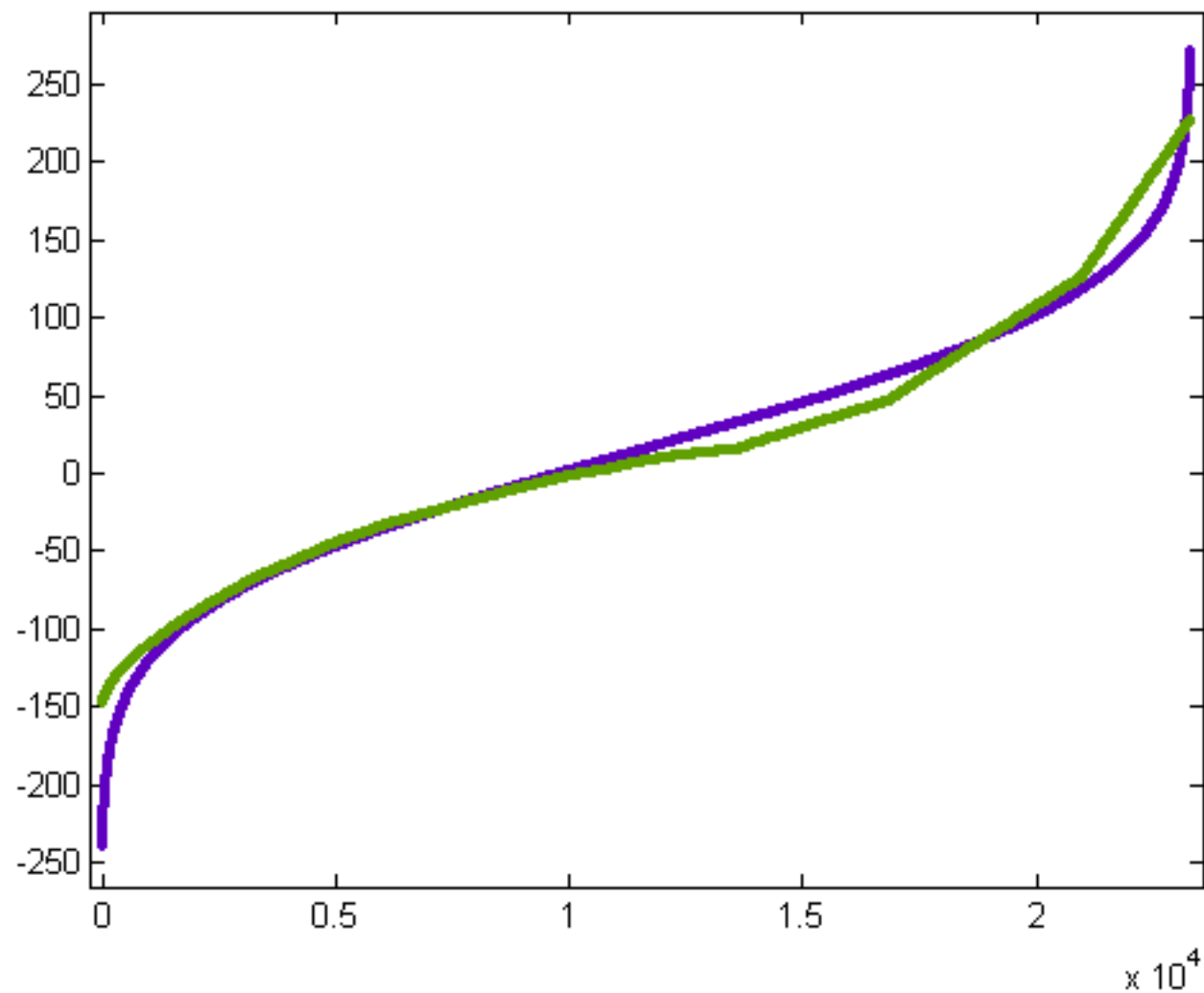


Figure 143

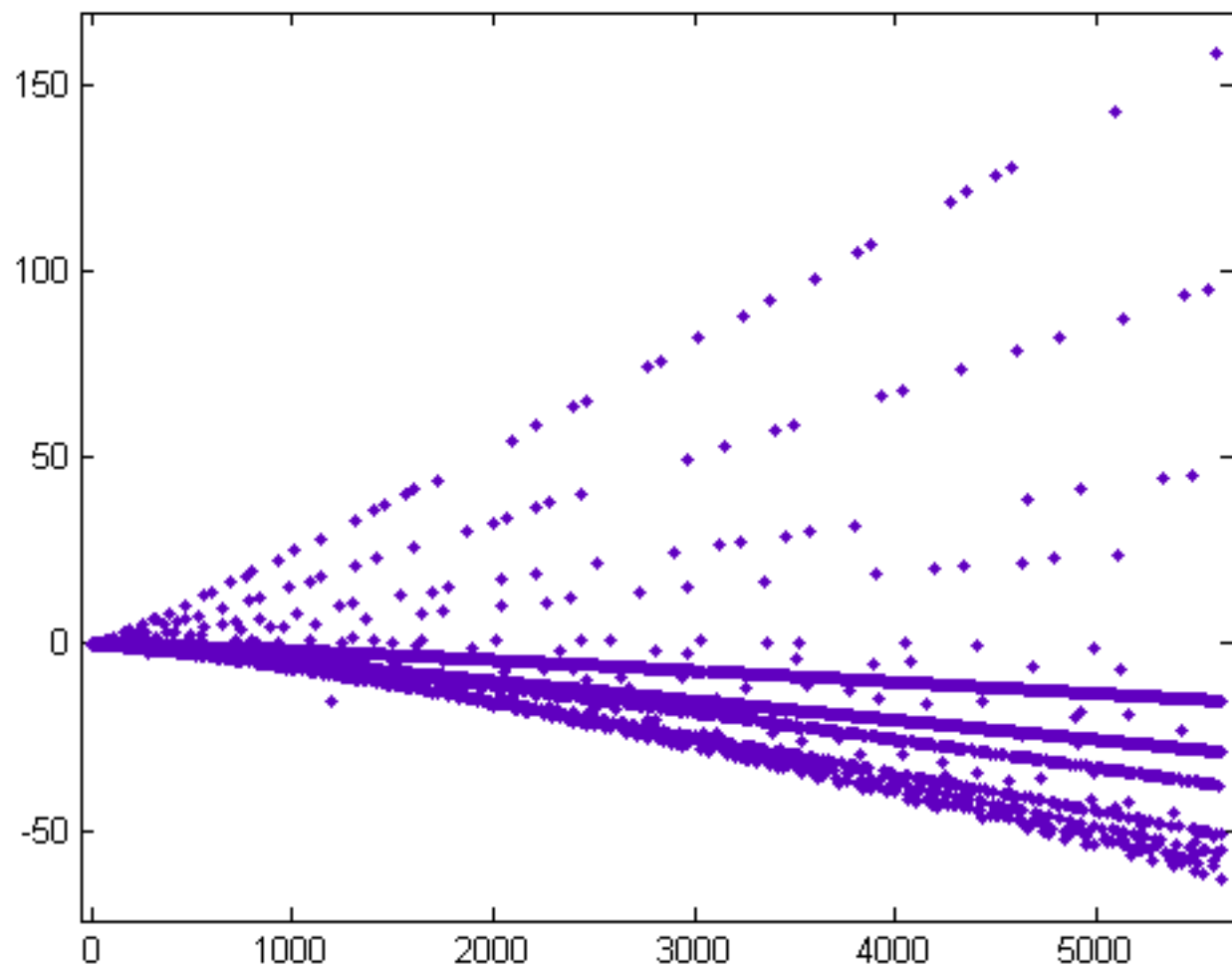


Figure 144

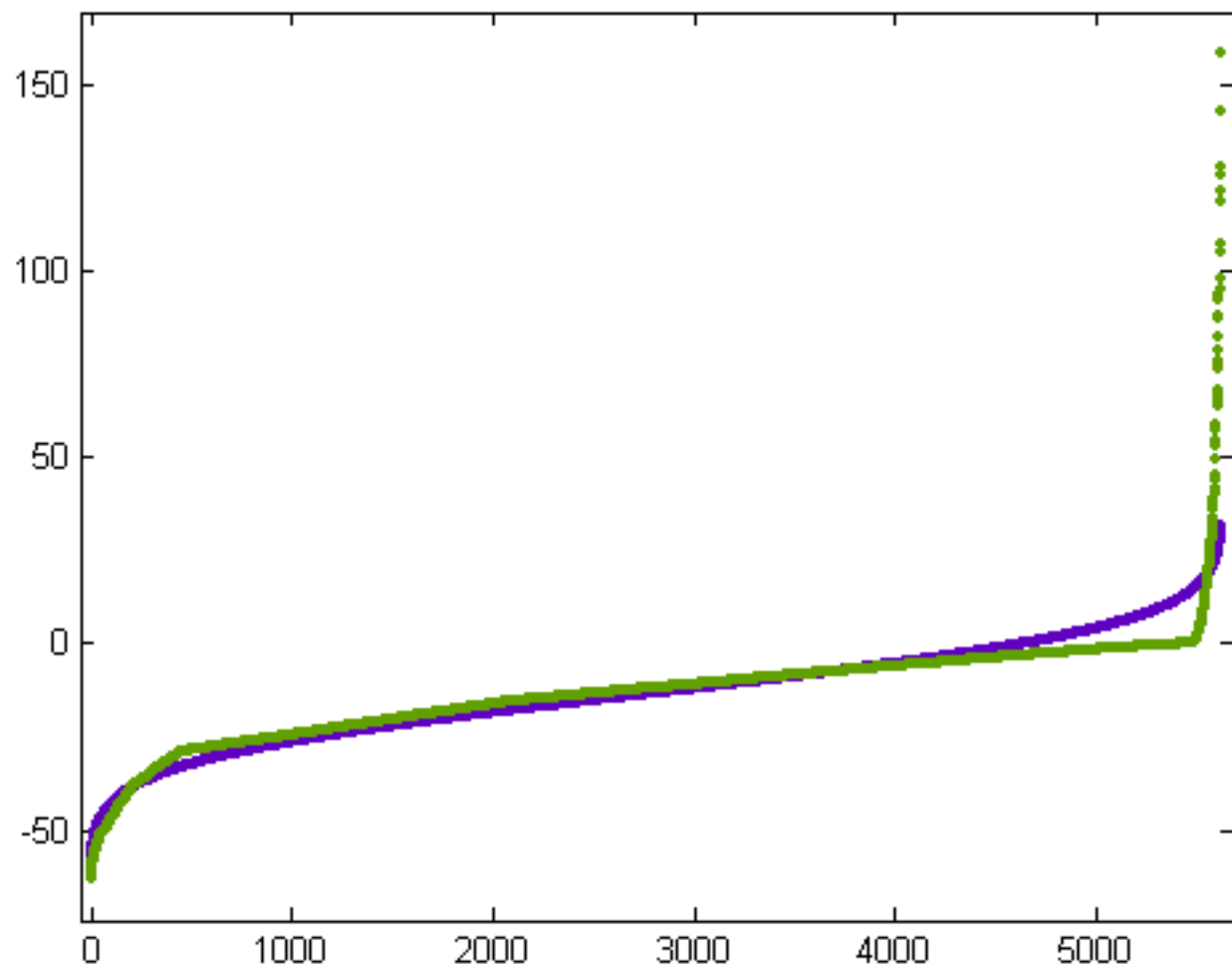


Figure 145

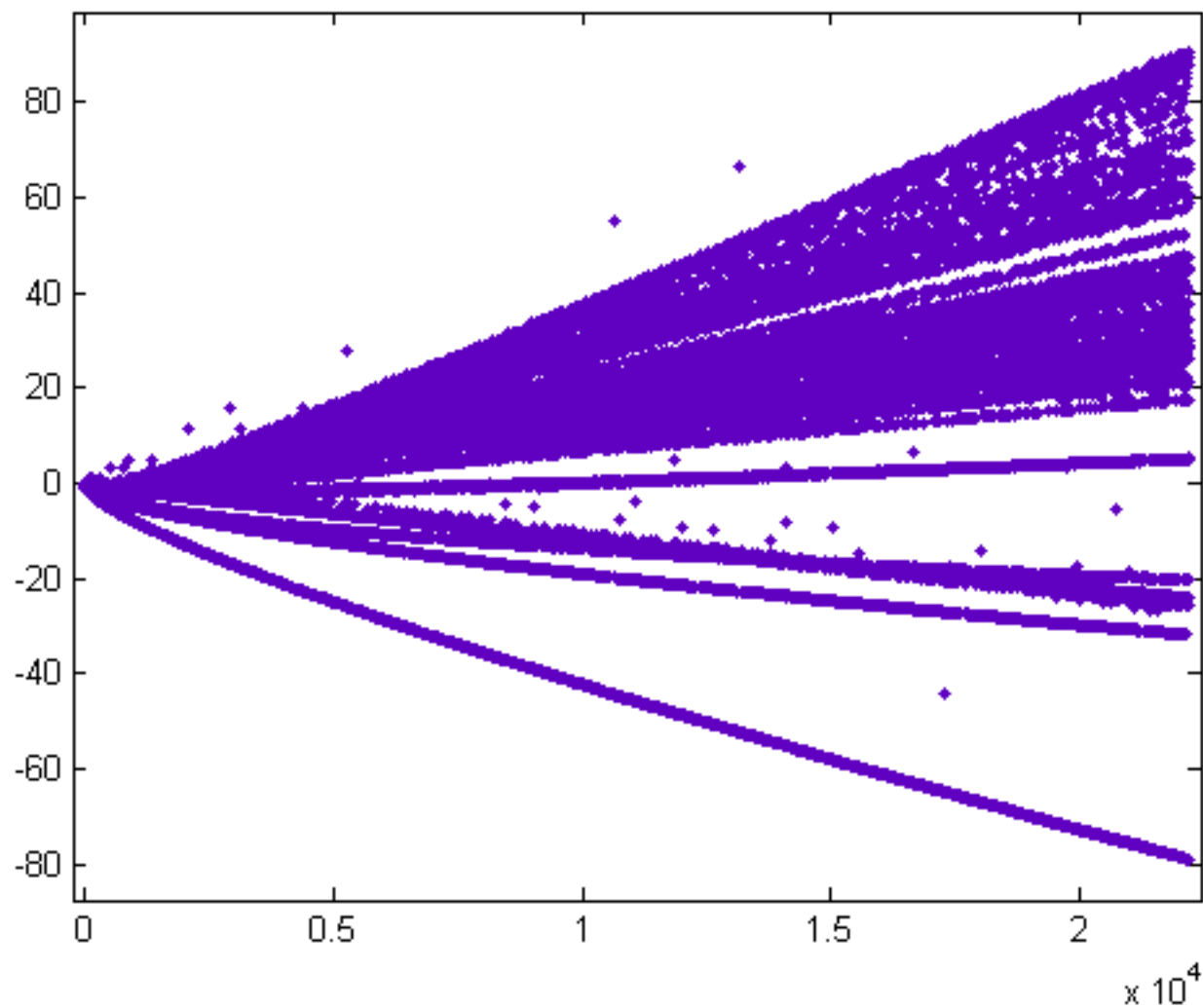


Figure 146

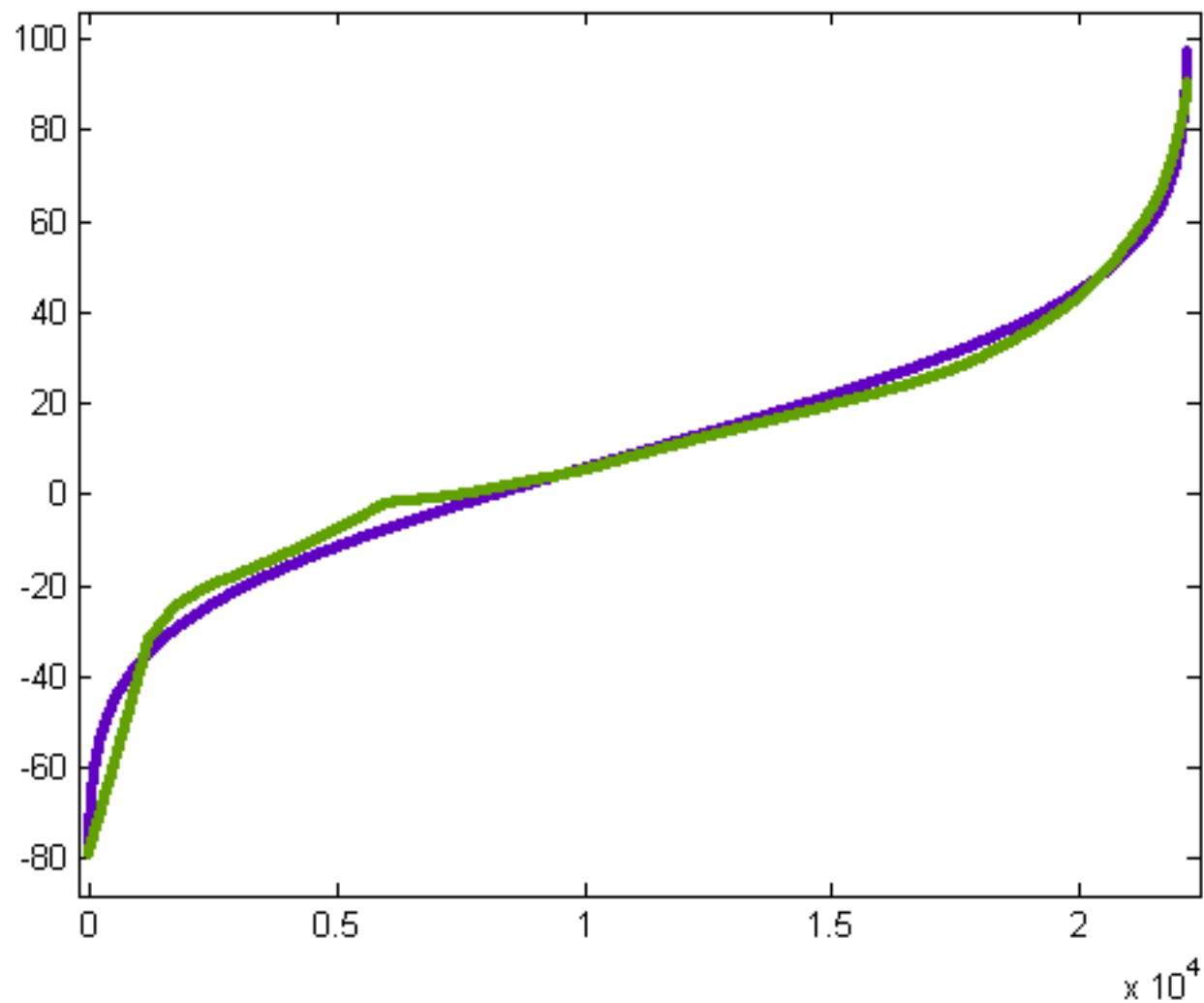


Figure 147

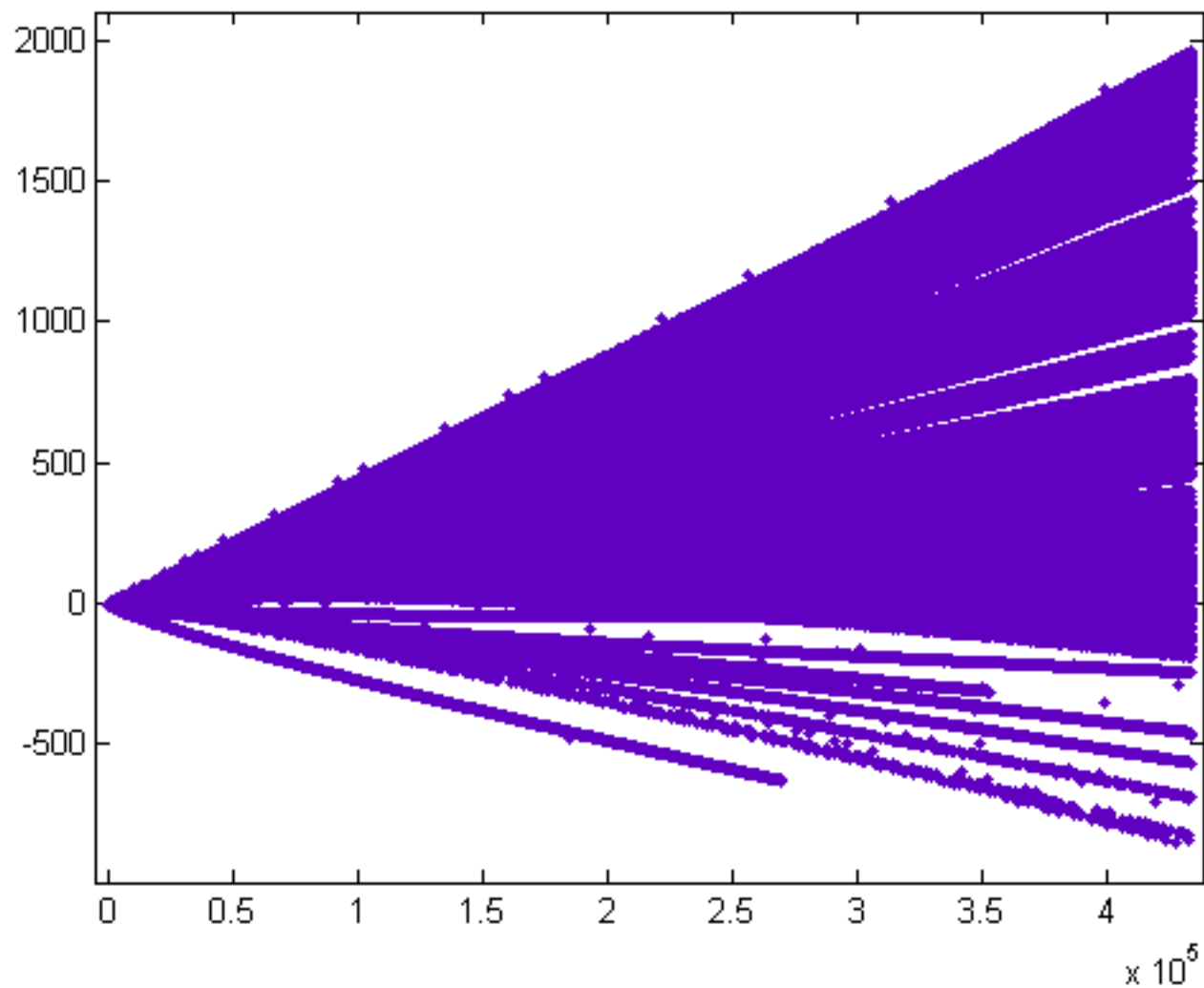


Figure 148

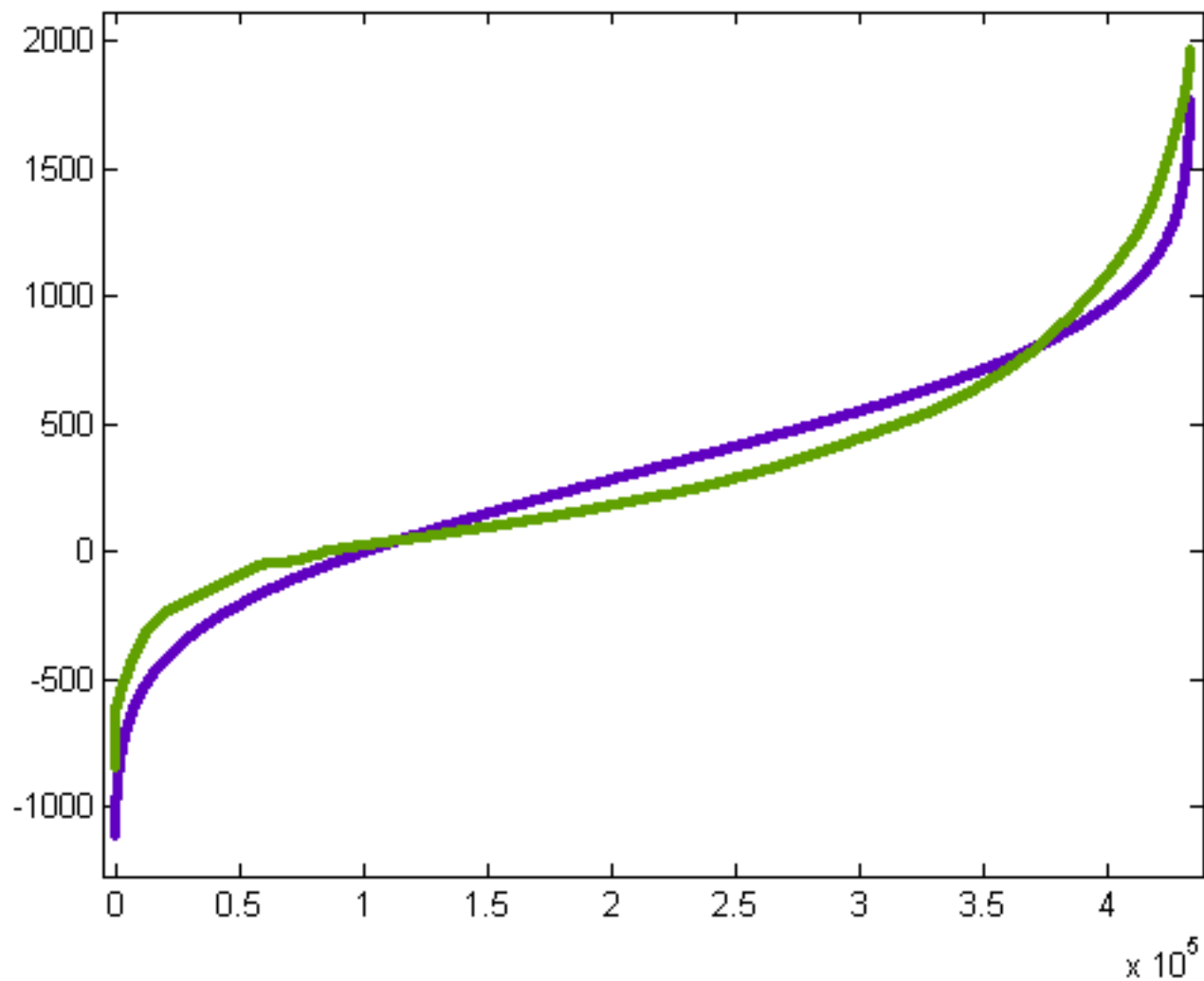


Figure 149

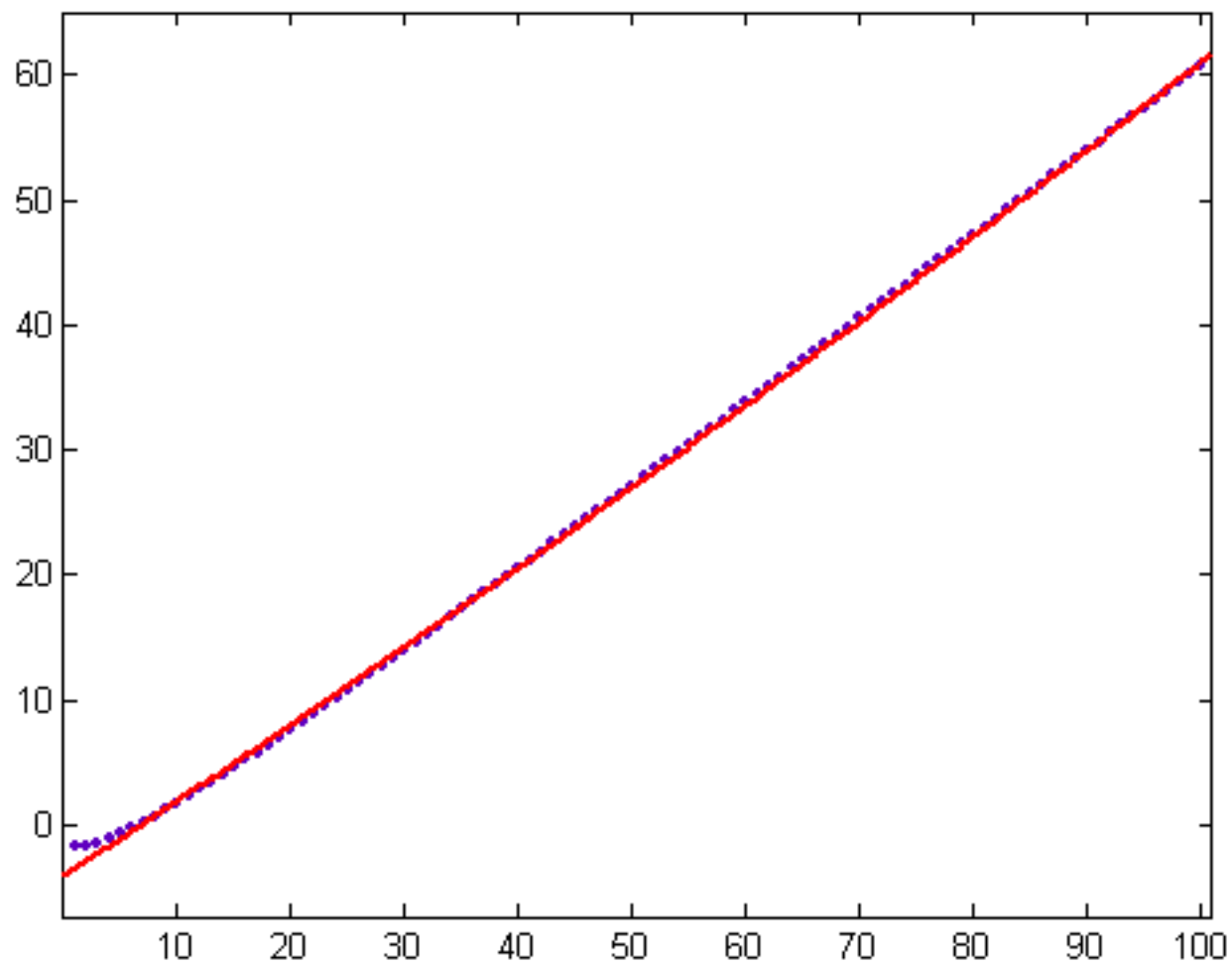


Figure 150

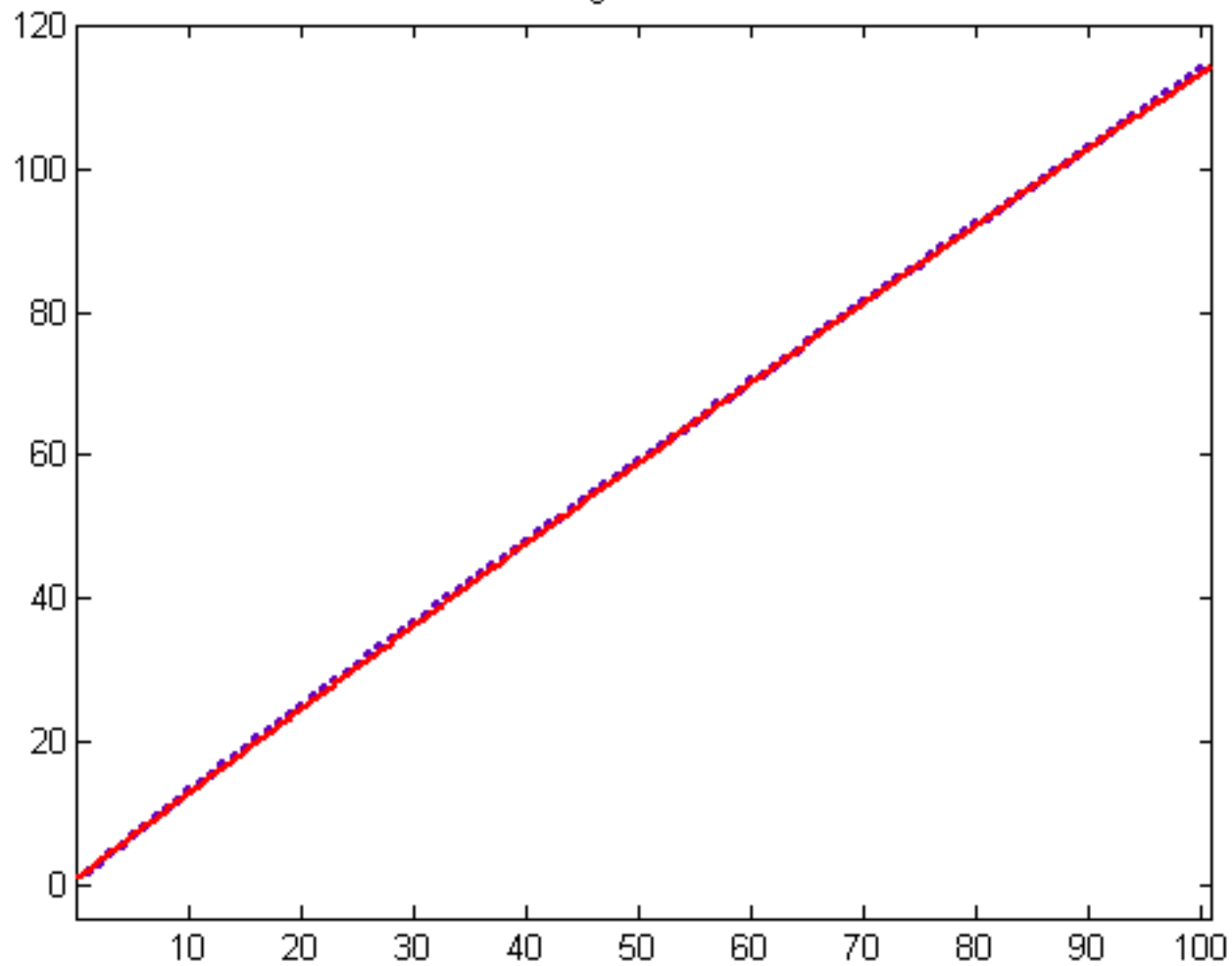


Figure 151

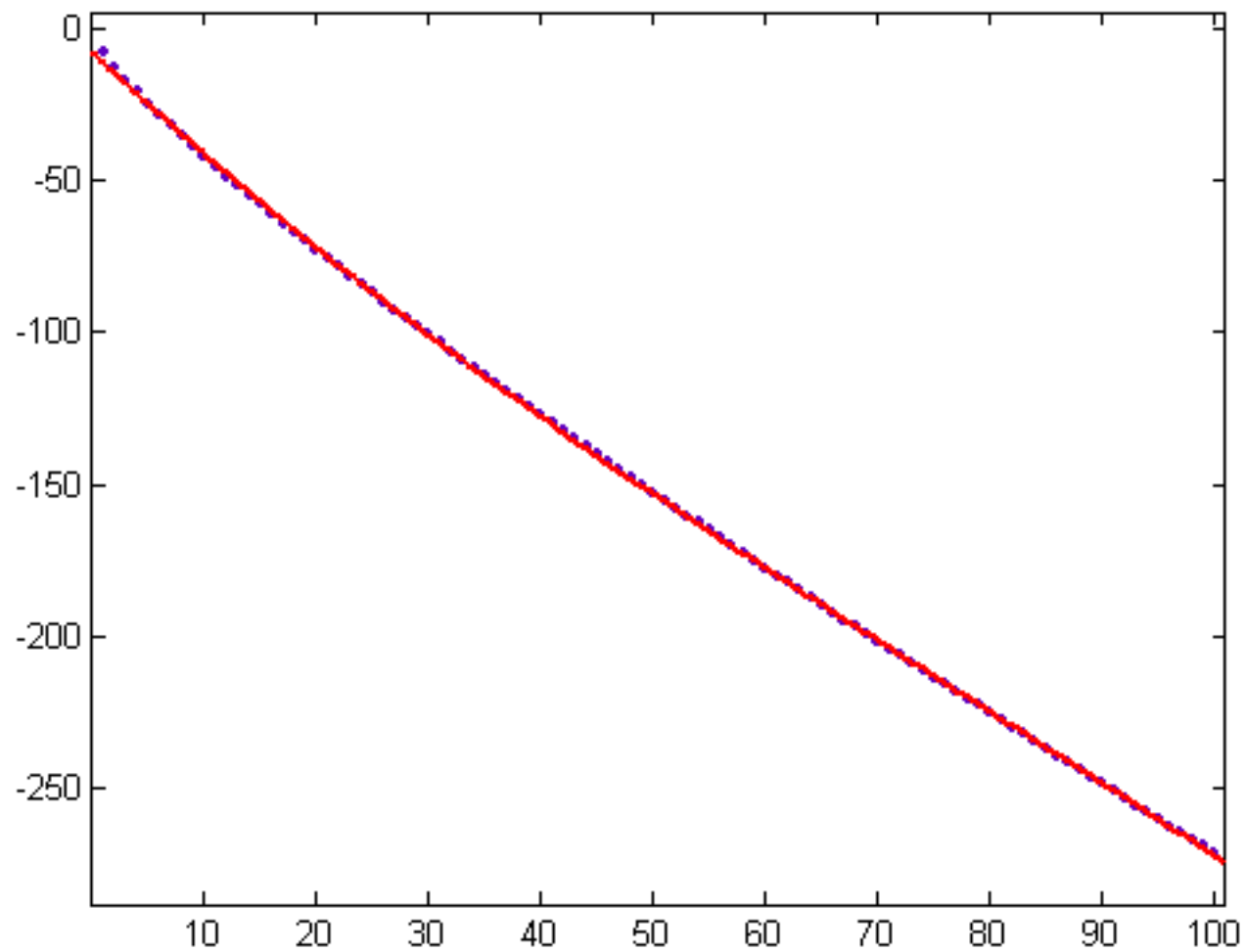


Figure 152

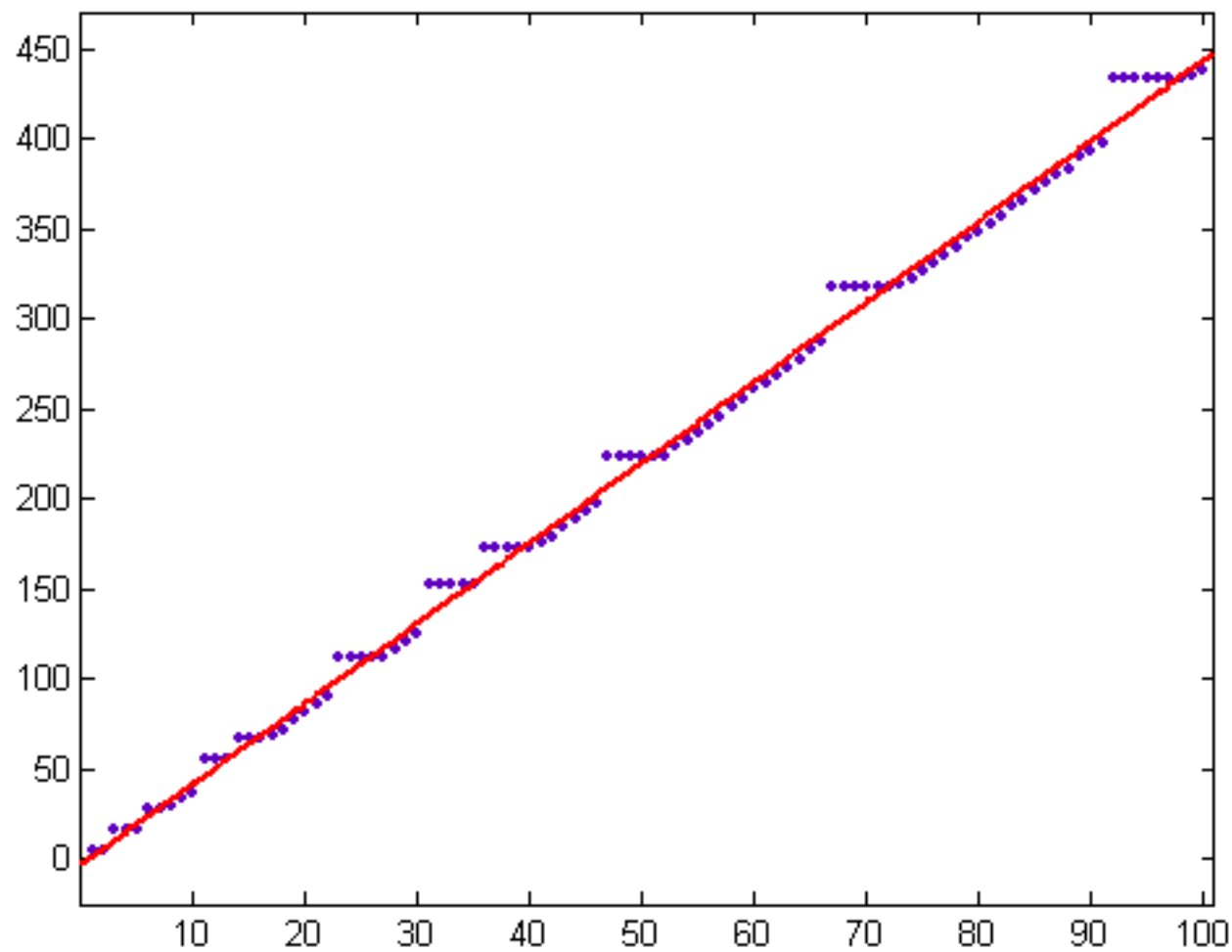


Figure 153

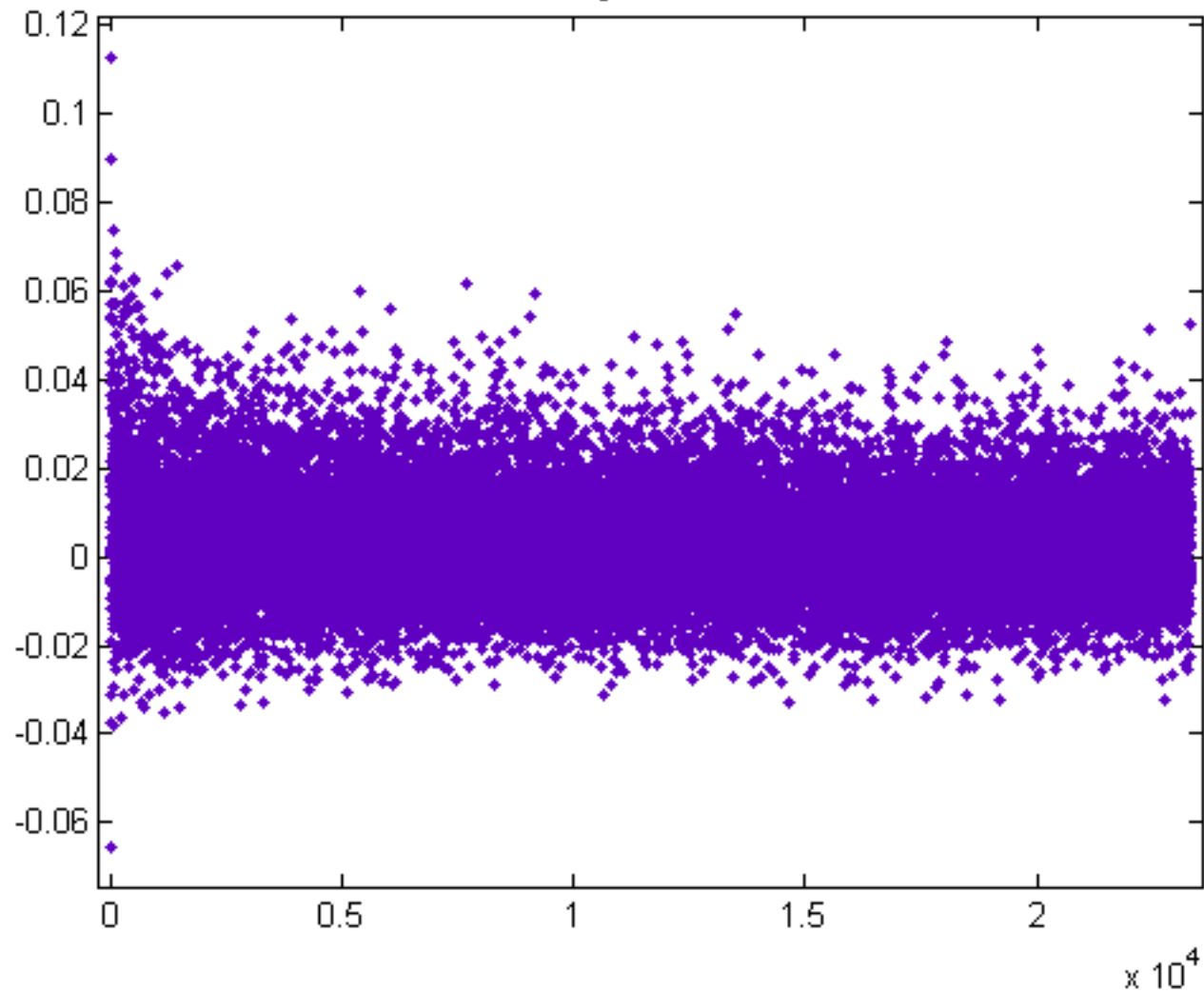


Figure 154

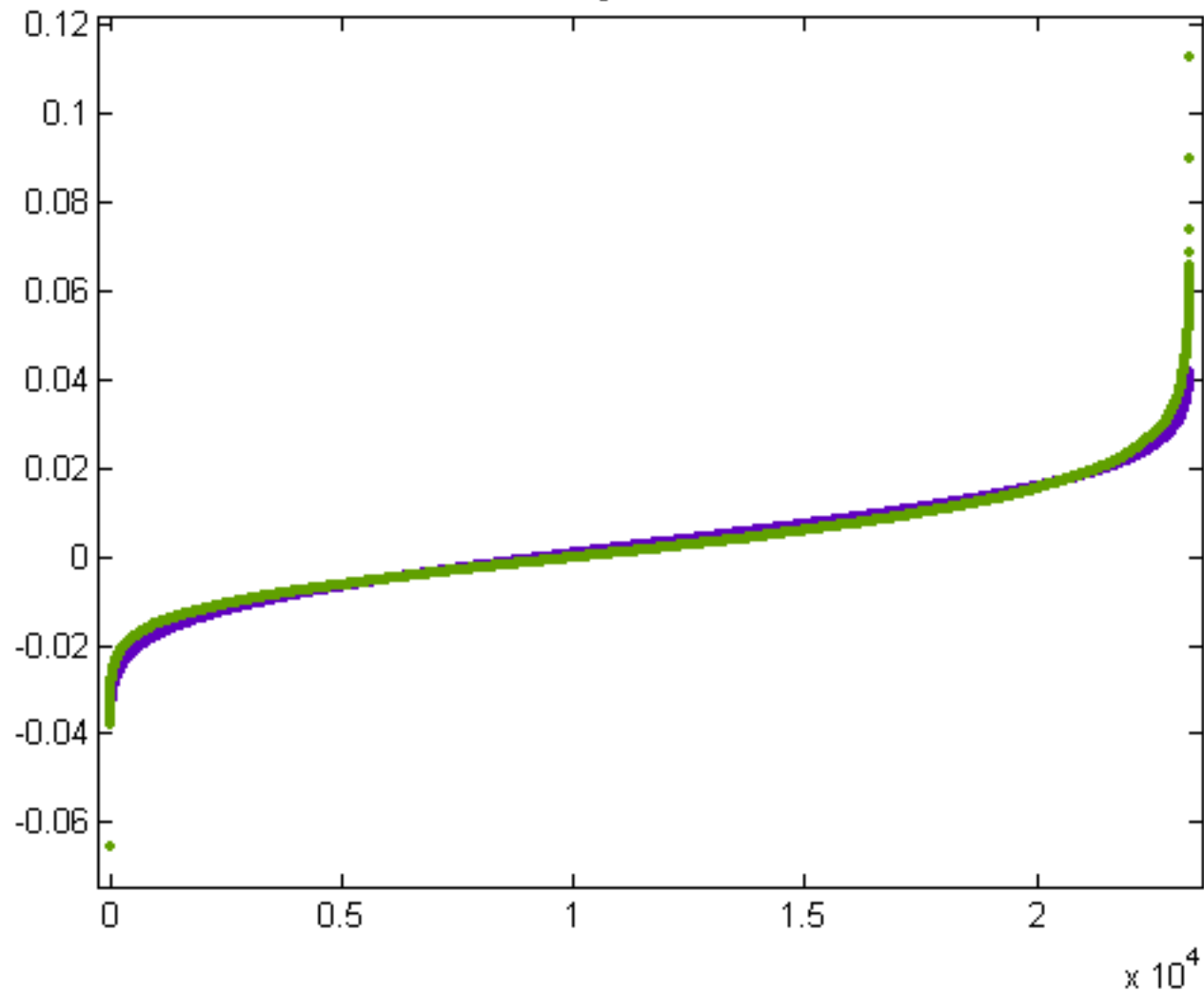


Figure 155

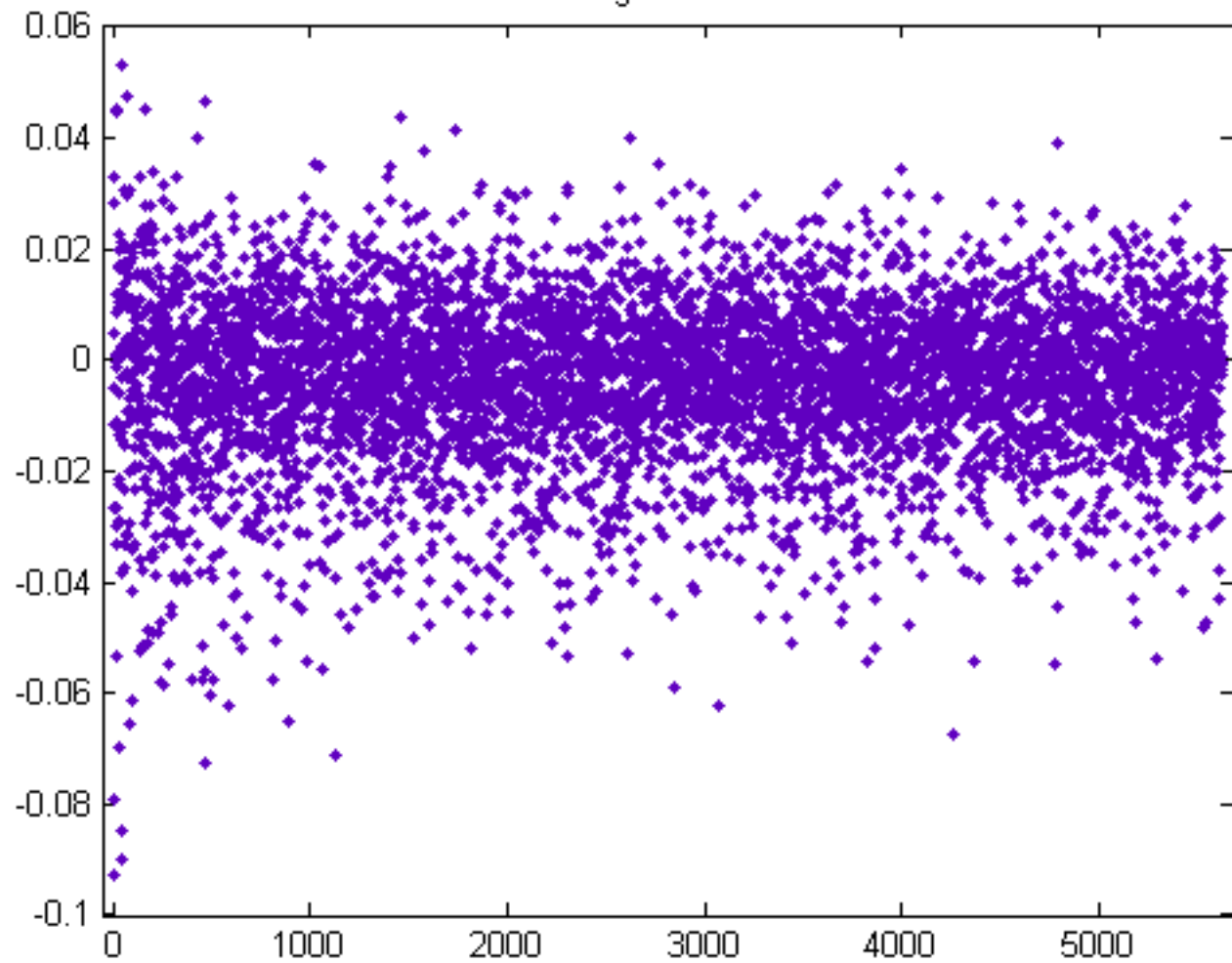


Figure 156

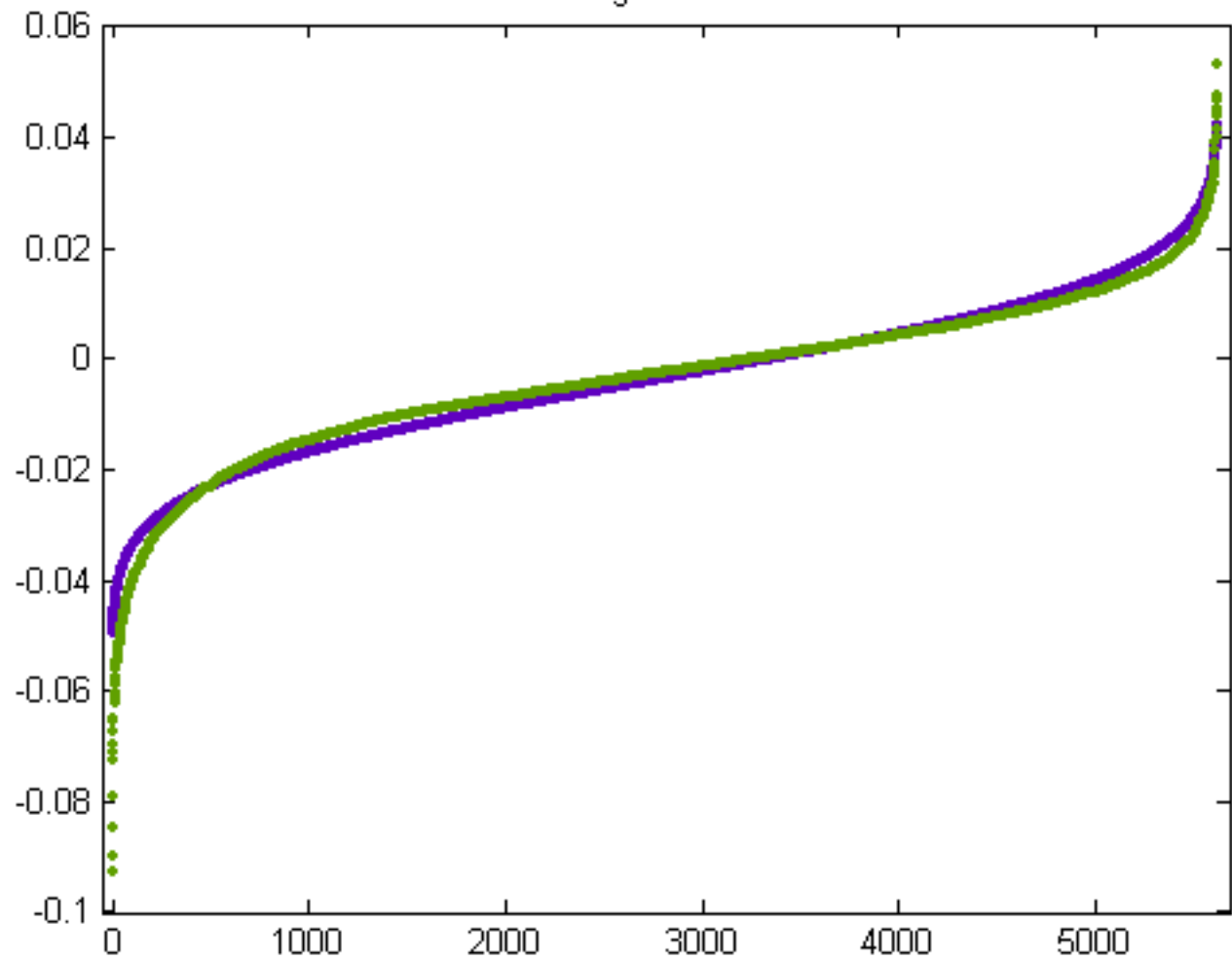


Figure 157

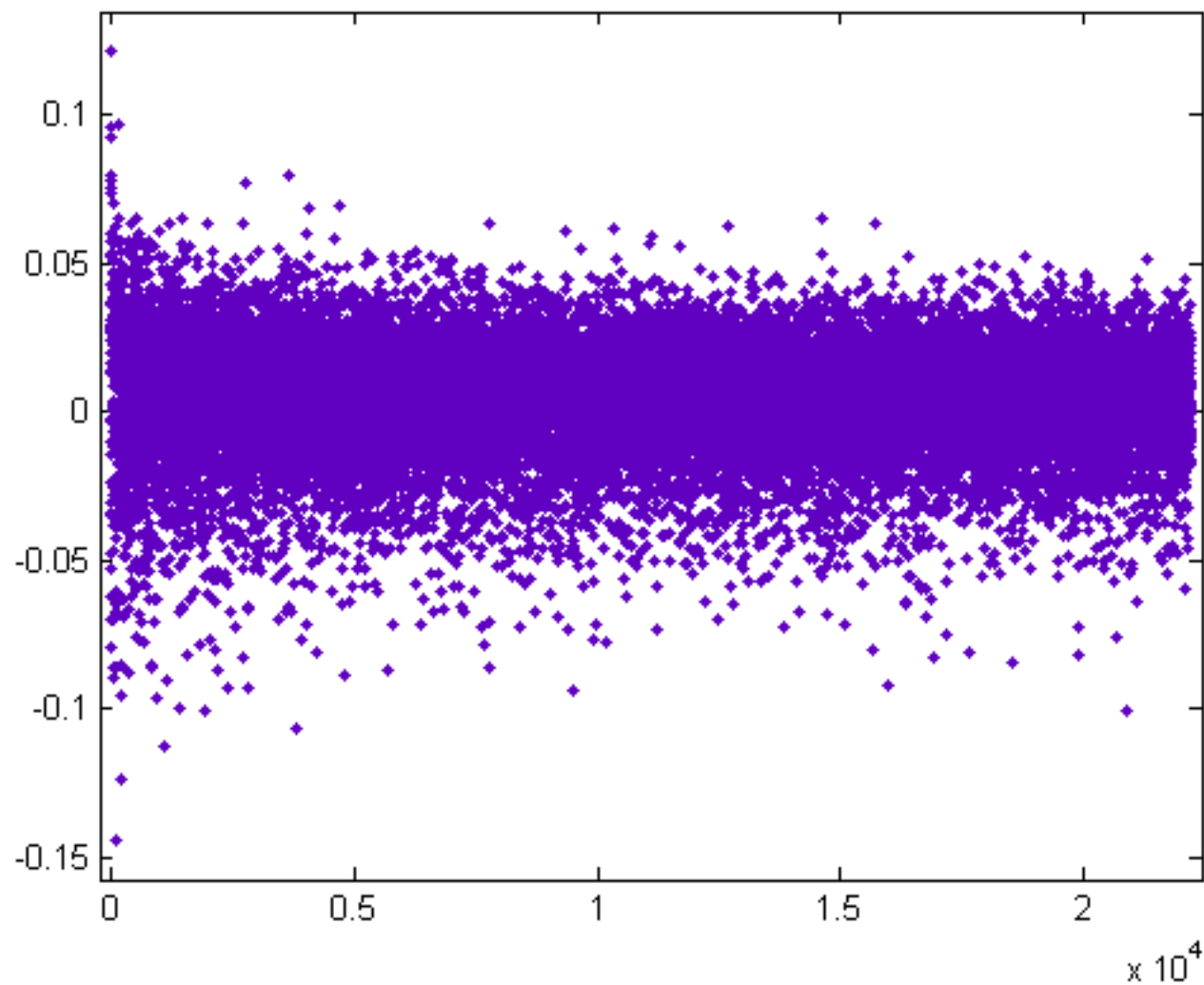


Figure 158

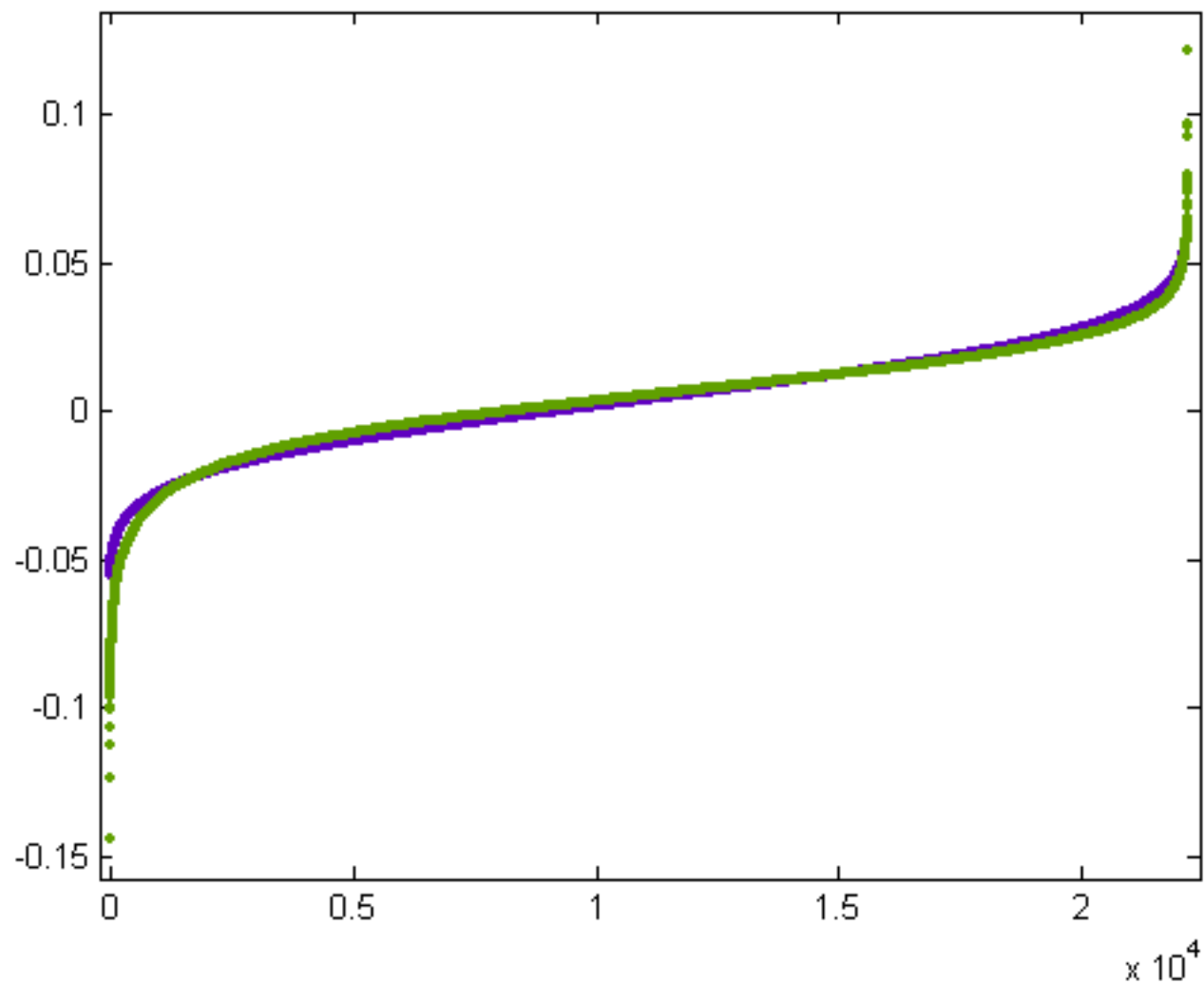


Figure 159

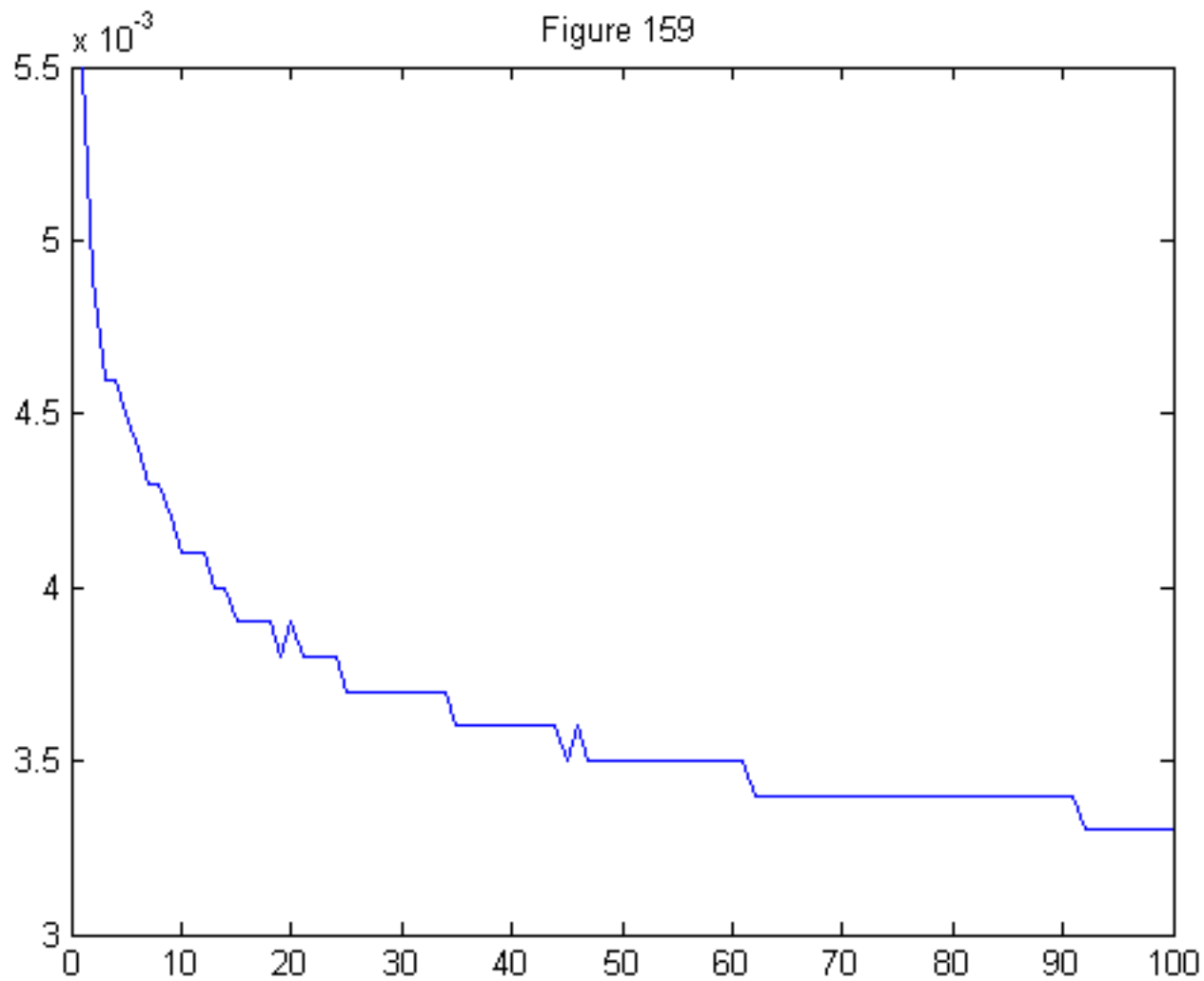


Figure 160

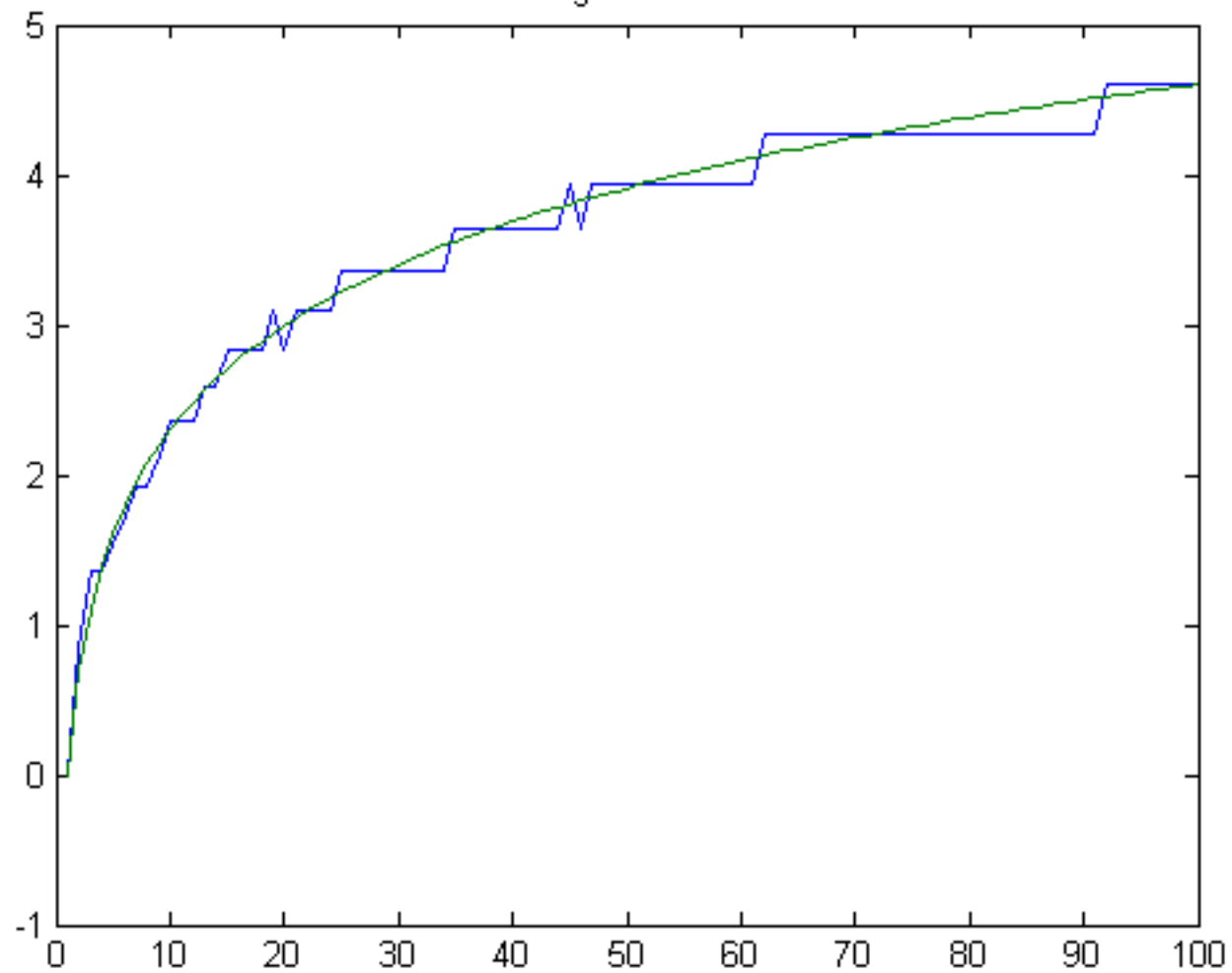


Figure 161

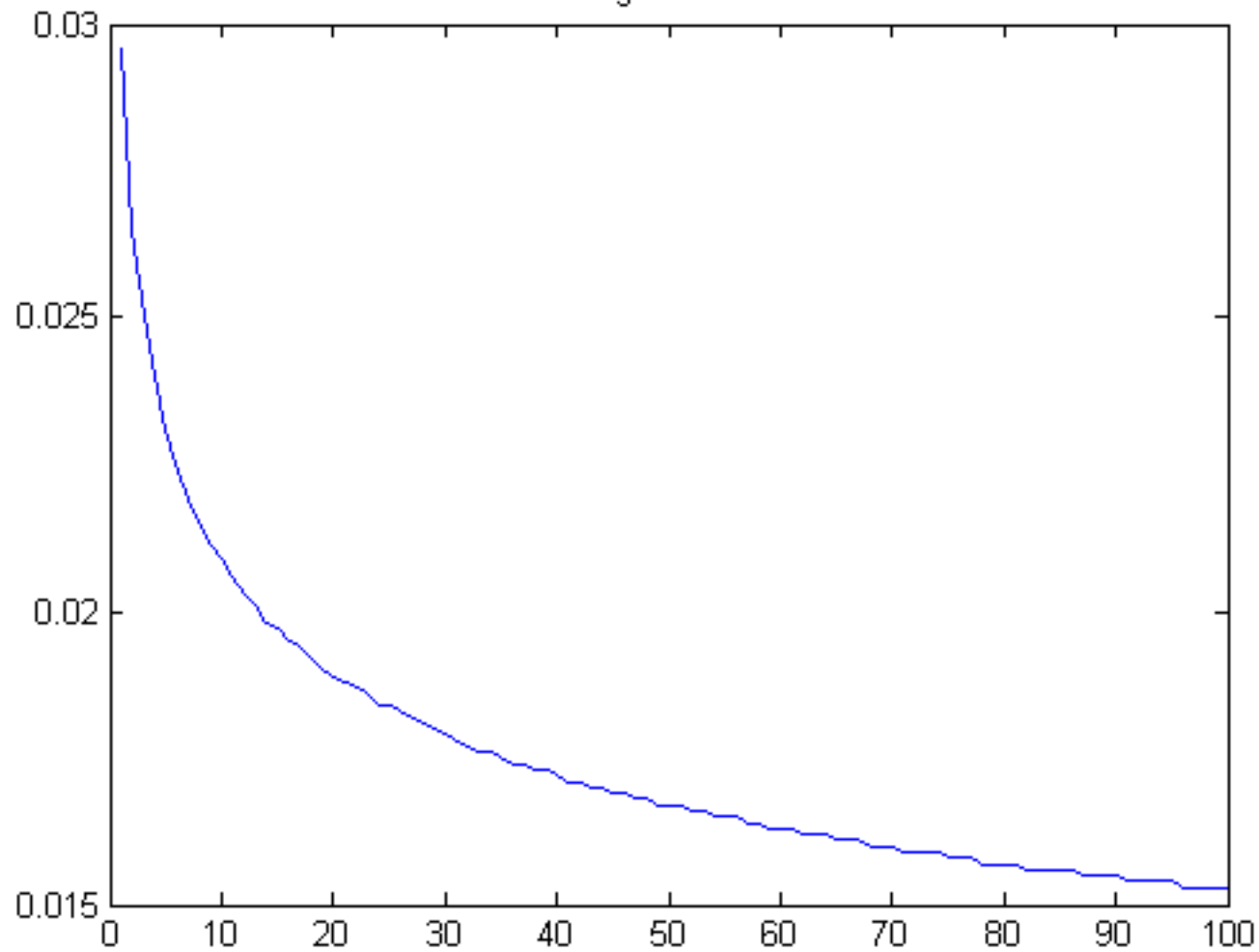


Figure 162

