Zeta Function Zeros and Standard Uniform Distributions

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Abstract

Statistical properties of the non-trivial zeros of the Riemann zeta function are investigated. These properties are compared to those of standard uniform distributions. Statistical properties of Dirichlet products involving perfectly uniform distributions are also investigated. This provides another way to characterize the distribution of the primes.

1 Introduction

Let θ_1, θ_2, θ_3, ... denote the imaginary parts of the nontrivial zeros of the Riemann zeta function. Let $\kappa_1(1)$, $\kappa_1(2)$, $\kappa_1(3)$, ..., denote $\log(\theta_1)$, $\log(\theta_2)$, $\log(\theta_3)$, ... and let $\kappa_n(x)$, $n = 2, 3, 4, ...$, denote these values and $n-1$ values that have been linearly interpolated between successive values. Let $\Lambda(i)$ denote the Mangoldt function. See Figure 1 for a plot of $\sum_{i=1}^{x} \kappa_1(\lfloor x/i \rfloor)$ and $\sum_{i=1}^{x} \kappa_1(\lfloor x/i \rfloor)\Lambda(i)$ (superimposed on each other) for $x = 1, 2, 3, ..., 1000$ (1001 zeta function zeros from Andrew Odlyzko's [1] tables were used). For a linear least-squares fit of $\sum_{i=1}^{x} \kappa_1(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 1000$, $p_1 = 3.09$ with a 95% confidence interval of (3.09, 3.09), $p_2 = -1.832$ with a 95% confidence interval of (-1.868, -1.797), SSE=80.33, R-square=1, and RMSE=0.2837. For a linear least-squares fit of $\sum_{i=1}^{x} \kappa_1(\lfloor x/i \rfloor)\Lambda(i)$ for $x = 1, 2, 3, ..., 1000$, $p_1 = 3.083$ with a 95% confidence interval of (3.081, 3.085), $p_2 = -7.408$ with a 95% confidence interval of (-8.703, -6.113), SSE=1.085e+5, R-square=0.9999, and RMSE=10.42.

Let η_1, η_2, η_3, ..., denote the ordered (increasing) elements of a standard uniform distribution. Let $\gamma_1(1)$, $\gamma_1(2)$, $\gamma_1(3)$, ..., denote $\log(\eta_1)$, $\log(\eta_2)$, $\log(\eta_3)$, ... and let $\gamma_n(x)$, $n = 2, 3, 4, ...$, denote these values and $n-1$ values that have been linearly interpolated between successive values. See Figure 2 for a plot of $\sum_{i=1}^{x} \gamma_1(\lfloor x/i \rfloor)$ and $\sum_{i=1}^{x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i)$ (superimposed on each other) for $x = 1, 2, 3, ..., 1000$ (1001 η values were used). For a linear least-squares fit of $\sum_{i=1}^{x} \gamma_1(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 1000$, $p_1 = -8.261$ with a 95% confidence interval of (-8.262, -8.261), $p_2 = -3.803$ with a 95% confidence interval of (-3.914, -3.691), SSE=803, R-square=1, and RMSE=0.897. For a linear least-squares fit of $\sum_{i=1}^{x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i)$ for $x = 1, 2, 3, ..., 1000$, $p_1 = -8.244$ with a 95% confidence interval of $(-8.255, -8.234)$, $p_2 = -6.998$ with a 95% confidence interval of $(-12.88, -1.115)$, SSE=2.239e+6, R-square=0.9996, and RMSE=47.37.

See Figure 3 for a plot of $(8.244/3.083)(-3.083x + \sum_{i=1}^{x} \kappa_1(\lfloor x/i \rfloor)\Lambda(i))$ and $-8.24x - \sum_{i=1}^{x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i)$ (superimposed on each other) for $x = 1, 2, 3, ..., 100$ (101 θ and η values were used). The peaks and valleys of the two curves occur at the same places and have roughly the same magnitudes. For a linear least-squares fit of $\sum_{i=1}^{x} \kappa_{20}(\lfloor x/i \rfloor)\Lambda(i)$ for $x = 1, 2, 3, ..., 20000$, $p_1 = 2.718$ with a 95% confidence interval of $(2.718, 2.718)$, $p_2 = -4.23$ with a 95% confidence interval of $(-5.902, -2.557)$, SSE=7.277e+7, R-square=1, and RMSE=60.32. For a linear least-squares fit of $\sum_{i=1}^{x} \gamma_{20}(\lfloor x/i \rfloor)\Lambda(i)$ for $x = 1, 2, 3, ..., 20000$, $p_1 = -9.813$ with a 95% confidence interval of $(-9.813, -9.812)$, $p_2 = -19.24$ with a 95% confidence interval of $(-25.73, -12.74)$, SSE=1.097+9, R-square=1, and RMSE=234.2. See Figure 4 for a plot of $(9.813/2.718)(-2.718x+\sum_{i=1}^{x} \kappa_{20}(\lfloor x/i \rfloor)\Lambda(i))$ and $-9.813x-\sum_{i=1}^{x} \gamma_{20}(\lfloor x/i \rfloor)\Lambda(i)$ (superimposed on each other) for $x = 1, 2, 3, ..., 100$ (six θ and η values were used). These curves have the same pattern as those in Figure 3.

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See Figure 5 for a plot of $\sum_{i|x} \kappa_1(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 20000$. The plot consists of about 30 different slowly increasing "curves" with gaps in them. (Note that Möbius inversion can be used to regenerate the κ values from these curves.) Curve #0 (numbering from the bottom) consists of a single element at $x = 1$ and having a value of $\log(\theta_1)$. Curve #1 consists of elements at $x = 2, 3, 5, ...$ (the primes) and having values of $\log(\theta_1) + \log(\theta_x)$. Curve #2 consists of elements at $x = 2^2, 3^2, 5^2, ...$ Curve #3 consists of elements at $x = 2^3, 3^3, 5^3, ...$ and x values that are the product of two distinct primes. Curve #4 consists of elements at $x = 2^4, 3^4, 5^4, ...$ Curve #5 consists of elements at $x = 2^5, 3^5, 5^5, ...$ and x values that are the product of the square of a prime and a different prime. In general, Curve #k consists of at least elements at $x = 2^k, 3^k, 5^k, ...$ Let $y_1, y_2, y_3, ..., y_{9592}$ equal the values of the curve at $x = 2, 3, 5, ..., 99991$ (100000 zeta function zeros were used). This curve has an initial value of 5.694206. See Figure 6 for a plot of $(y_z - 5.694206 + 0.944206) - 0.995\log(z)$ for $z = 1, 2, 3, ..., 9592$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $0.995\log(z)$. Let $y_1, y_2, y_3, ..., y_{65}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 313^2$ (having an initial value of 9.109467). See Figure 7 for a plot of $(y_z - 9.109467 + 2.0) - 3.132\log(z)$ for $z = 1, 2, 3, ..., 65$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $3.132\log(z)$. Let $y_1, y_2, y_3, ..., y_{14}$ equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 43^3$ (having an initial value of 12.87824). $y_z - 12.87824 + 4.0$ is increasing at almost the same rate as $6.60\log(z)$

for $z = 1, 2, 3, ..., 14$. Let y_1, y_2, y_3, ..., y_7 equal the values of the curve at $x = 2^4, 3^4, 5^4, ..., 17^4$ (having an initial value of 17.08413). $y_z - 17.08413 + 8.0$ is increasing at almost the same rate as $12.19 \log(z)$ for $z = 1, 2, 3, ..., 7$. See Figure 8 for a plot of the logarithm coefficients (0.995, 3.132, 6.60, and 12.19). For a quadratic least-squares fit of these coefficients, $p_1 = 0.8633$ with a 95% confidence interval of $(-0.2604, 1.987)$, $p_2 = -0.611$ with a 95% confidence interval of $(-6.319, 5.097)$, $p_3 = 0.7823$ with a 95% confidence interval of $(-5.474, 7.039)$, SSE=0.03128, R-square=0.9996, and RMSE=0.1769.

See Figure 9 for a plot of $\sum_{i|x} \kappa_1(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 5000$. Numbering from the bottom, the first four major curves in Figure 9 occur at x where x equals 2 times the primes other than 2, where x equals 3 times the primes other than 3, where x equals 2^2 times the primes other than 2, and where x equals 5 times the primes other than 5. Two other curves that intersect these curves occur at x where x equals the square of a prime and where x equals the cube of a prime. Let y_1, y_2, y_3, ..., y_{65} equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 313^2$ (100000 zeta function zeros were used). See Figure 10 for a plot of y_z and $1.213242 \log(z)^2$ (superimposed on each other) for $z = 1, 2, 3, ..., 65$ (the absolute values of the differences between the two curves are less than 0.31). Let y_1, y_2, y_3, ..., y_{14} equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 43^3$. See Figure 11 for a plot of y_z for $z = 1, 2, 3, ..., 14$. For a quadratic least-squares fit of the curve, SSE=4.576, R-square=0.9959, and RMSE=0.645.

Let y_1, y_2, y_3, ..., y_{65} equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 313^2$. Let w_2, w_3, w_4, ..., w_{5133} equal the values of the curve at $x = 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, ..., 2 \cdot 49999$ and let $w_1 = 2y_1$. $w_{5133} = 9.81135$ and $9.81135/\log(5133) = 1.1484$. See Figure 12 for a plot of $w_z - 1.1484 \log(z)$ for $z = 1, 2, 3, ..., 5133$. For $z > 1$, the absolute values of the elements of this curve are less than 0.0083. Let w_1, w_3, w_4, ..., w_{3569} equal the values of the curve at $x = 3 \cdot 2, 3 \cdot 5, 3 \cdot 7, ..., 3 \cdot 33331$ and let $w_2 = 2y_2$. $w_{3569} = 14.29311$ and $14.29311/\log(3569) = 1.7473$. See Figure 13 for a plot of $w_z - 1.7473 \log(z)$ for $z = 1, 2, 3, ..., 3569$. For $z > 1$, the absolute values of the elements of this curve are less than 0.15. Let w_1, w_2, w_4, ..., w_{2262} equal the values of the curve at $x = 5 \cdot 2, 5 \cdot 3, 5 \cdot 7, ..., 5 \cdot 19997$ and let $w_3 = 2y_3$. $w_{2262} = 19.88788$ and $19.88788/\log(2262) = 2.5748$. See Figure 14 for a plot of $w_z - 2.5748 \log(z)$ for $z = 1, 2, 3, ..., 2262$. For $z > 1$, the absolute values of the elements of this curve are less than 0.22. In general, this process can be continued for curves corresponding to prime multiples of primes. For example, let w_1, w_2, w_3, ..., w_{64}, w_{66} equal the values of the curve at $x = 313 \cdot 2, 313 \cdot 3, 313 \cdot 5, ..., 313 \cdot 311, 313 \cdot 317$ and let $w_{65} = 2y_{65}$. $w_{66} = 42.40156$ and $42.40156/\log(66) = 10.1205$. See Figure 15 for a plot of $w_z - 10.1205 \log(z)$ for $z = 1, 2, 3, ..., 66$. For $z > 1$, the absolute values of the elements of this curve are less than 0.77. Let c_1, c_2, c_3, ..., c_{26} denote the coefficients of the above logarithms (1.1484, 1.7473, 2.5748, ..., 7.9536) corresponding to the curves for 2 times the primes, 3 times the primes, 5 times the primes, ..., 101 times the primes. See Figure 16 for a plot of $2.4328 \log(z)$ and c_z (superimposed on each other) for $z = 1, 2, 3, ..., 26$. (For a quadratic least-squares fit of the coefficients, SSE=1.275, R-square=0.9864, and RMSE=0.2355.)

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See Figure 17 for a plot of $\sum_{i|x} \kappa_{200}(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 200$ (the κ values are perfectly distributed between $\log(\theta_1)$ and $\log(\theta_2)$). Except for being more linear, the different curves are the same as for $\sum_{i|x} \kappa_1(\lfloor x/i \rfloor)$. Let $y_1, y_2, y_3, ..., y_{78498}$ equal the values of $\sum_{i|x} \kappa_{1000000}(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 1000000$ at $x = 2, 3, 5, ..., 999983$ (having an initial value of 5.297269). See Figure 18 for a plot of $y_z - 5.297269$ and $z \log(z)/2230000$ (superimposed on each other) for $z = 1, 2, 3, ..., 78498$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $z \log(z)/2230000$ (the absolute values of the differences between the two curves are less than $8 \cdot 10^{-4}$). Let $y_1, y_2, y_3, ..., y_{160}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 997^2$ (having an initial value of 7.945905). See Figure 19 for a plot of $y_z - 7.945905$ and $z^2 \log(z)^2/1872050$ (superimposed on each other) for $z = 1, 2, 3, ..., 160$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $z^2 \log(z)^2/1872050$ (the absolute values of the differences between the two curves are less than 0.02). Let $y_1, y_2, y_3, ..., y_{25}$ equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 97^3$ (having an initial value of 10.59454). See Figure 20 for a plot of $y_z - 10.59454$ and $z^3 \log(z)^3/1480000$ (superimposed on each other) for $z = 1, 2, 3, ..., 25$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $z^3 \log(z)^3/1480000$. Let $y_1, y_2, y_3, ..., y_{11}$ equal the values of the curve at $x = 2^4, 3^4, 5^4, ..., 31^4$ (having an initial value of 13.24318). See Figure 21 for a plot of $y_z - 13.24318$ and $z^4 \log(z)^4/1000000$ for $z = 1, 2, 3, ..., 11$. The plot shows that the normalized curve of y values is increasing at roughly the same rate as $z^4 \log(z)^4/1000000$. See Figure 22 for a plot of the coefficients (1/2230000, 1/1872050, 1/1480000, and 1/1000000). For a quadratic least-squares fit of these coefficients, SSE=8.073e−16, R-square=0.9954, and RMSE=2.841e−8.

See Figure 23 for a plot of $\sum_{i|x} \kappa_{1000}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 1000$. Let $y_1, y_2, y_3, ..., y_{65}$ equal the values of $\sum_{i|x} \kappa_{100000}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 100000$ at $x = 2^2, 3^2, 5^2, ..., 313^2$. See Figure 24 for a plot of y_z, $z = 1, 2, 3, ..., 65$. For a quadratic least-squares fit of these quantities, SSE=4.948e−7, R-square=0.9984, and RMSE=8.933e−5. Let $y_1, y_2, y_3, ..., y_{14}$ equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 47^3$. See Figure 25 for a plot of y_z, $z = 1, 2, 3, ..., 14$. For a cubic least-squares fit of these quantities, SSE=9.471e−6, R-square=0.9925, and RMSE=0.0009732.

Let $y_1, y_2, y_3, ..., y_{65}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 313^2$. Let $w_2, w_3, w_4, ..., w_{5133}$ equal the values of $\sum_{i|x} \kappa_{100000}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$

for $x = 1, 2, 3, ..., 100000$ at $x = 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, ..., 2 \cdot 49999$ and let $w_1 = 2y_1$. $w_{5133} = 0.1376052$ and $0.1376052/5133/\log(5133) = 3.13783\text{e}{-6}$. See Figure 26 for a plot of $0.00000313783 z \log(z)$ and w_z (superimposed on each other) for $z = 1, 2, 3, ..., 5133$ (the absolute values of the differences between the two curves are less than 5e−4). Let $w_1, w_3, w_4, ..., w_{3569}$ equal the values of the curve at $x = 3 \cdot 2, 3 \cdot 5, 3 \cdot 7, ..., 3 \cdot 33331$ and let $w_2 = 2y_2$. $w_{3569} = 0.1454279$ and $0.1454279/3569/\log(3569) = 4.98133\text{e}{-6}$. See Figure 27 for a plot of $0.00000498133 z \log(z)$ and w_z (superimposed on each other) for $z = 1, 2, 3, ..., 3569$ (the absolute values of the differences between the two curves are less than 6e−4). Let $w_1, w_2, w_4, ..., w_{2262}$ equal the values of the curve at $x = 5 \cdot 2, 5 \cdot 3, 5 \cdot 7, ..., 5 \cdot 19997$ and let $w_3 = 2y_3$. $w_{2262} = 0.1279006$ and $0.1279006/2262/\log(2262) = 7.32044\text{e}{-6}$. See Figure 28 for a plot of $w_z - 0.00000732044 z \log(z)$ for $z = 1, 2, 3, ..., 2262$ (the absolute values of the elements of this curve are less than 6e−4). Let $w_1, w_2, w_3, w_5, ..., w_{1676}$ equal the values of the curve at $x = 7 \cdot 2, 7 \cdot 3, 7 \cdot 5, 7 \cdot 11 ..., 7 \cdot 14281$ and let $w_4 = 2y_4$. $w_{1676} = 0.1105269$ and $0.1105269/1676/\log(1676) = 8.88272\text{e}{-6}$. See Figure 29 for a plot of $w_z - 0.00000888272 z \log(z)$ for $z = 1, 2, 3, ..., 1676$ (the absolute values of the elements of this curve are less than 6e−4). The corresponding coefficients of $z \log(z)$ for 11 times the primes, 13 times the primes, 17 times the primes, and 19 times the primes are 1.094132e−5, 1.171728e−5, 1.293219e−5, and 1.358418e−5 respectively. Let $c_1, c_2, c_3, ..., c_8$ denote the coefficients of $z \log(z)$ corresponding to the curves for 2 times the primes, 3 times the primes, 5 times the primes, ..., 19 times the primes. See Figure 30 for a plot of c_z for $z = 1, 2, 3, ..., 8$. For a quadratic least-squares fit of these values, SSE=2.877e−13, R-square=0.9971, and RMSE=2.399e−7.

Let $y_1, y_2, y_3, ..., y_{168}$ equal the values of $\sum_{i|x} \kappa_{1000000}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 1000000$ at $x = 2^2, 3^2, 5^2, ..., 997^2$. Let $w_2, w_3, w_4, ..., w_{41538}$ equal the values of $\sum_{i|x} \kappa_{1000000}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 1000000$ at $x = 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, ..., 2 \cdot 499979$ and let $w_1 = 2y_1$. $w_{41538} = 0.1375669$ and $0.1375669/41538/\log(41538) = 3.1142\text{e}{-7}$. See Figure 31 for a plot of $w_z - 0.00000031142 z \log(z)$ for $z = 1, 2, 3, ..., 41538$ (the absolute values of the elements of this curve are less than 3e−4). Let $w_1, w_3, w_4, ..., w_{28665}$ equal the values of the curve at $x = 3 \cdot 2, 3 \cdot 5, 3 \cdot 7, ..., 3 \cdot 333331$ and let $w_2 = 2y_2$. $w_{28665} = 0.1453684$ and $0.1453684/28665/\log(28665) = 4.9411\text{e}{-7}$. See Figure 32 for a plot of $w_z - 0.00000049411 z \log(z)$ for $z = 1, 2, 3, ..., 28665$ (the absolute values of the elements of this curve are less than 3e−4). Let $w_1, w_2, w_4, ..., w_{17984}$ equal the values of the curve at $x = 5 \cdot 2, 5 \cdot 3, 5 \cdot 7, ..., 5 \cdot 199999$ and let $w_3 = 2y_3$. $w_{17984} = 0.127787$ and $0.127787/17984/\log(17984) = 7.2526\text{e}{-7}$. See Figure 33 for a plot of $w_z - 0.00000072526 z \log(z)$ for $z = 1, 2, 3, ..., 17984$ (the absolute values of the elements of this curve are less than 3e−4). Note that the coefficients of $z \log(z)$ are about a tenth of the above coefficients and the approximation errors are less.

Let y_1 equal the value of $\sum_{i|x} \kappa_8(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3,$

..., 8 at $x = 2^2$. Let w_2 equal the value of $\sum_{i|x} \kappa_8(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 8$ at $x = 2 \cdot 3$ and let $w_1 = 2y_1$. $w_2 = 0.1232939$ and $0.1232939/2/\log(2) = 0.08893774905$. The absolute values of $w_z - 0.08893774905 z\log(z)$ for $z = 1$ and 2 are less than 0.07. Let y_1 equal the value of $\sum_{i|x} \kappa_{16}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 16$ at $x = 2^2$. Let w_2, w_3, and w_4 equal the values of $\sum_{i|x} \kappa_{16}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 16$ at $x = 2 \cdot 3$, $2 \cdot 5$, and $2 \cdot 7$ and let $w_1 = 2y_1$. $w_4 = 0.1514511$ and $0.1514511/4/\log(4) = 0.02731221886$. The absolute values of $w_z - 0.02731221886z\log(z)$ for $z = 1$, 2, 3 and 4 are less than 0.035. Let y_1 equal the value of $\sum_{i|x} \kappa_{32}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 32$ at $x = 2^2$. Let w_2, w_3, w_4, w_5, and w_6 equal the values of $\sum_{i|x} \kappa_{32}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 32$ at $x = 2 \cdot 3$, $2 \cdot 5$, $2 \cdot 7$, $2 \cdot 11$, and $2 \cdot 13$ and let $w_1 = 2y_1$. $w_6 = 0.1349922$ and $0.1349922/6/\log(6) = 0.01255676355$. The absolute values of $w_z - 0.01255676355z\log(z)$ for $z = 1, 2, 3, 4, 5$, and 6 are less than 0.018. Continuing in this manner up to $\sum_{i|x} \kappa_{65536}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 65536$ gives coefficients of 0.00569696197, 0.00272394602, 0.00134264501, 0.00064356496, 0.00031303628, 0.00015628883, 0.00007768926, 0.00003857753, 0.00001932131, 0.00000959629, and 0.00000479728 (each successive coefficient is less than half of the previous coefficient). The corresponding least upper bounds of the approximation errors are 0.013, 0.008, 0.010, 0.004, 0.004, 0.0025, 0.0025, 0.0016, 0.0007, 0.0010, and 0.0005. The approximation errors usually decrease. The anomaly for $\sum_{i|x} \kappa_{256}(\lfloor x/i \rfloor)\Lambda(i) - \log(\theta_1)\log(x)$ for $x = 1, 2, 3, ..., 256$ is due to the large gap between the successive primes 113 and 127.

4 Dirichlet Products Involving γ

See Figure 34 for a plot of $\sum_{i|x} \gamma_1(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 1000$. The different curves are the same as for $\sum_{i|x} \kappa_1(\lfloor x/i \rfloor)$. For example, Curve #3 (numbering from the top) consists of elements at $x = 2^3, 3^3, 5^3, ...$ and x values that are the product of two distinct primes. See Figure 35 for a plot of these values. Let $y_1, y_2, y_3, ..., y_{78498}$ equal the values of the curve at $x = 2, 3, 5, ..., 999983$ (for a standard uniform distribution having 1000001 elements). The initial value is -25.68024. See Figure 36 for a plot of $(y_z + 25.68024 - 0.5) - 1.0819701\log(z)$ for $z = 1, 2, 3, ..., 78498$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $1.0819701\log(z)$. Let $y_1, y_2, y_3, ..., y_{168}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 997^2$ (having an initial value of -37.71227). See Figure 37 for a plot of $(y_z + 37.71227 + 1.0) - 3.6763725\log(z)$ for $z = 1, 2, 3, ..., 168$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $3.6763725\log(z)$. Let $y_1, y_2, y_3, ..., y_{25}$ equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 97^3$ (having an initial value of -48.99217). See Figure 38 for a plot of $(y_z + 48.99217 + 3.5) - 7.9434565\log(z)$ for $z = 1, 2, 3, ..., 25$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $7.9434565\log(z)$. Let $y_1, y_2, y_3, ..., y_{11}$ equal the values of the curve at $x = 2^4, 3^4, 5^4, ..., 31^4$ (having an initial value of

-59.65474). See Figure 39 for a plot of $(y_z + 59.65474 + 7.0) - 13.770952 \log(z)$ for $z = 1, 2, 3, ..., 11$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $13.770952 \log(z)$. Let $y_1, y_2, y_3, ..., y_6$ equal the values of the curve at $x = 2^5, 3^5, 5^5, ..., 13^5$ (having an initial value of -69.73988). See Figure 40 for a plot of $(y_z + 69.73988 + 13.0) - 22.101327 \log(z)$ for $z = 1, 2, 3, ..., 6$. The plot shows that the normalized curve of y values is increasing at almost the same rate as $22.101237 \log(z)$. See Figure 41 for a plot of the logarithm coefficients (1.0819701, 3.6763725, 7.9434565, 13.770952, and 22.101237). These coefficients are roughly equal to the corresponding coefficients for the zeta function zeros. For a quadratic least-squares fit of these coefficients, $p_1 = 0.9309$ with a 95% confidence interval of (0.6941, 1.168), $p_2 = -0.3719$ with a 95% confidence interval of (-1.82, 1.076), $p_3 = 0.591$ with a 95% confidence interval of (-1.309, 2.491), SSE=0.0848, R-square=0.9997, and RMSE=0.2059.

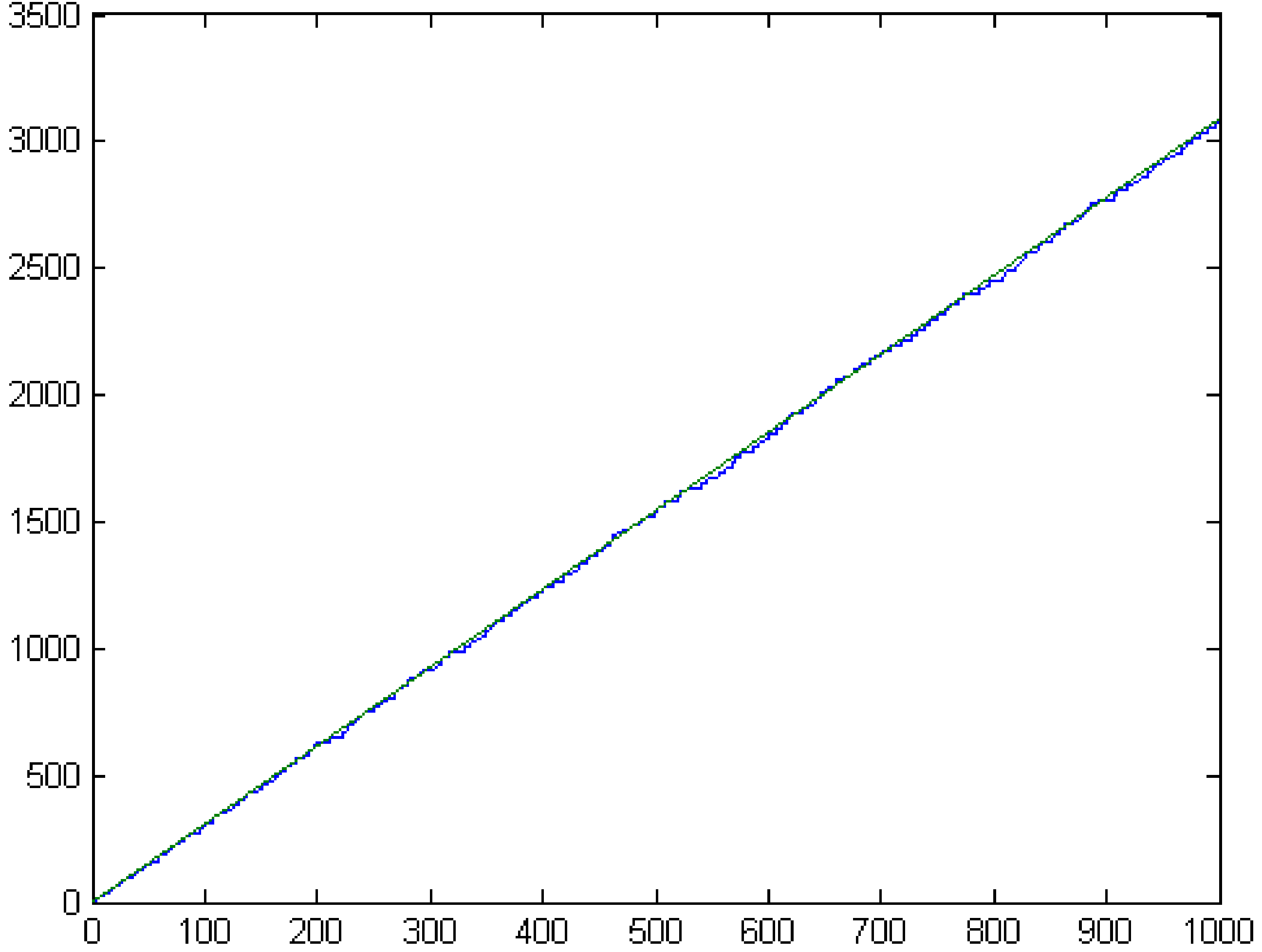
See Figure 42 for a plot of $\sum_{i|x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i) - \log(0.0000398896)\log(x)$ for $x = 1, 2, 3, ..., 1000$ (3.98896e-5 is the first element in a sorted standard uniform distribution having 1001 elements). The different curves are the same as the corresponding curves for the zeta function zeros. The first element in a sorted standard uniform distribution having 1000001 elements is 2.2932941e-6. The plot of $\sum_{i|x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i) - \log(0.0000022932941)\log(x)$ for $x = 1, 2, 3, ..., 1000000$ has similar curves. Let $y_1, y_2, y_3, ..., y_{168}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 997^2$. See Figure 43 for a plot of y_z and $1.615 \log(z)^2$ (superimposed on each other) for $z = 1, 2, 3, ..., 168$ (the absolute values of the differences between the two curves are less than 1.4). Let $y_1, y_2, y_3, ..., y_{25}$ equal the values of the curve at $x = 2^3, 3^3, 5^3, ..., 97^3$. See Figure 44 for a plot of y_z for $z = 1, 2, 3, ..., 25$. For a quadratic least-squares fit of the curve, SSE=12.86, R-square=0.9981, and RMSE=0.7646.

The first element in a sorted standard uniform distribution having 100001 elements is 5.28008163e-6. The plot of $\sum_{i|x} \gamma_1(\lfloor x/i \rfloor)\Lambda(i) - \log(0.00000528008163)\log(x)$ for $x = 1, 2, 3, ..., 100000$ has similar curves. Let $y_1, y_2, y_3, ..., y_{65}$ equal the values of the curve at $x = 2^2, 3^2, 5^2, ..., 313^2$. Let $w_2, w_3, w_4, ..., w_{5133}$ equal the values of the curve at $x = 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, ..., 2 \cdot 49999$ and let $w_1 = 2y_1$. See Figure 45 for a plot of $(w_z - 2.3) - 2.031370 \log(z)$ for $z = 1, 2, 3, ..., 5133$. For $z > 1$, the absolute values of the elements of this curve are less than 0.0083. Let $w_1, w_3, w_4, ..., w_{3569}$ equal the values of the curve at $x = 3 \cdot 2, 3 \cdot 5, 3 \cdot 7, ..., 3 \cdot 33331$ and let $w_2 = 2y_2$. See Figure 46 for a plot of $(w_z - 3.5) - 2.951208 \log(z)$ for $z = 1, 2, 3, ..., 3569$. For $z > 1$, the absolute values of the elements of this curve are less than 0.15. Let $w_1, w_2, w_4, ..., w_{2262}$ equal the values of the curve at $x = 5 \cdot 2, 5 \cdot 3, 5 \cdot 7, ..., 5 \cdot 19997$ and let $w_3 = 2y_3$. See Figure 47 for a plot of $(w_z - 4.7) - 4.288876 \log(z)$ for $z = 1, 2, 3, ..., 2262$. For $z > 1$, the absolute values of the elements of this curve are less than 0.22. In general, this process can be continued for curves corresponding to prime multiples of primes. For 7 times the primes, the w values are normalized by subtracting 5.4, for 11 times the primes, the w values are normalized by subtracting 5.8, for 13 times the primes, the w values are normalized by subtracting 6.0, for 17 times the primes, the w values are normalized by subtracting 6.6, and for 19 times the primes, the w values are normalized by subtracting 6.8. (Such normalization is not required for the corresponding zeta function curves.) Let c_1, c_2, c_3, ..., c_8 denote the coefficients of the above logarithms (2.031370, 2.951208, 4.288876, 4.856559, 5.804920, 6.209113, 7.179045, and 7.535560) corresponding to the curves for 2 times the primes, 3 times the primes, 5 times the primes, ..., 19 times the primes. See Figure 48 for a plot of c_z for $z = 1, 2, 3, ..., 8$. For a quadratic least-squares fit of these values, SSE=0.1528, R-square=0.9943, and RMSE=0.1748.

References

[1] Odlyzko, A. M., "www.dtc.umn.edu/~odlyZko/zeta_tables/index.html"

Figure 1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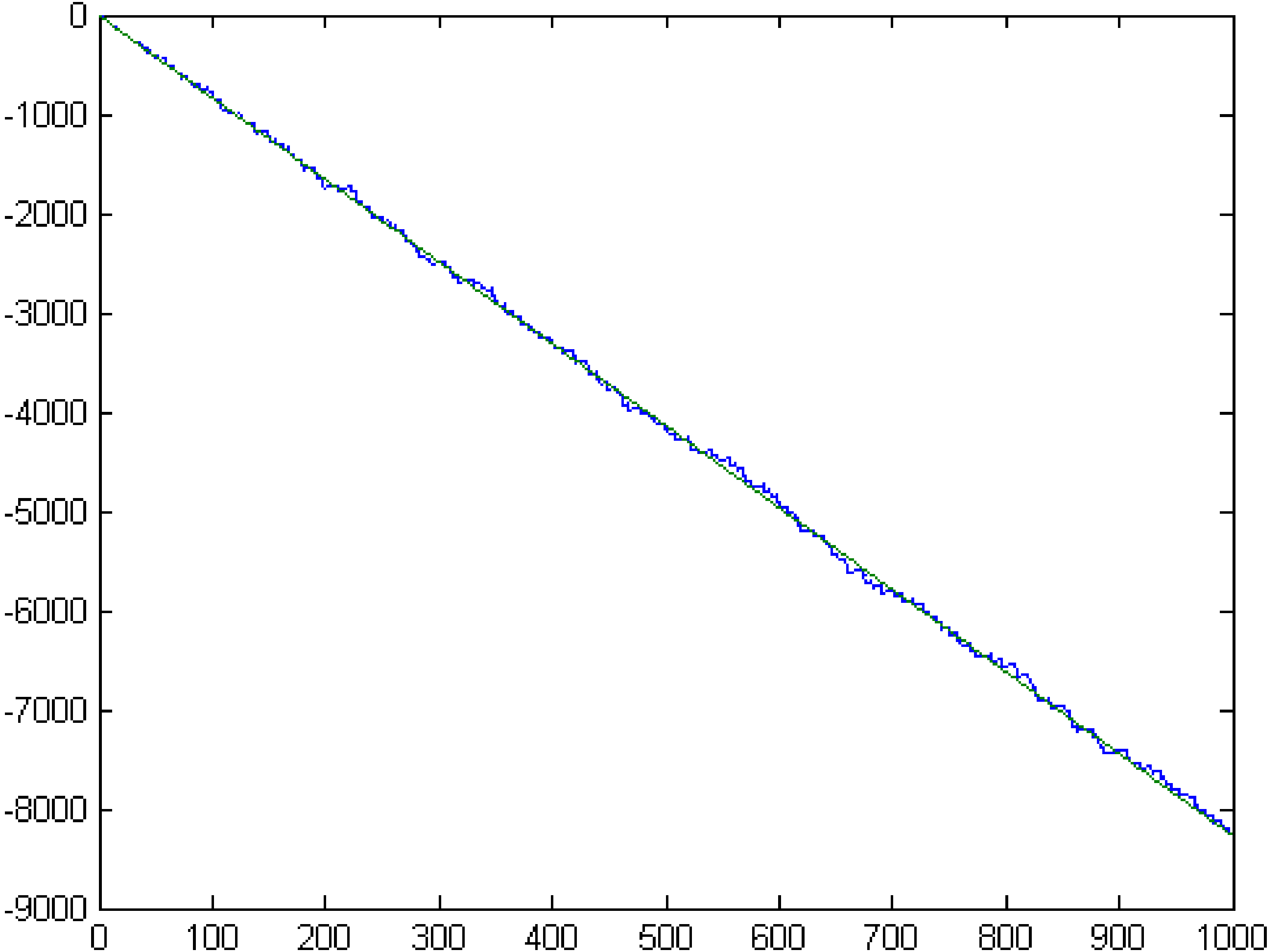
900
1000

Figure 2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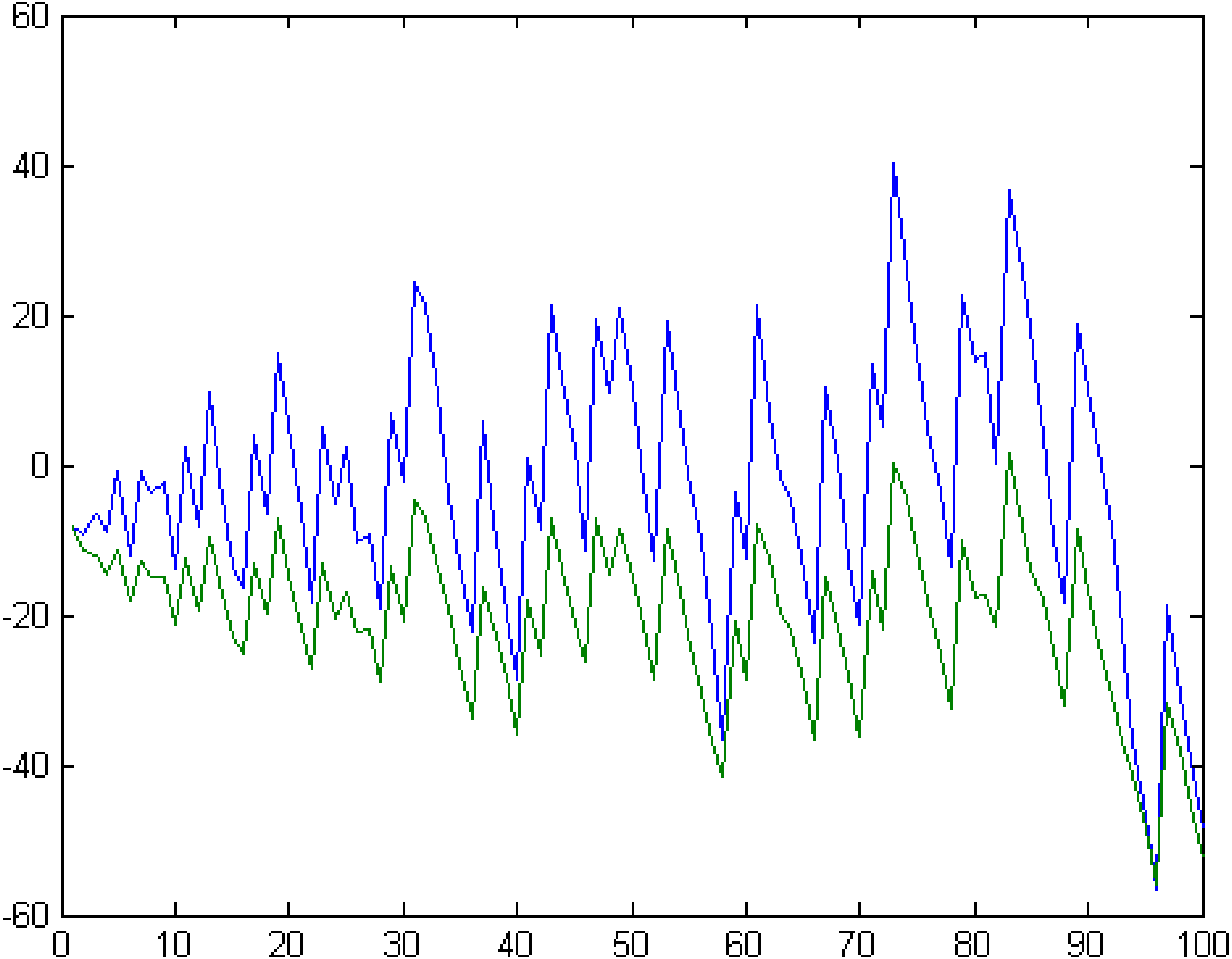
700
800
900
1000

Figure 3



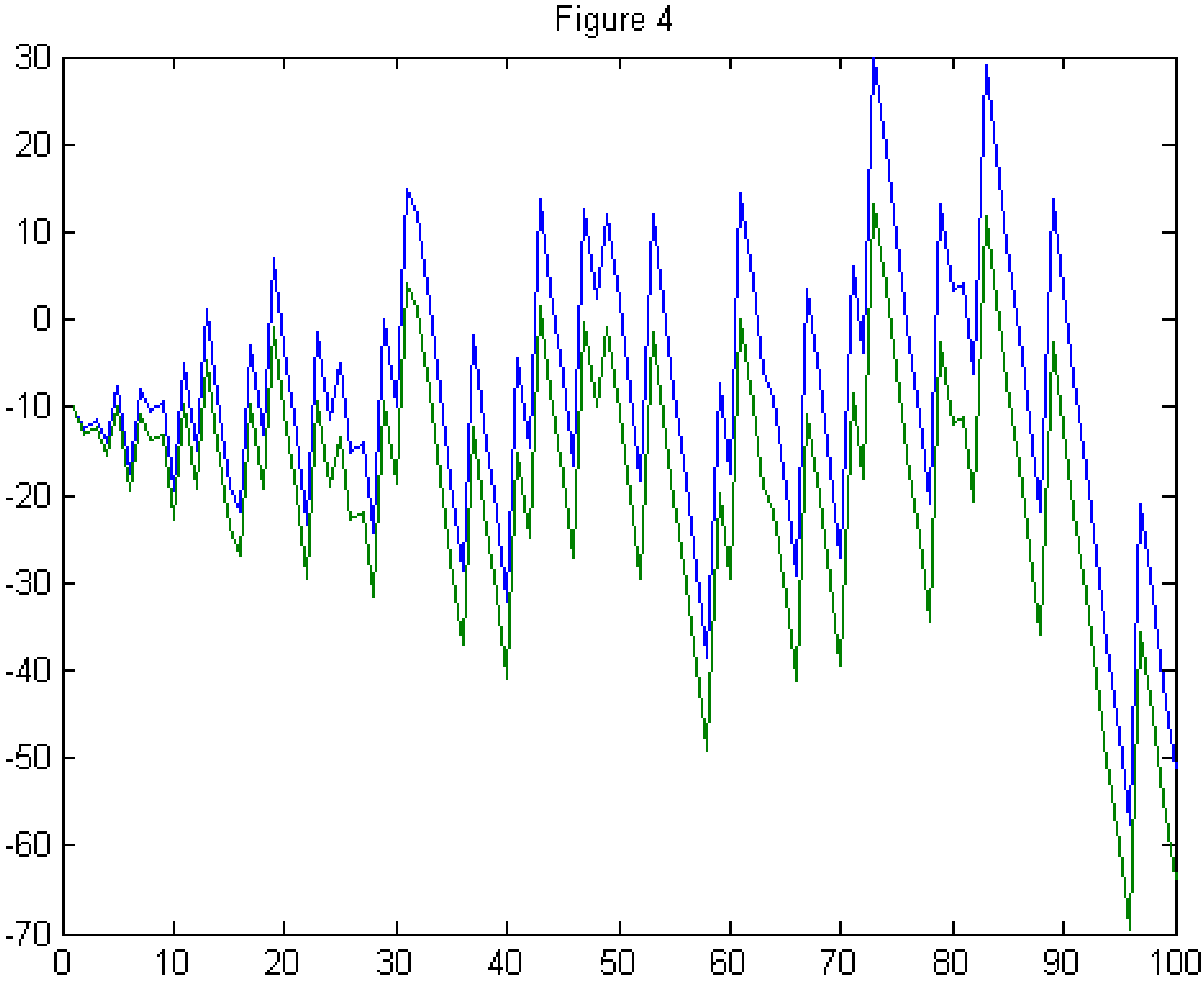


Figure 4
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10
0
-10
-20
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-50
-60
-70
0
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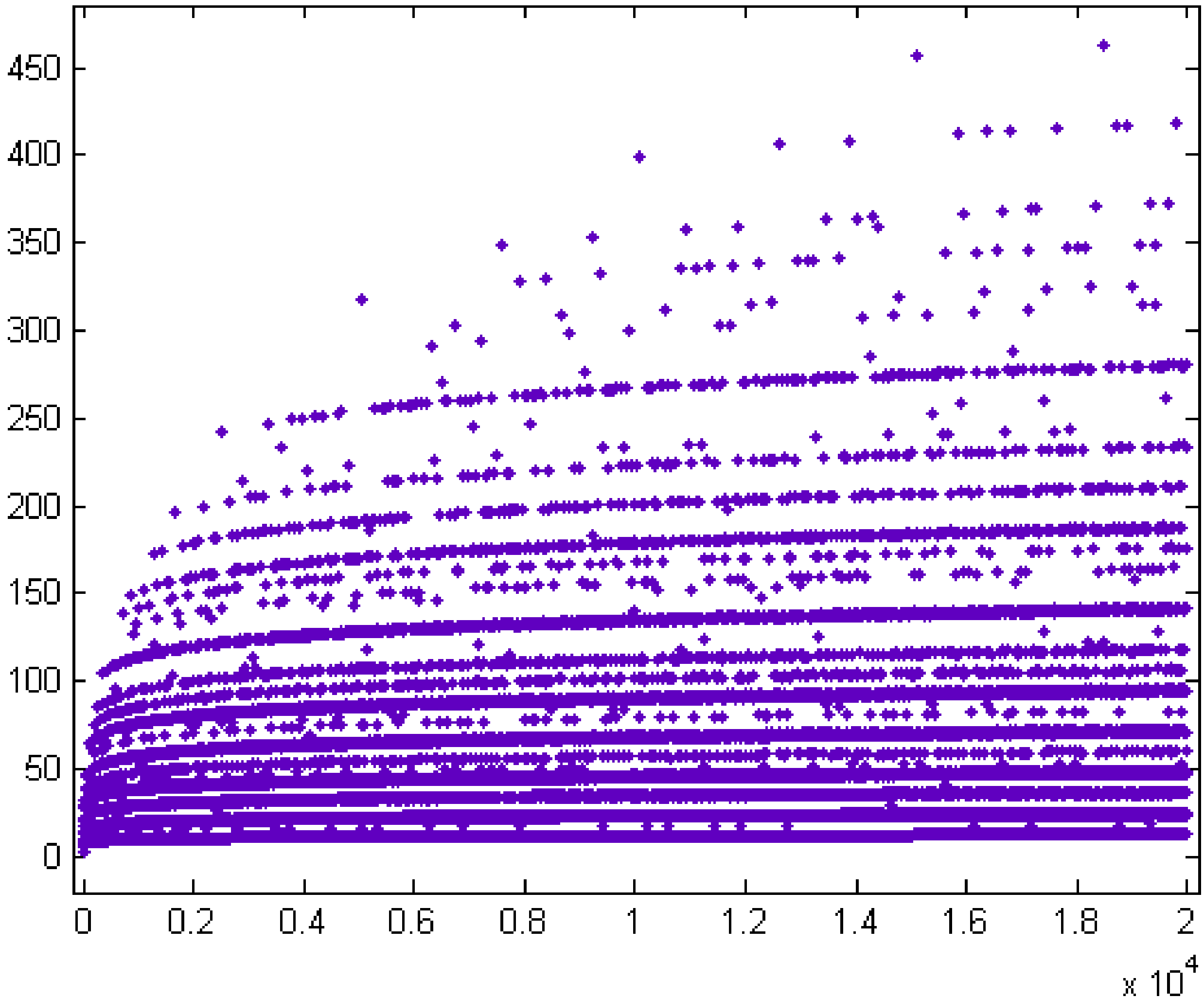
70
80
90
100

Figure 5



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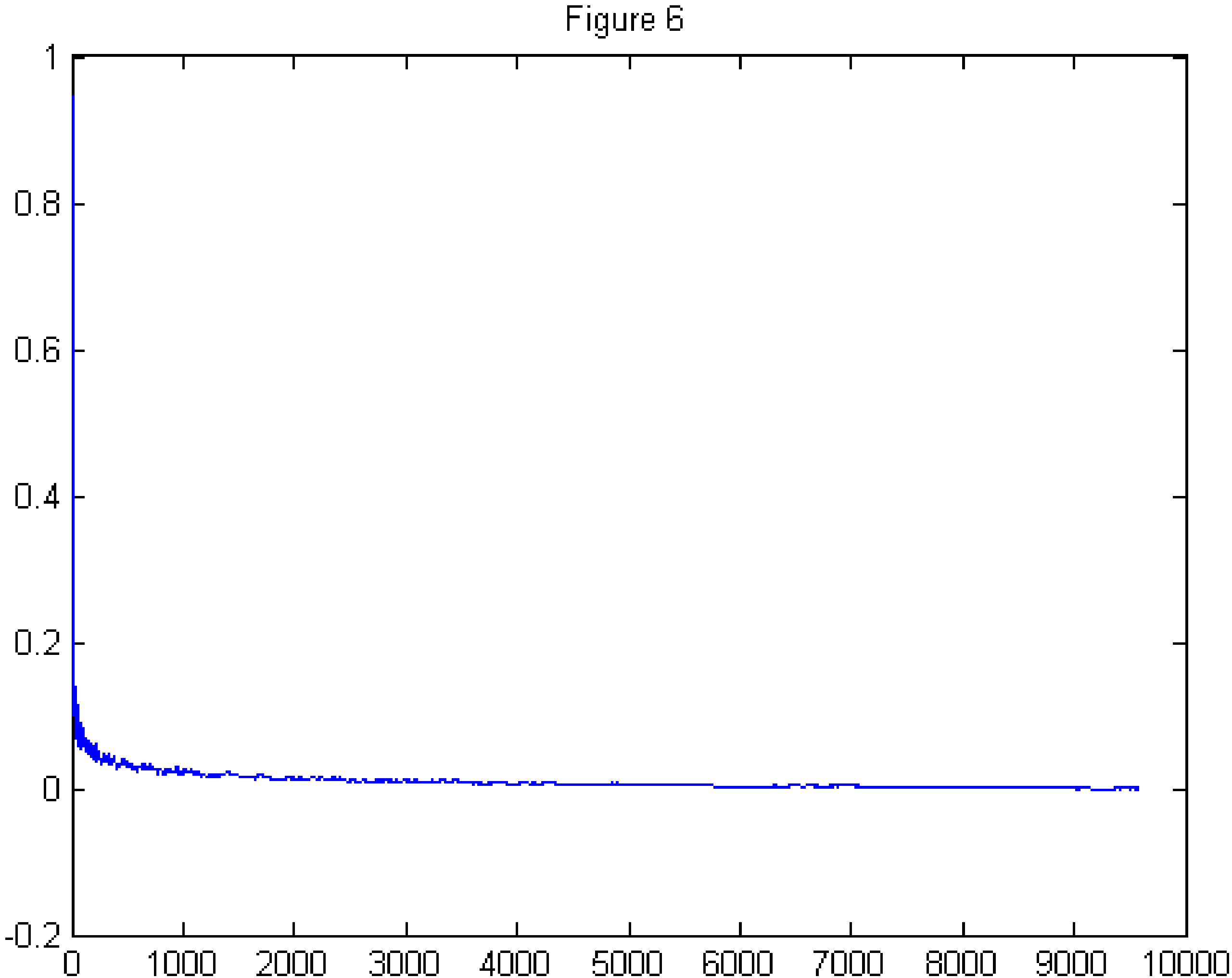
400
350
300
250
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^4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0.6
0.4
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-0.2
0
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4000
5000
6000
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8000
9000
10000

Figure 7

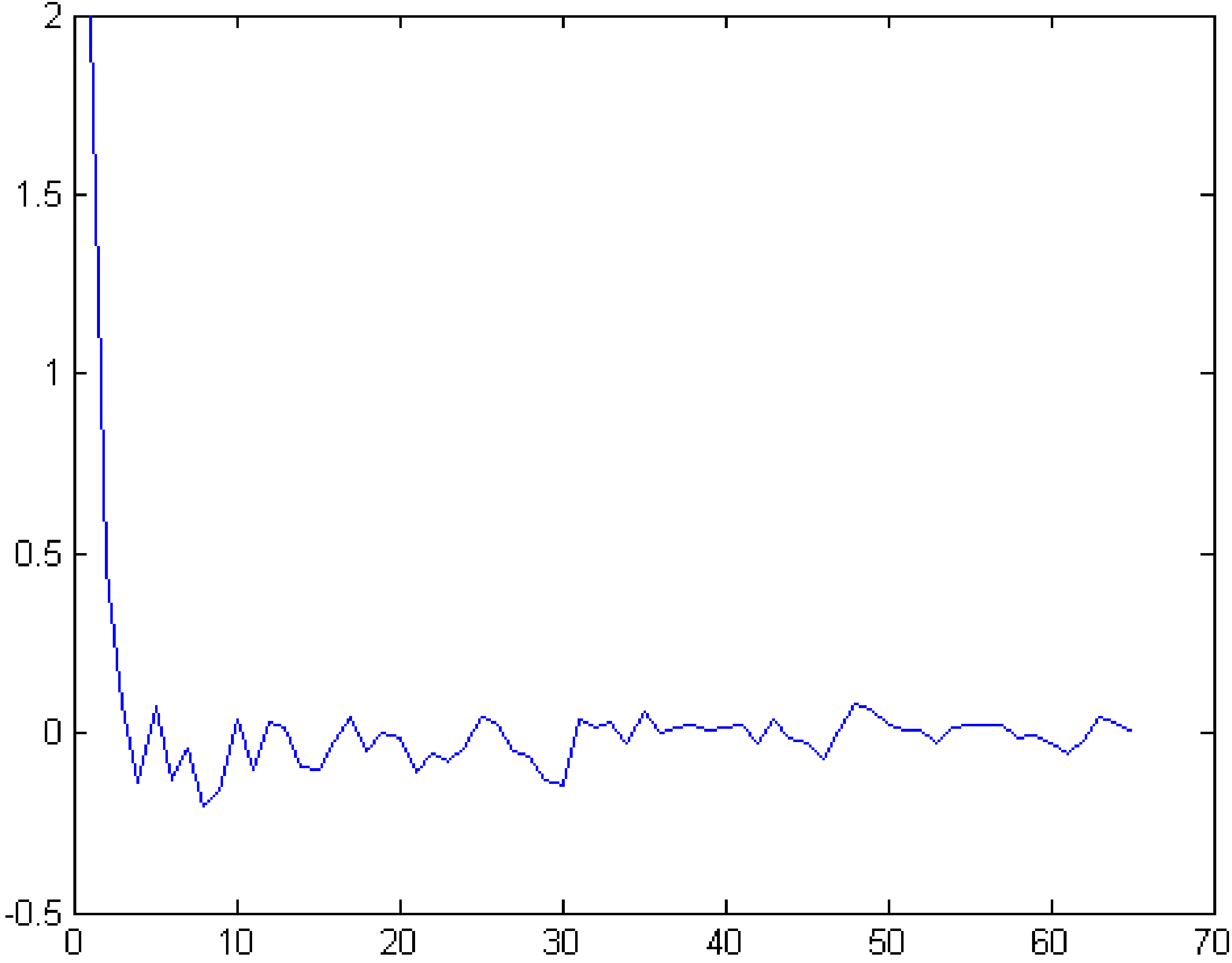


Figure 8

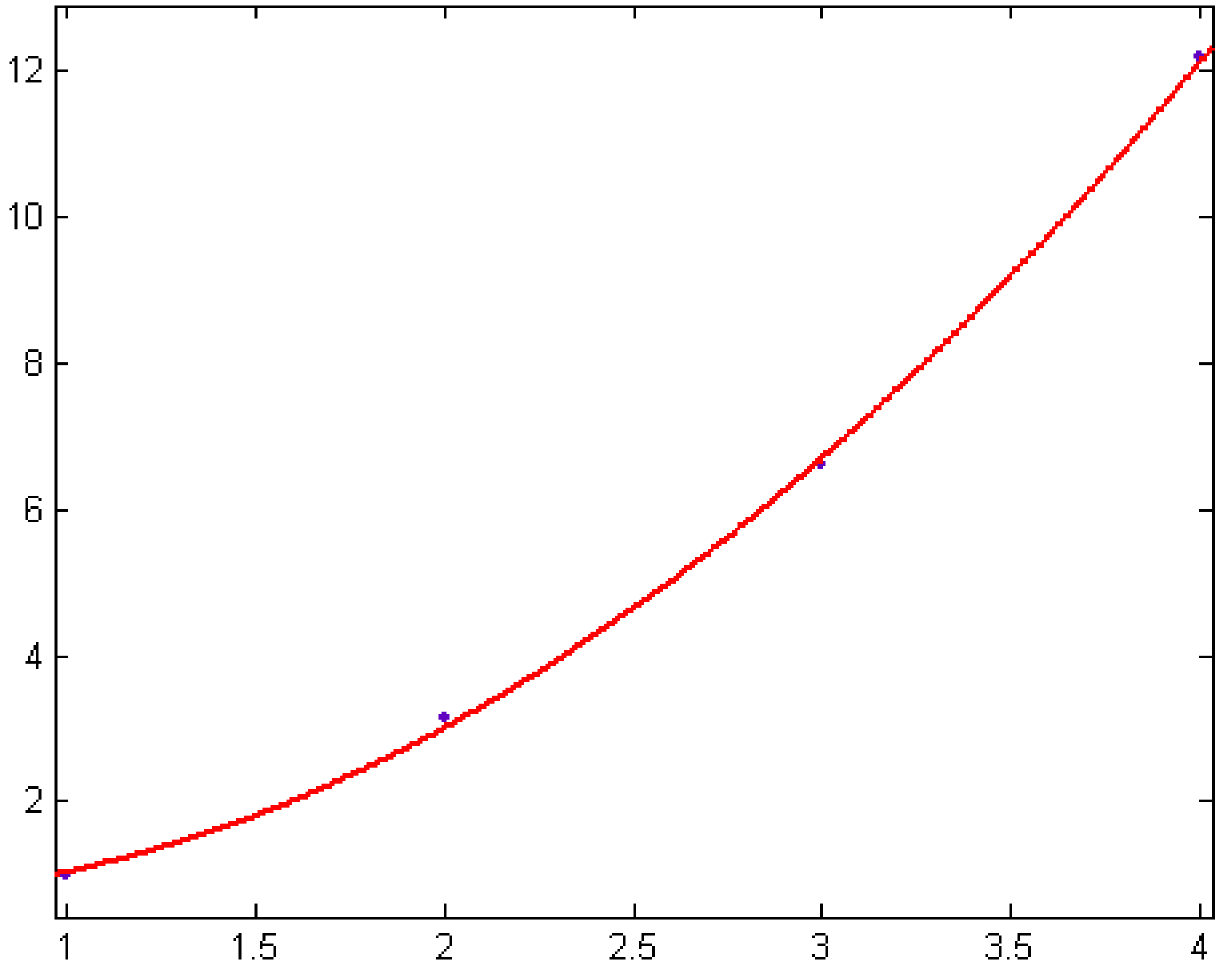
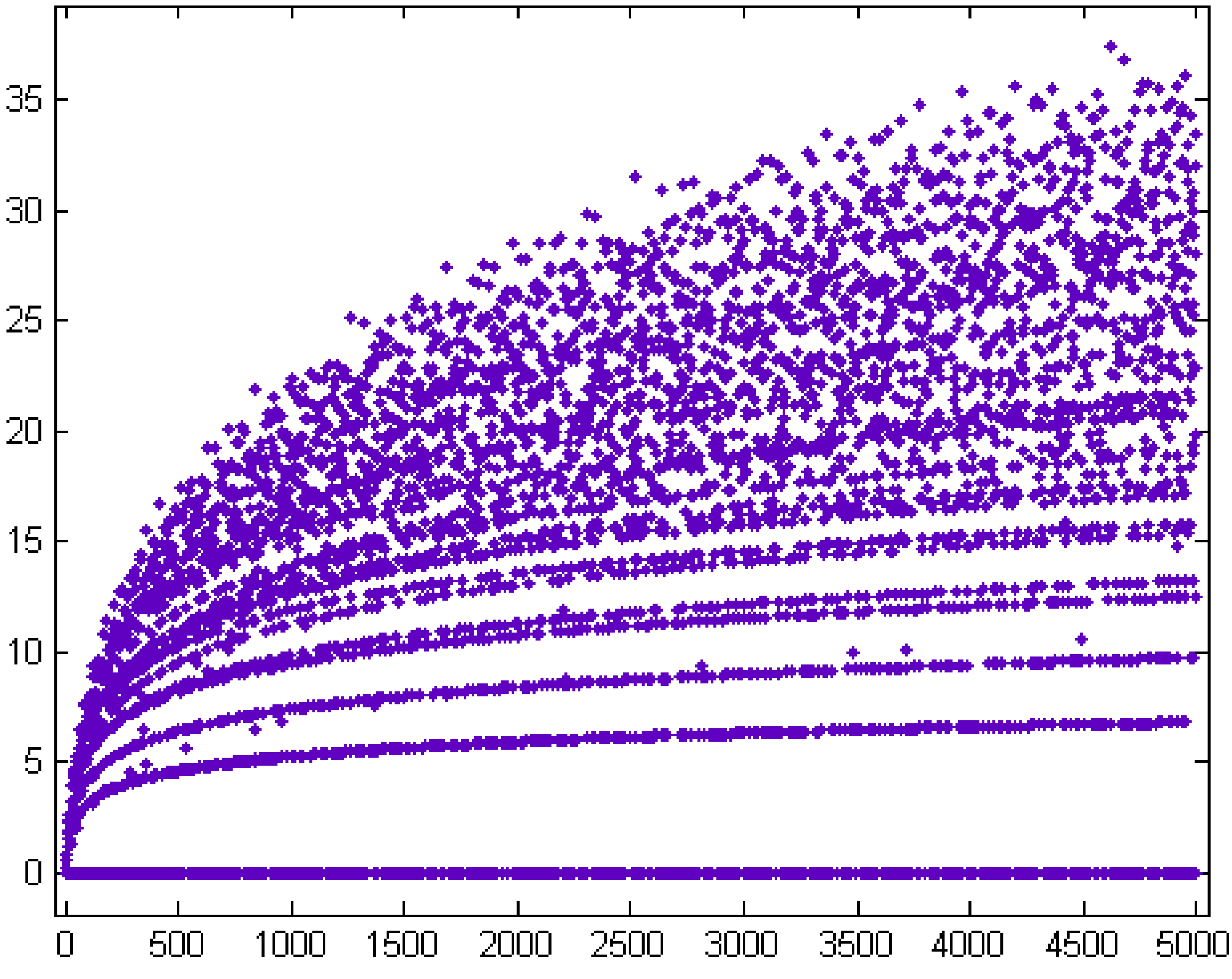


Figure 9



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4000
4500
5000

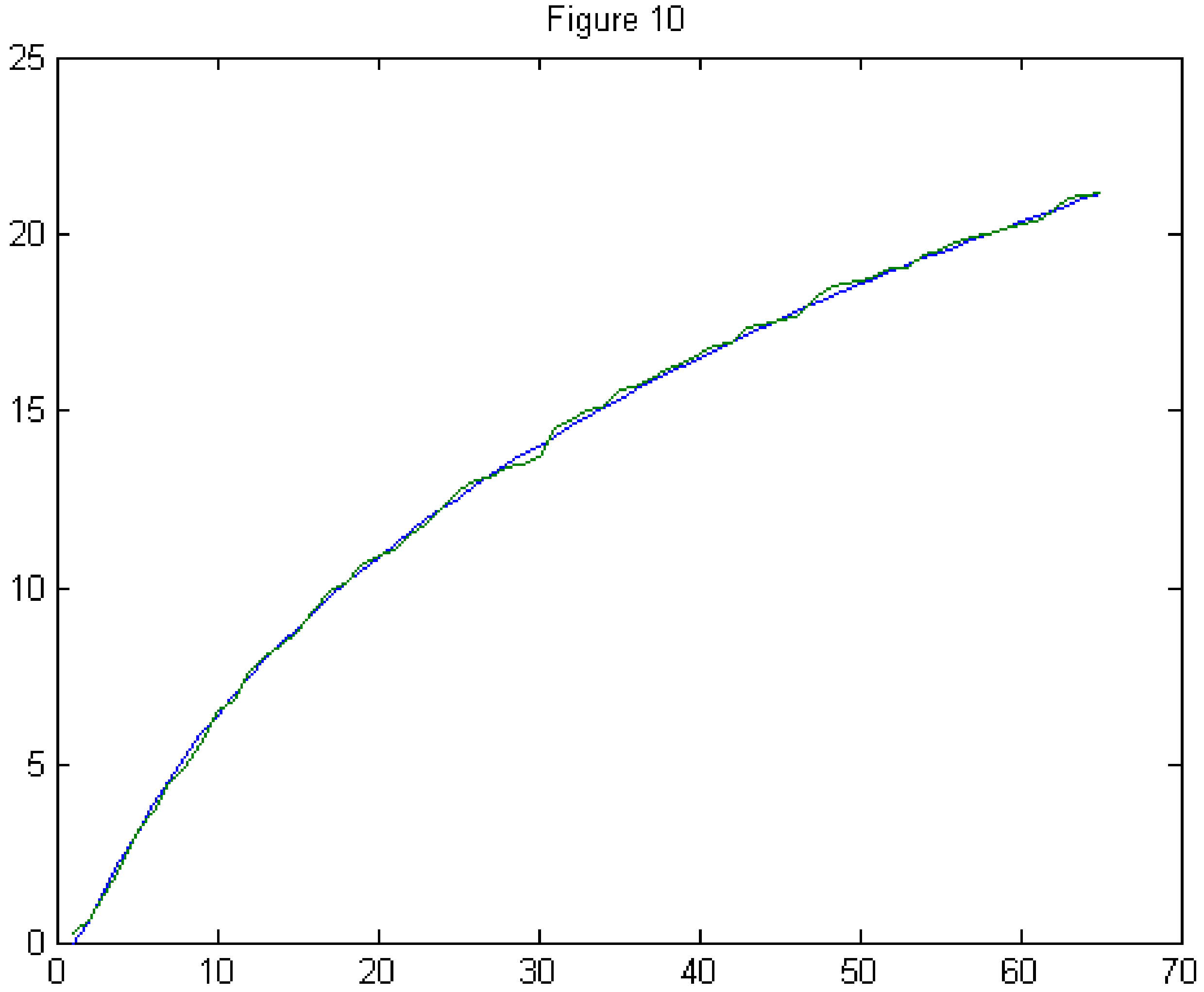

Figure 10
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15
10
5
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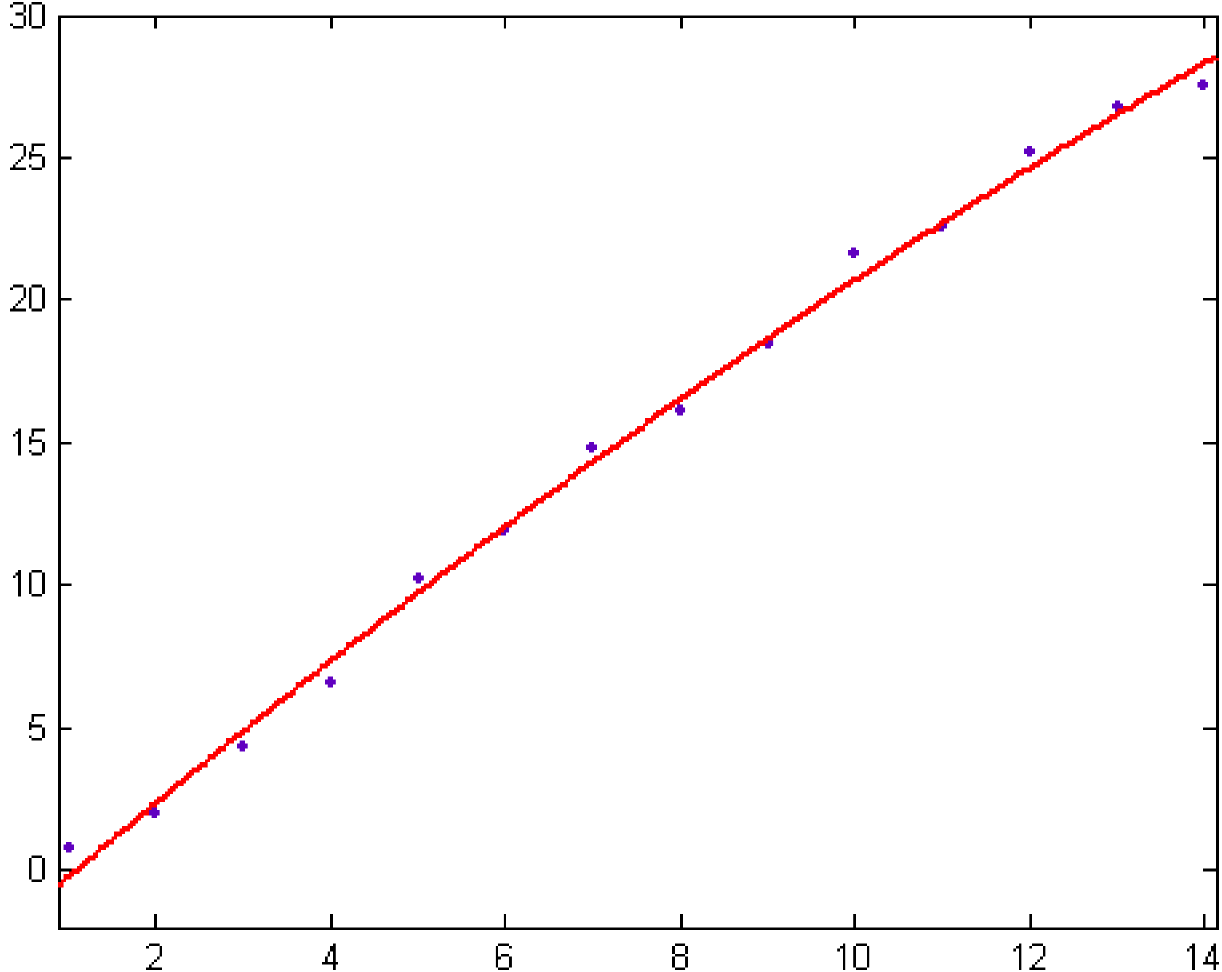
10
20
30
40
50
60
70

Figure 11



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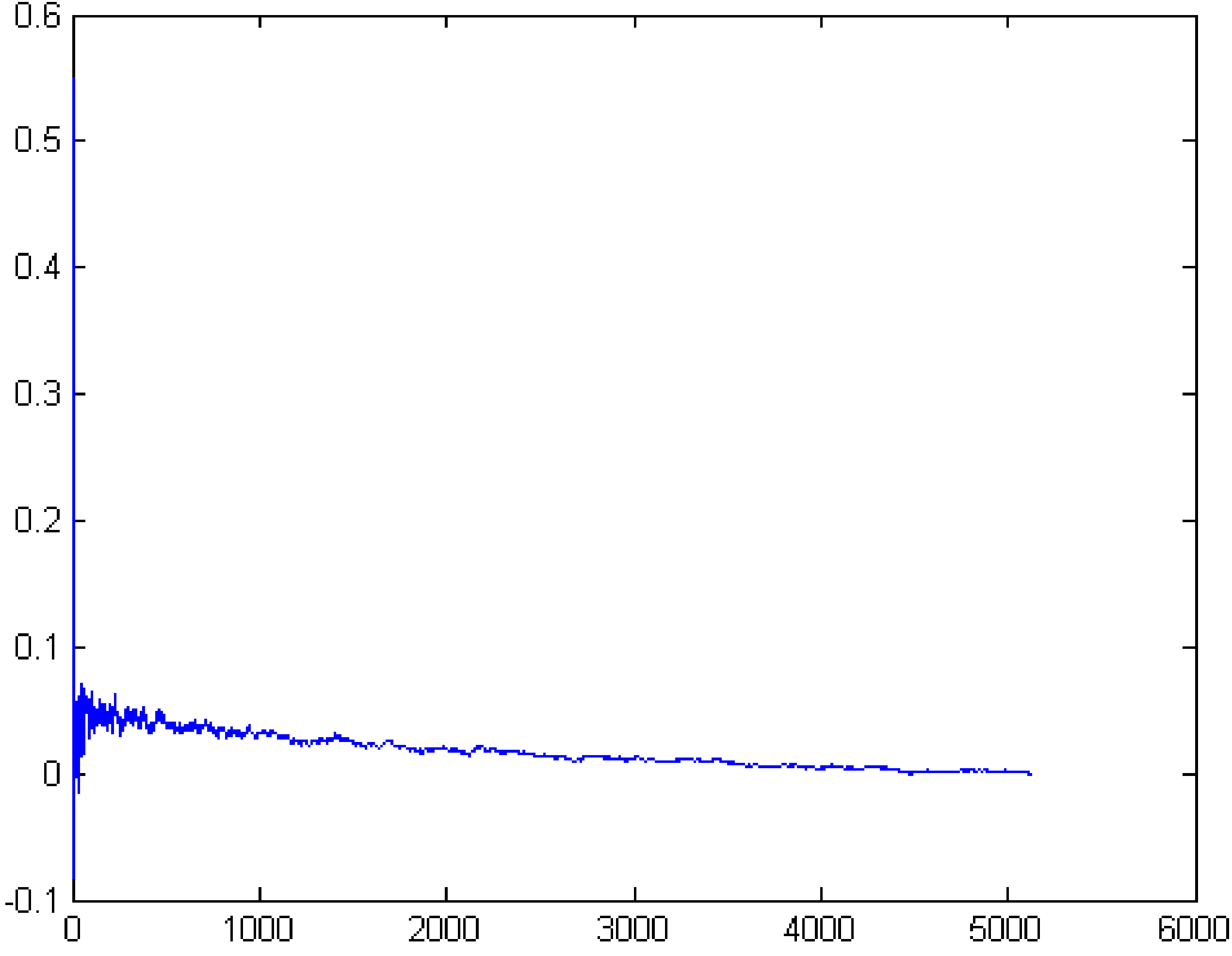

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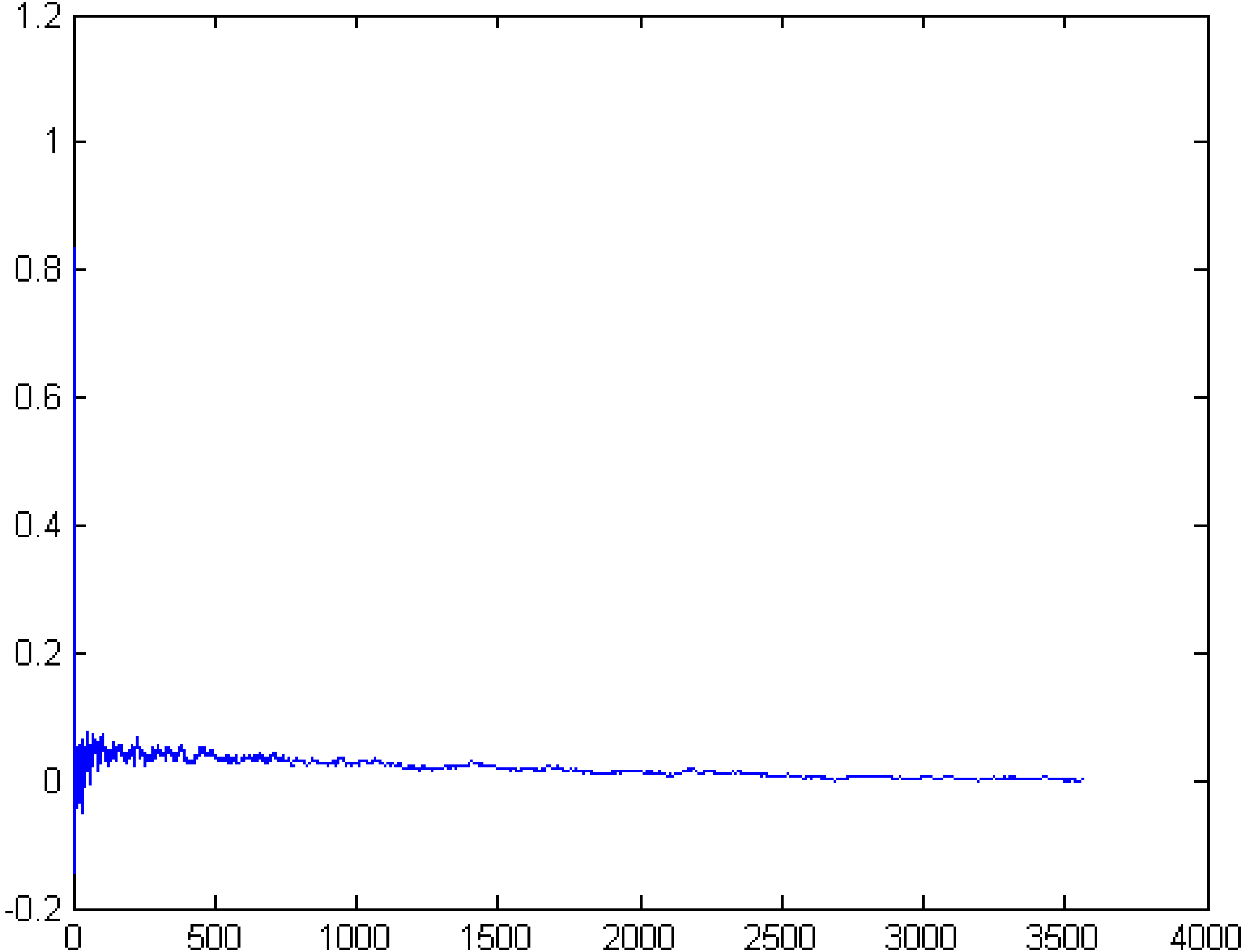

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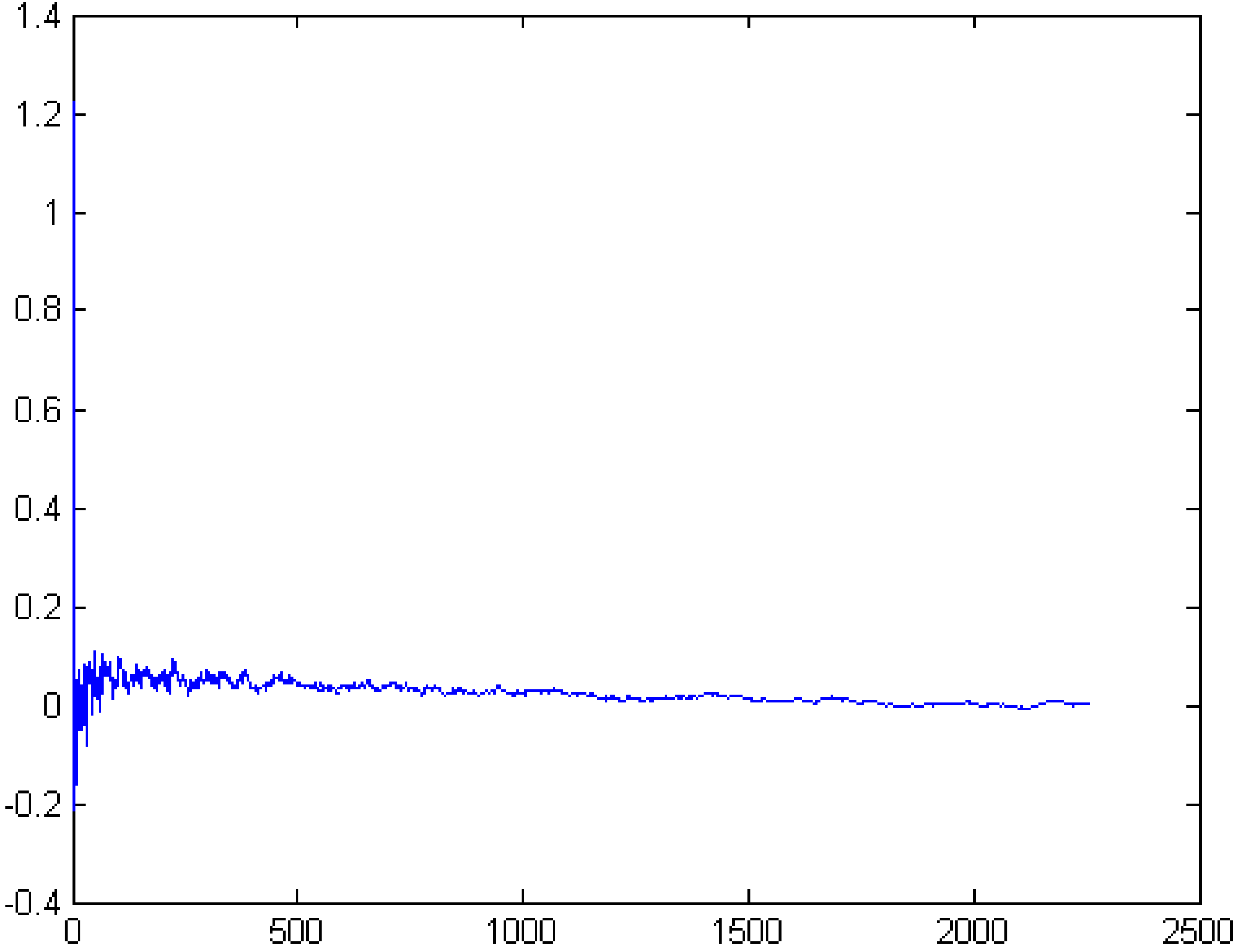

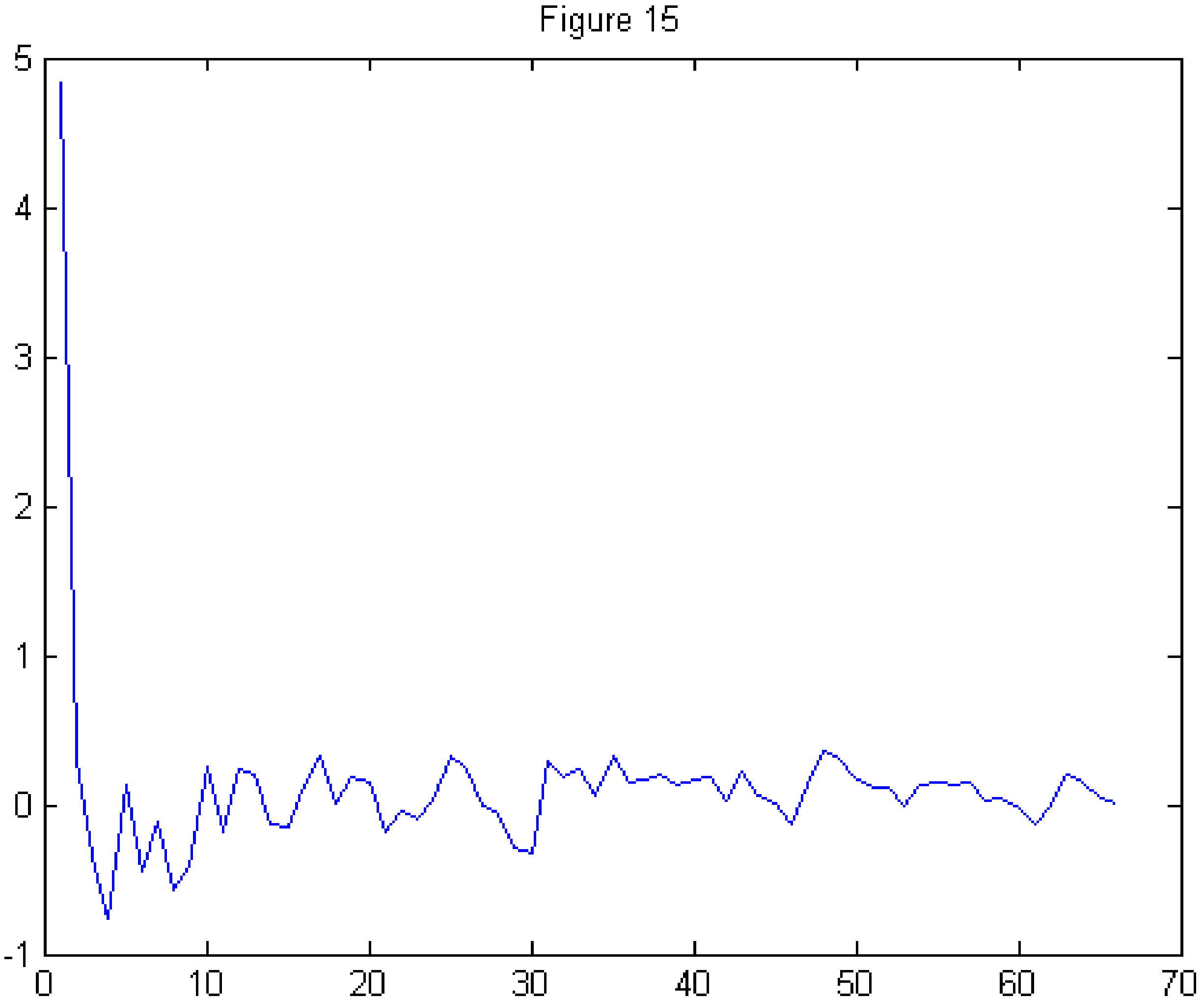
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3
2
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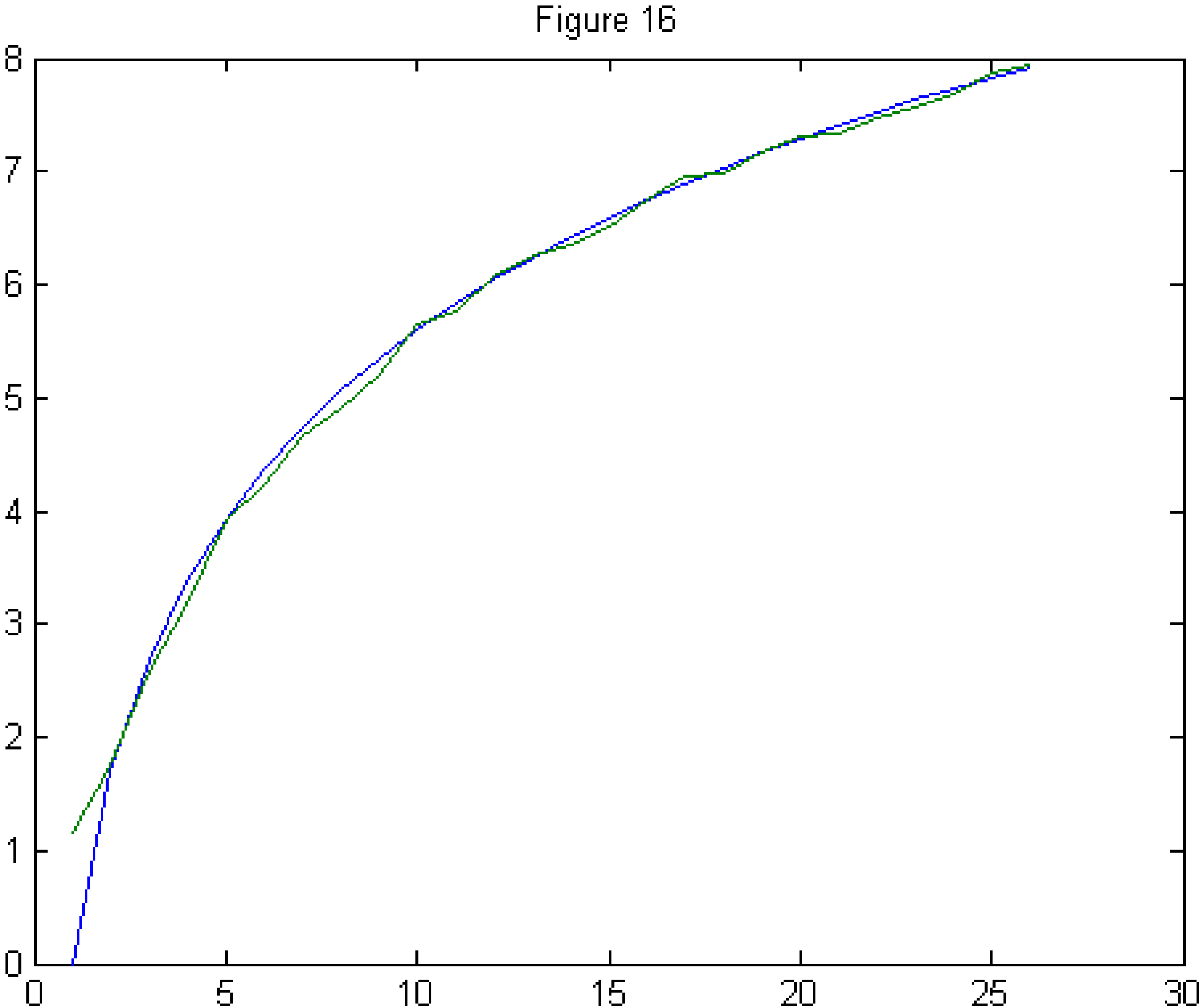
-1
0
10
20
30
40
50
60
70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7
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0
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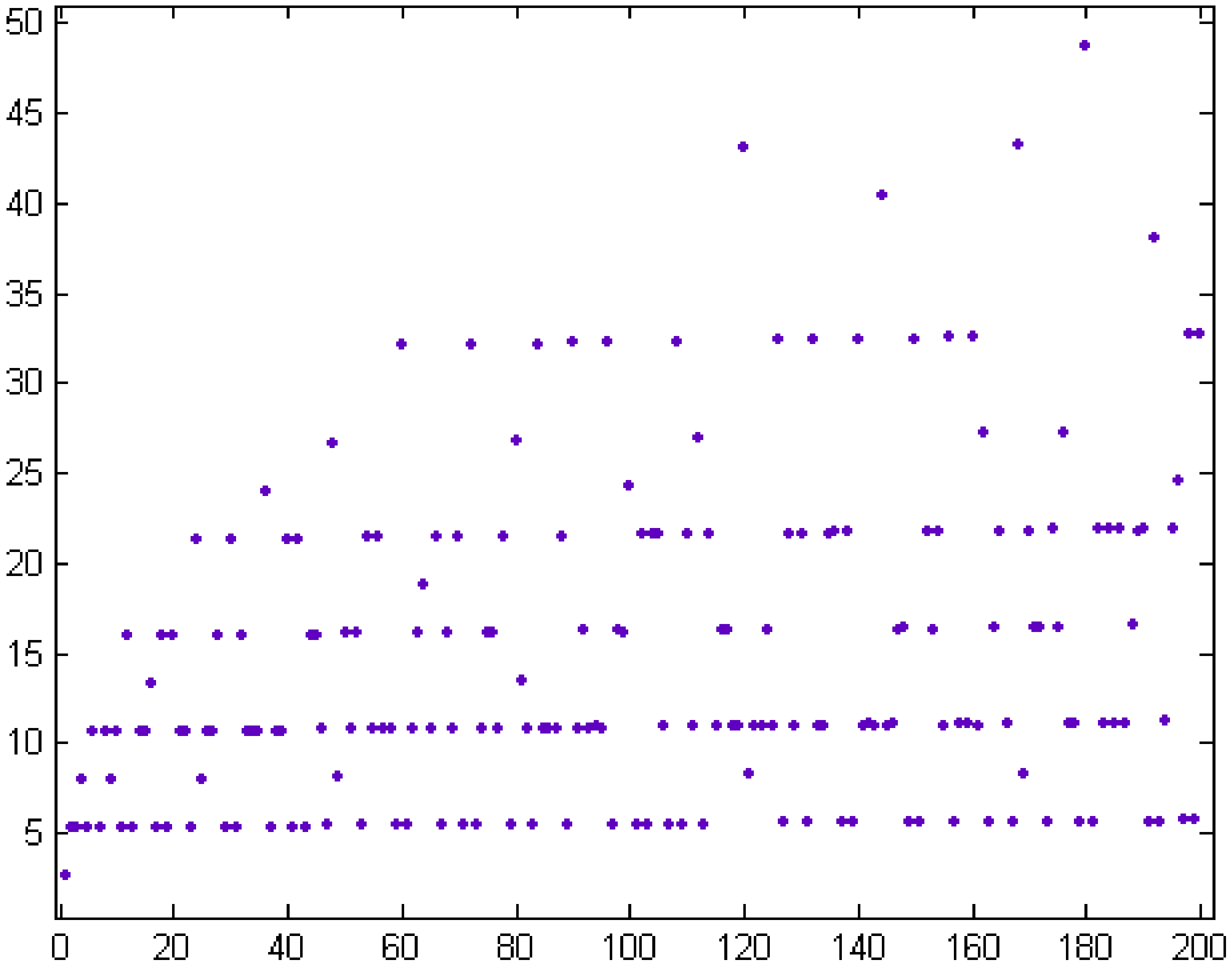
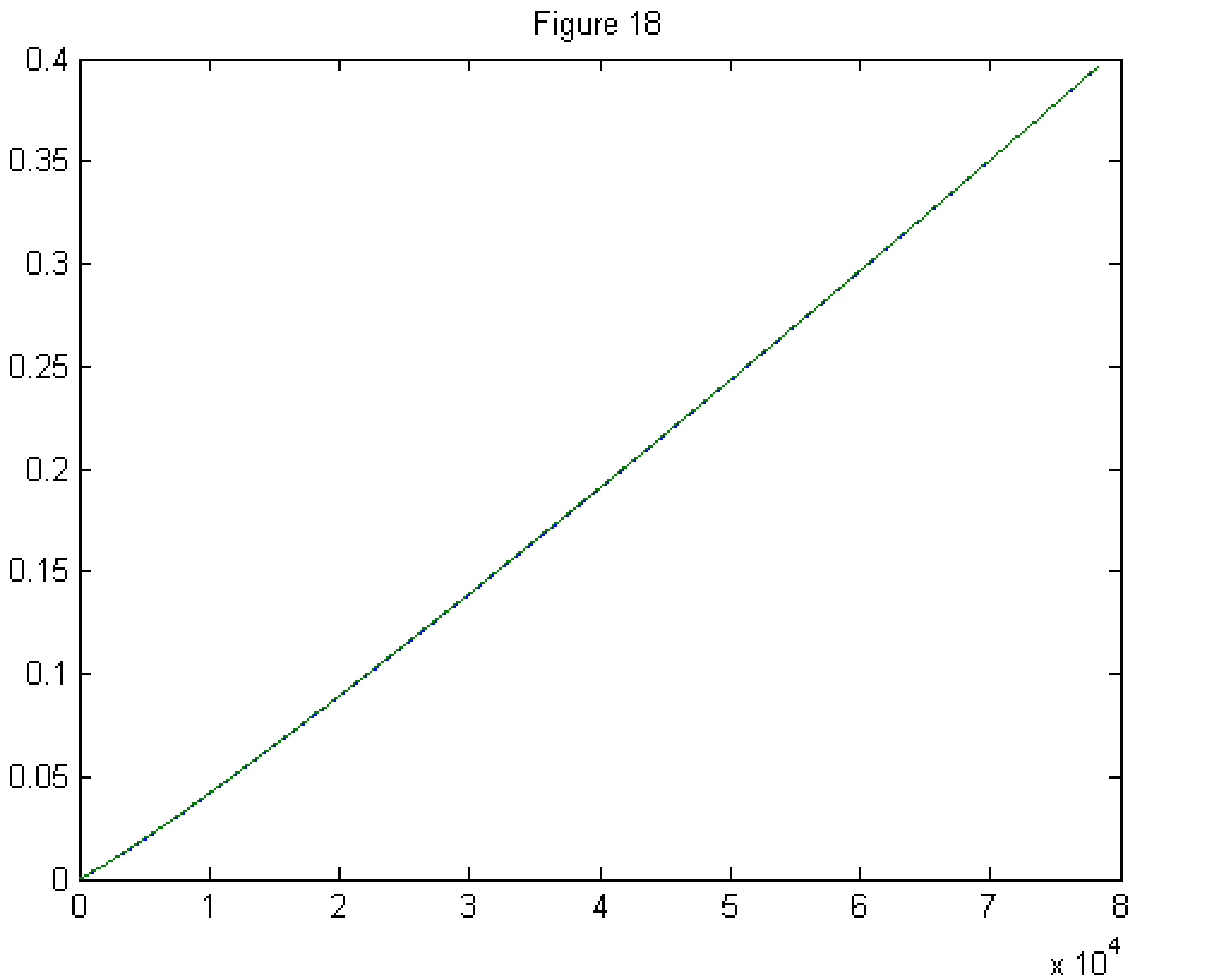
10
15
20
25
30

Figure 17

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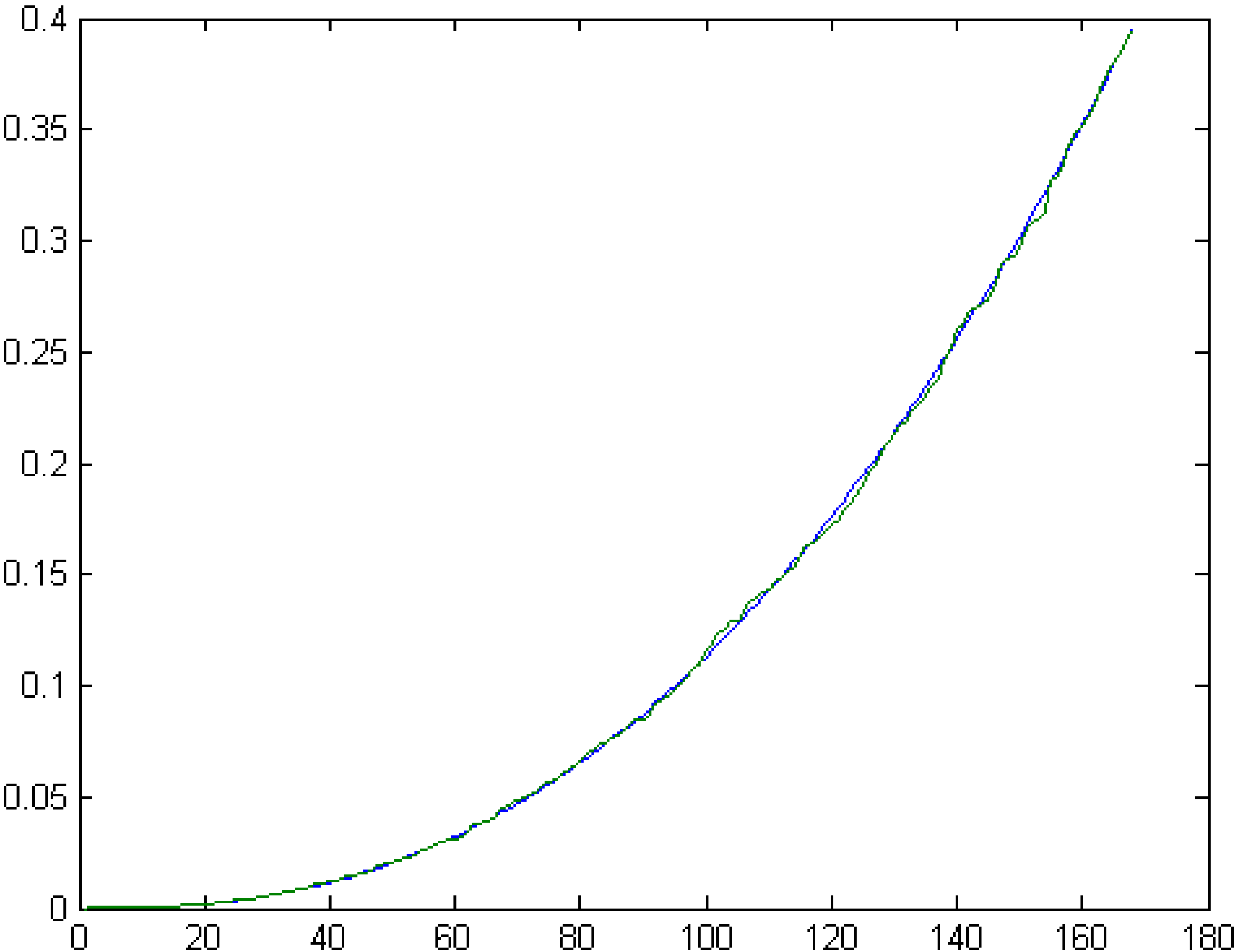
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
x 10^4

Figure 19



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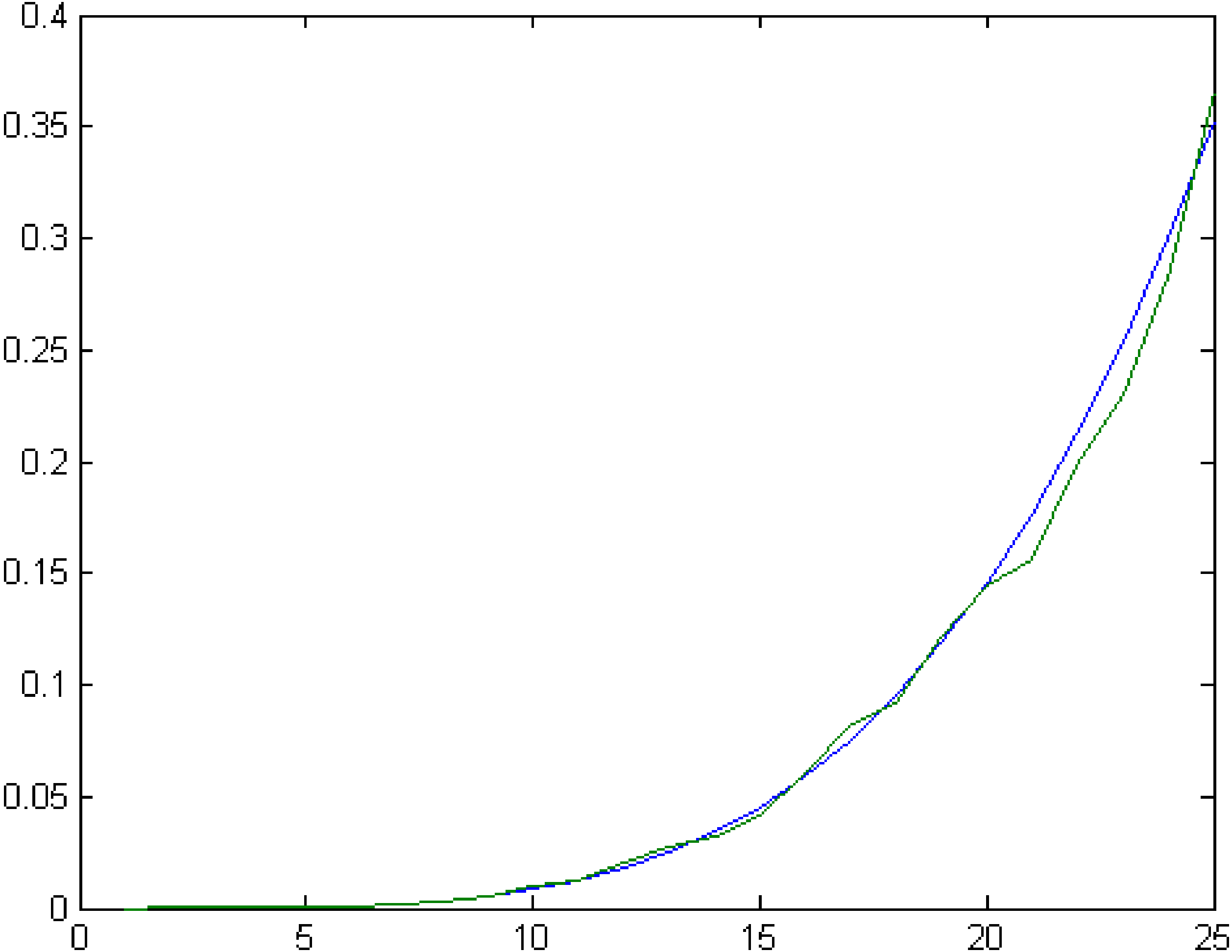

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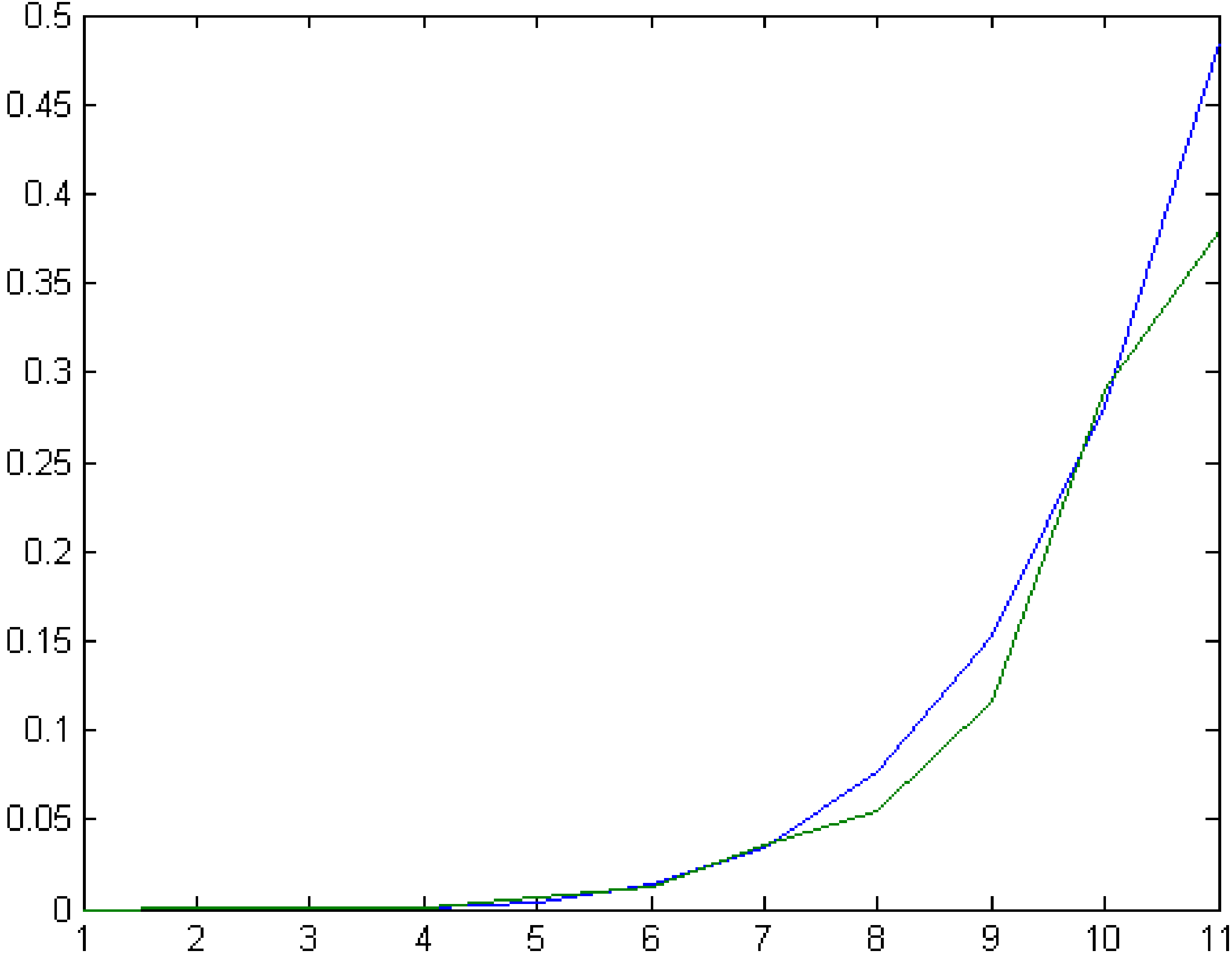


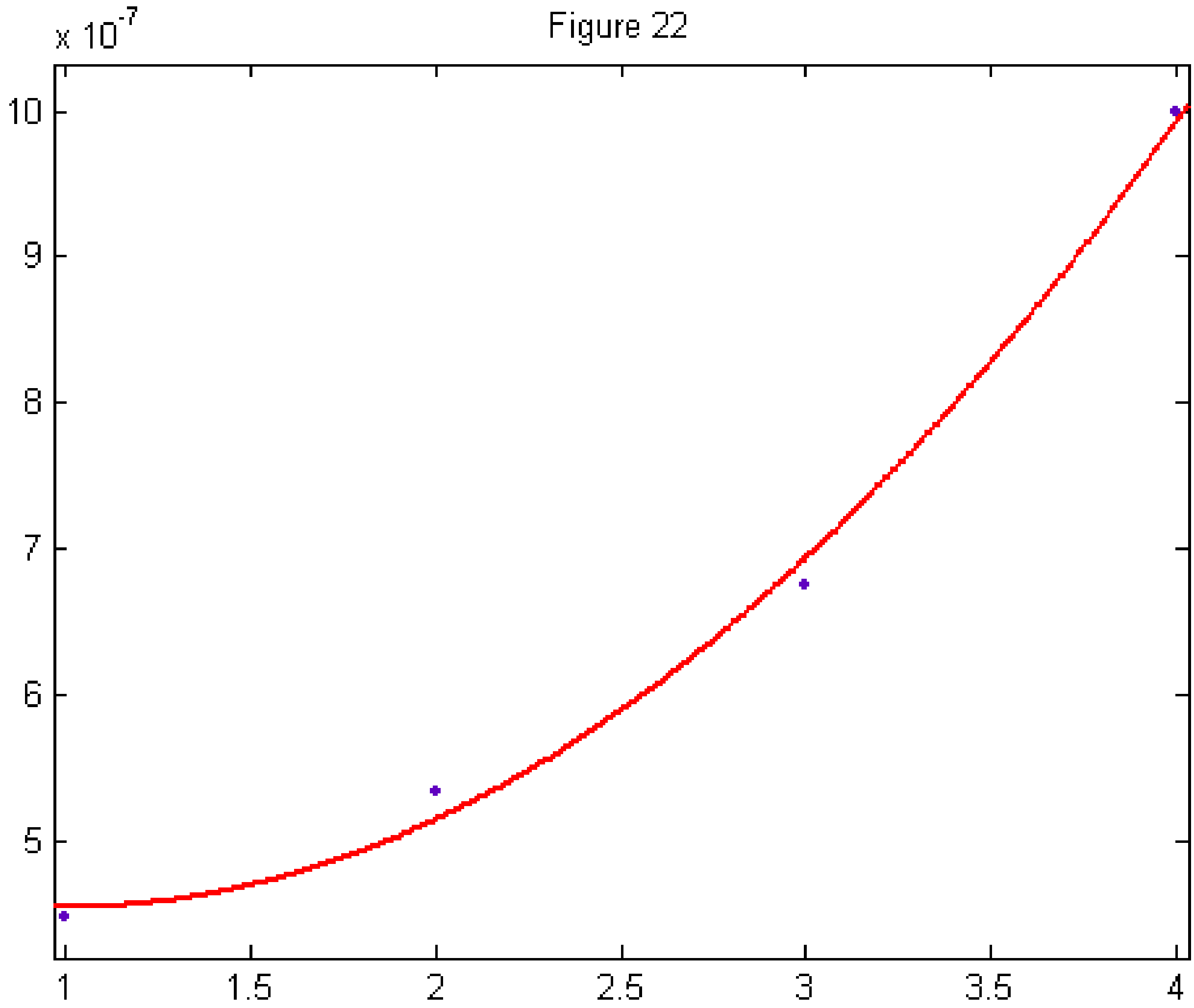
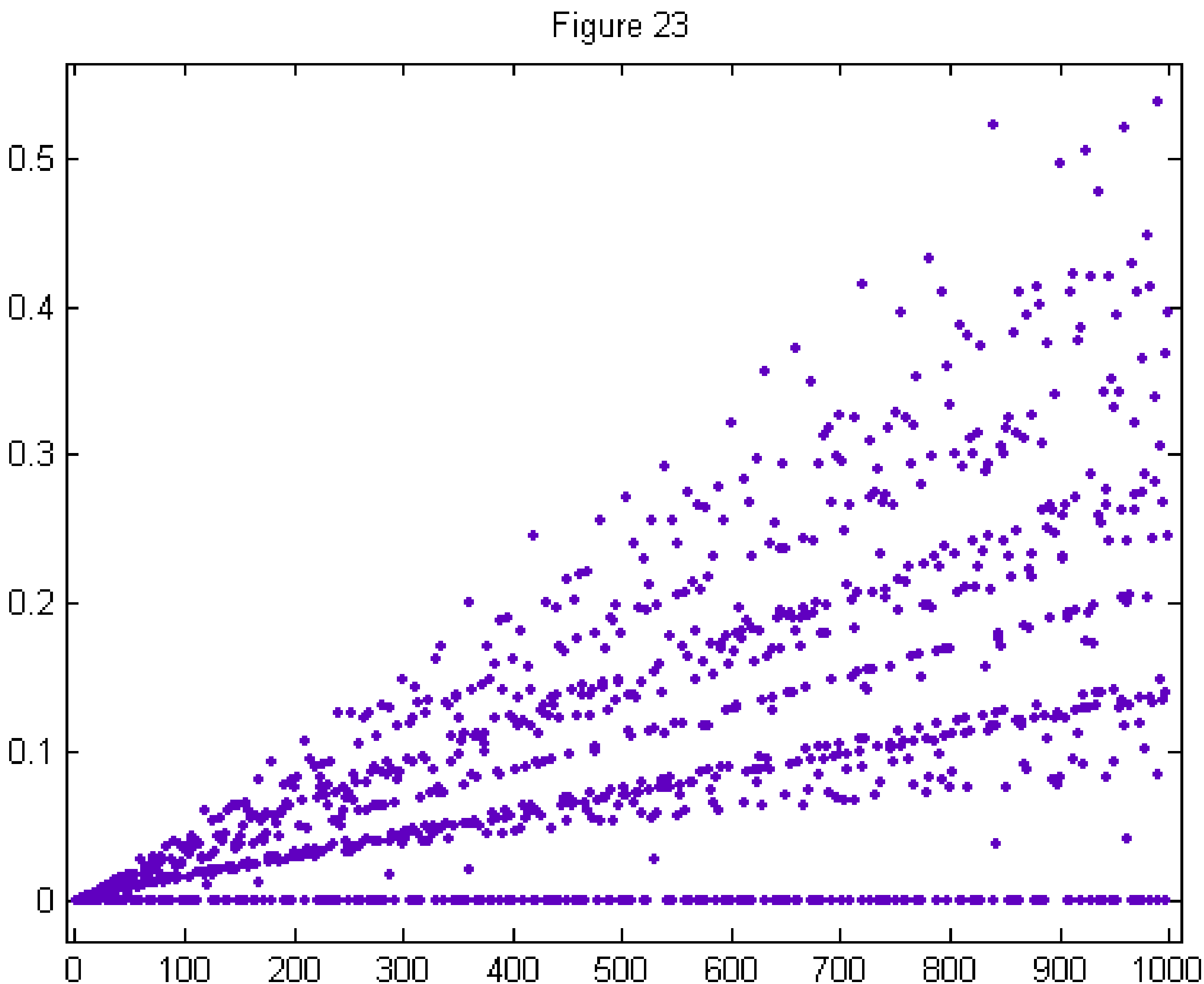


Figure 22
x 10^-7
10
9
8
7
6
5
1
1.5
2
2.5
3
3.5
4



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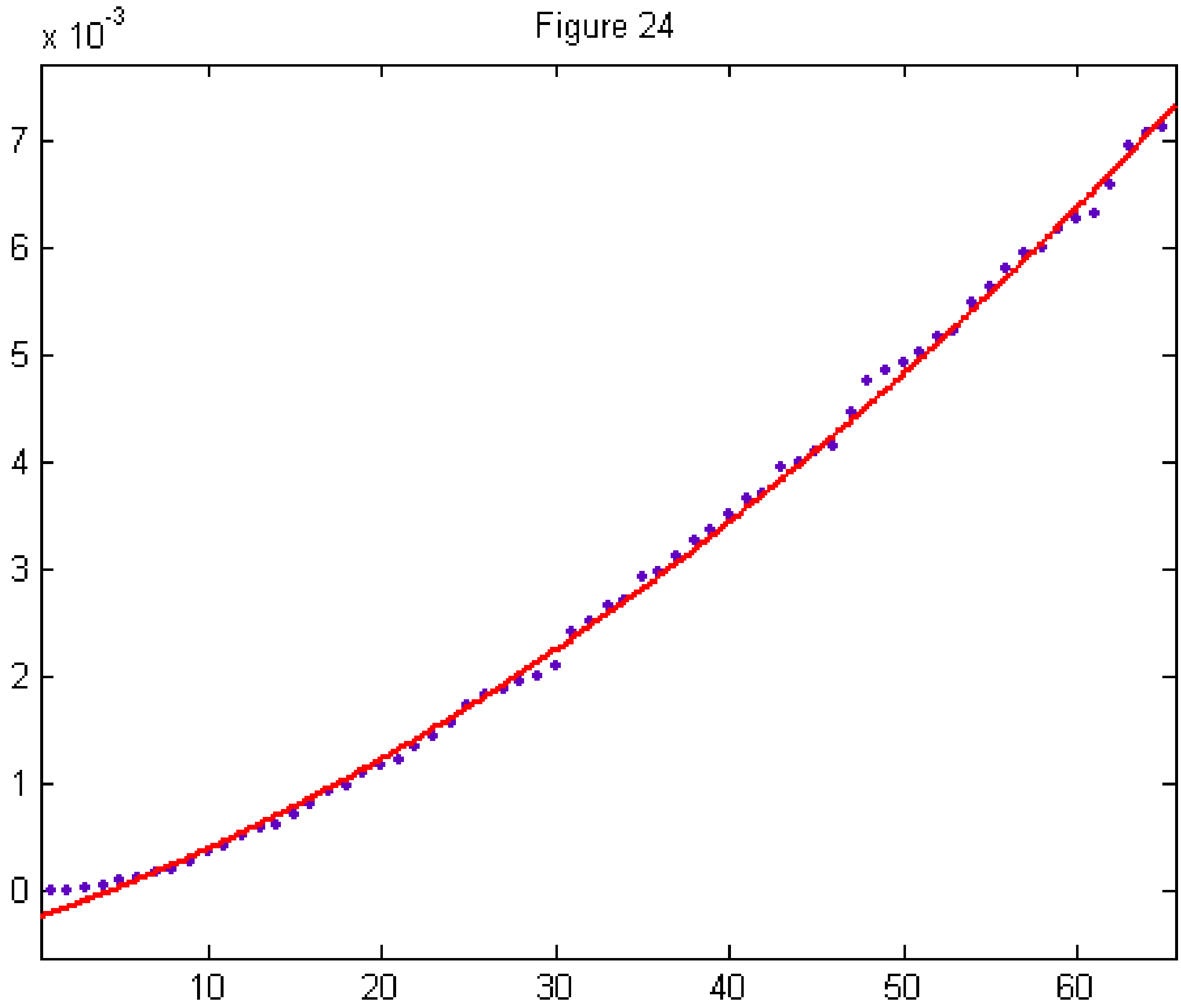
0.4
0.3
0.2
0.1
0
0
100
200
300
400
500
600
700
800
900
1000



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x 10^-3
7
6
5
4
3
2
1
0
10
20
30
40
50
60

Figure 25

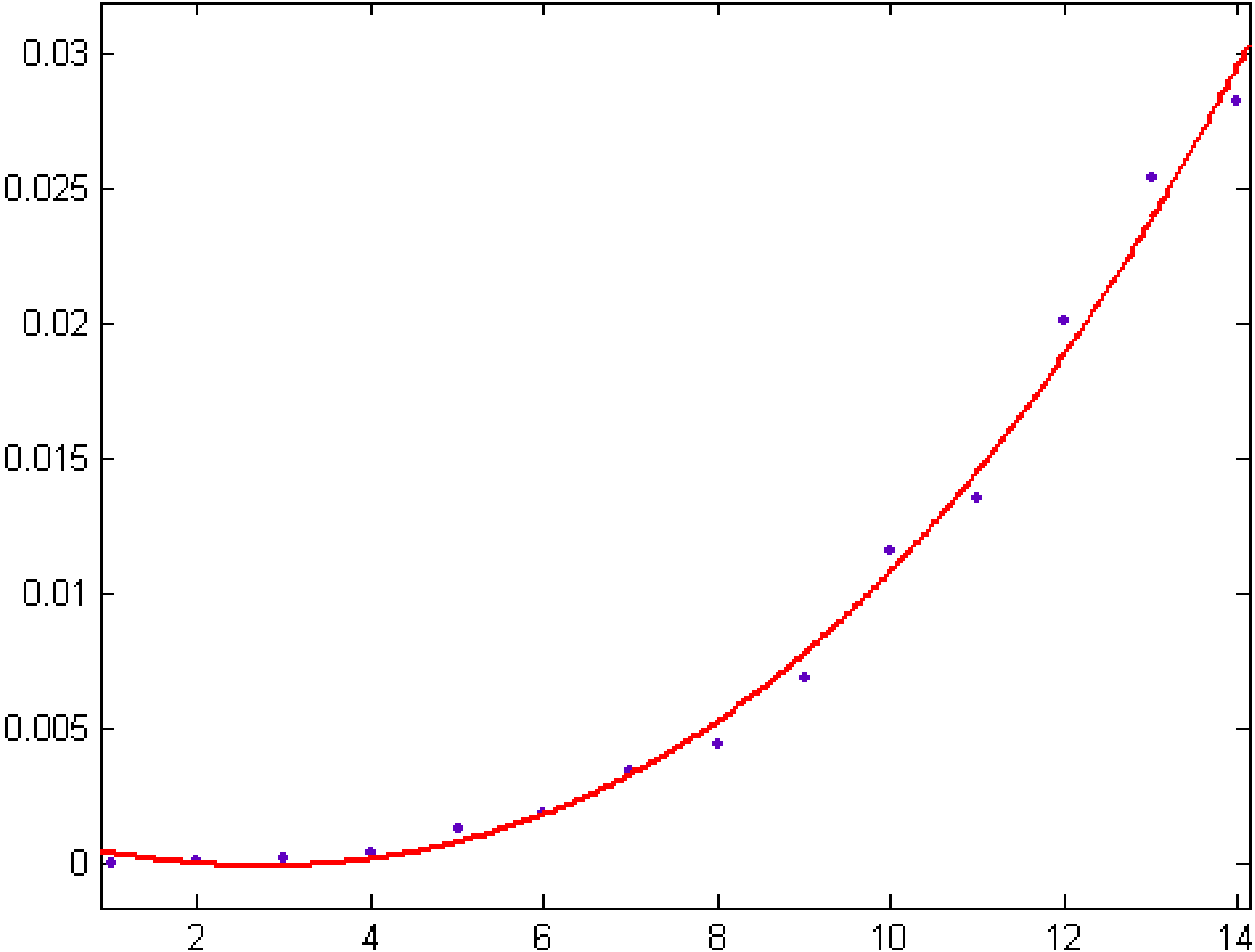
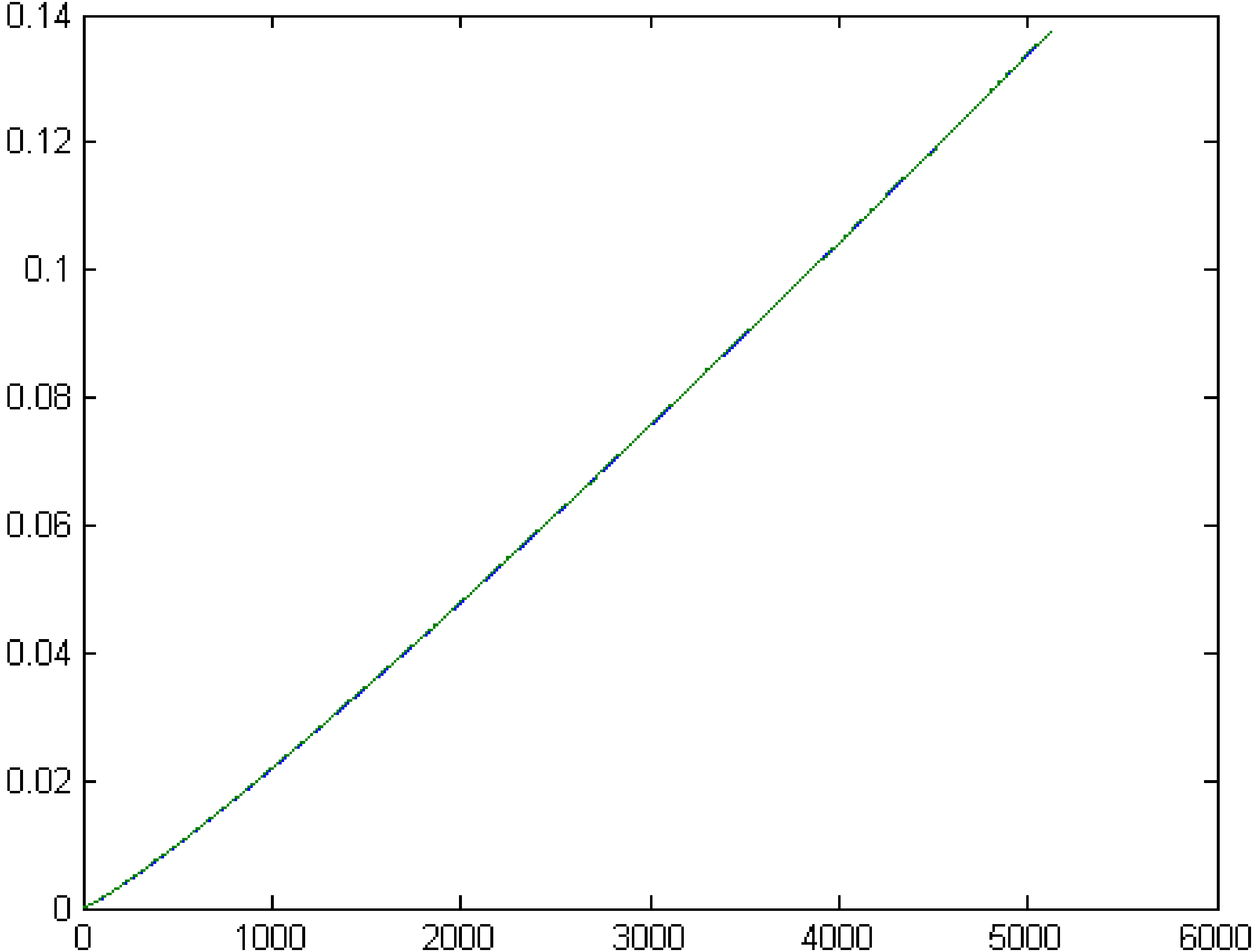
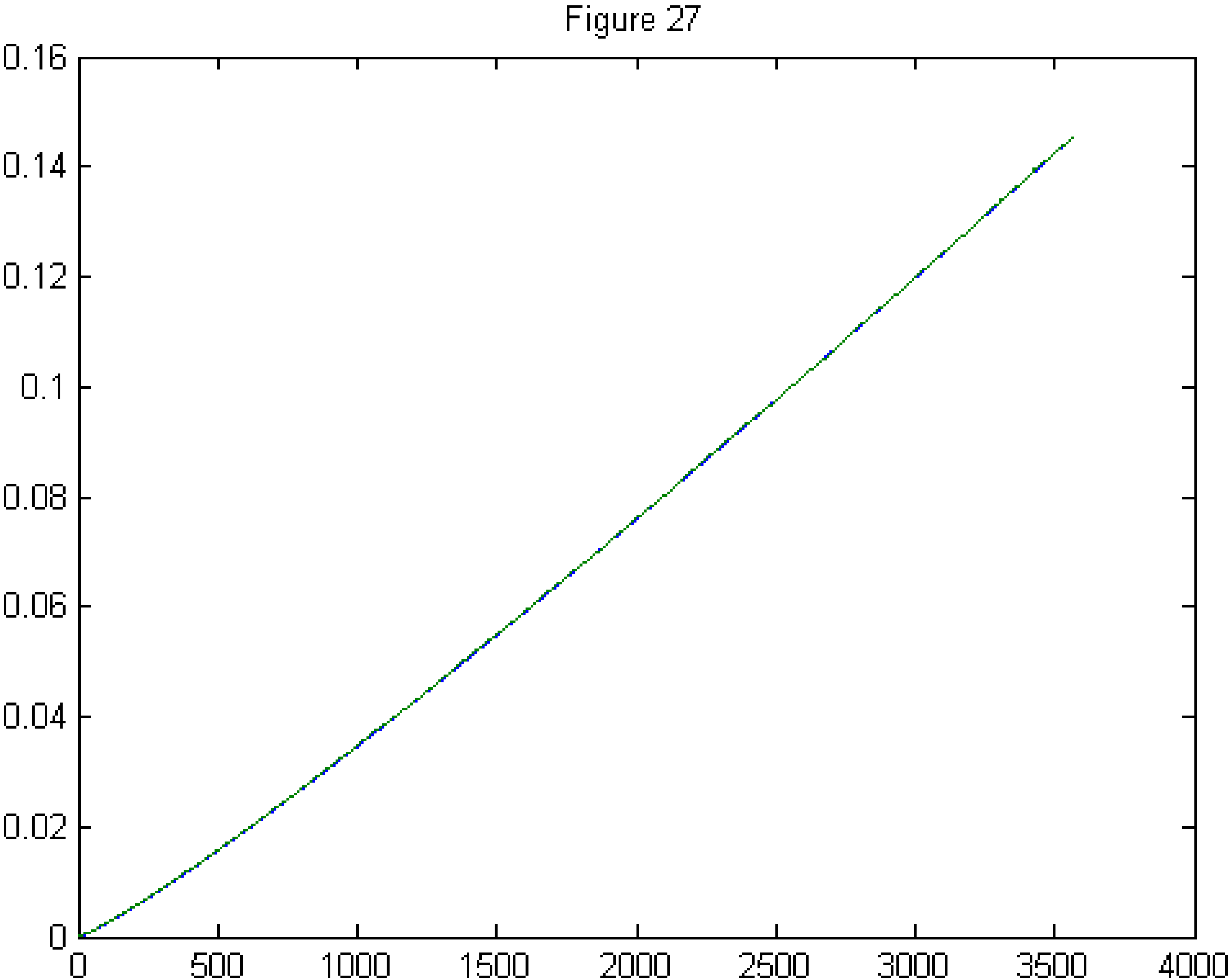


Figure 26

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0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
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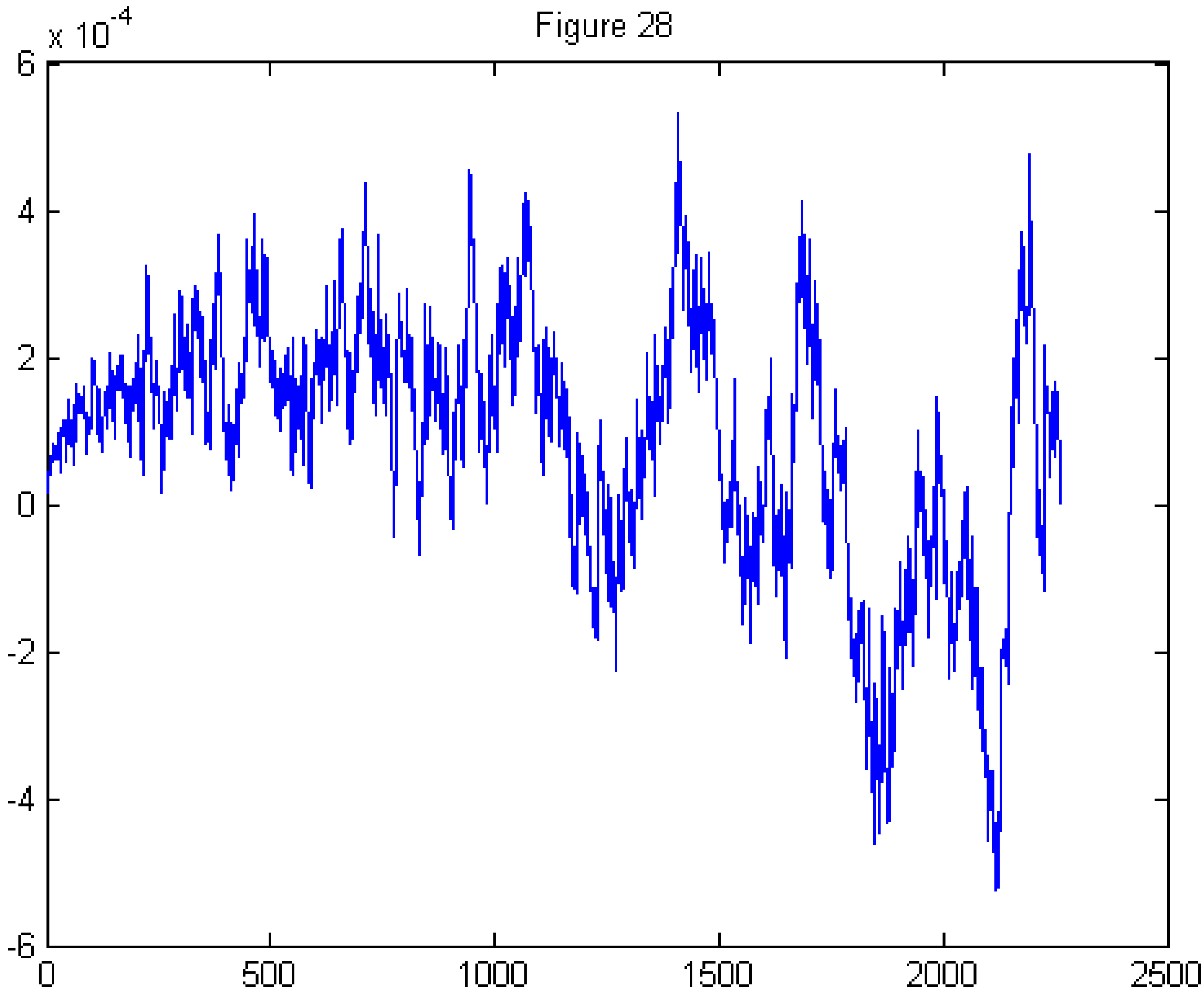
1000
1500
2000
2500
3000
3500
4000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x 10^-4
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2
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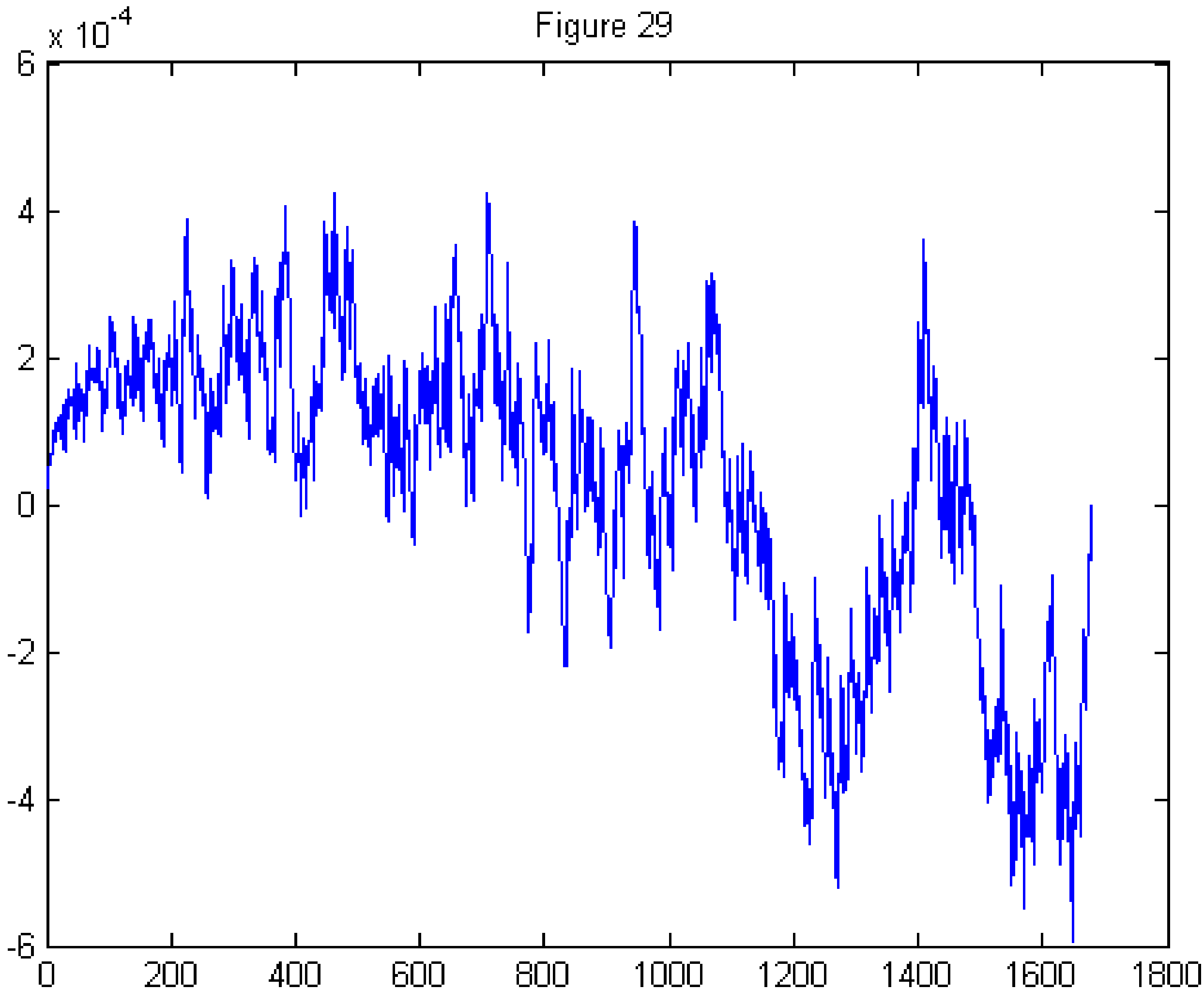
-2
-4
-6
0
500
1000
1500
2000
2500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x 10^-4
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4
2
0
-2
-4
-6
0
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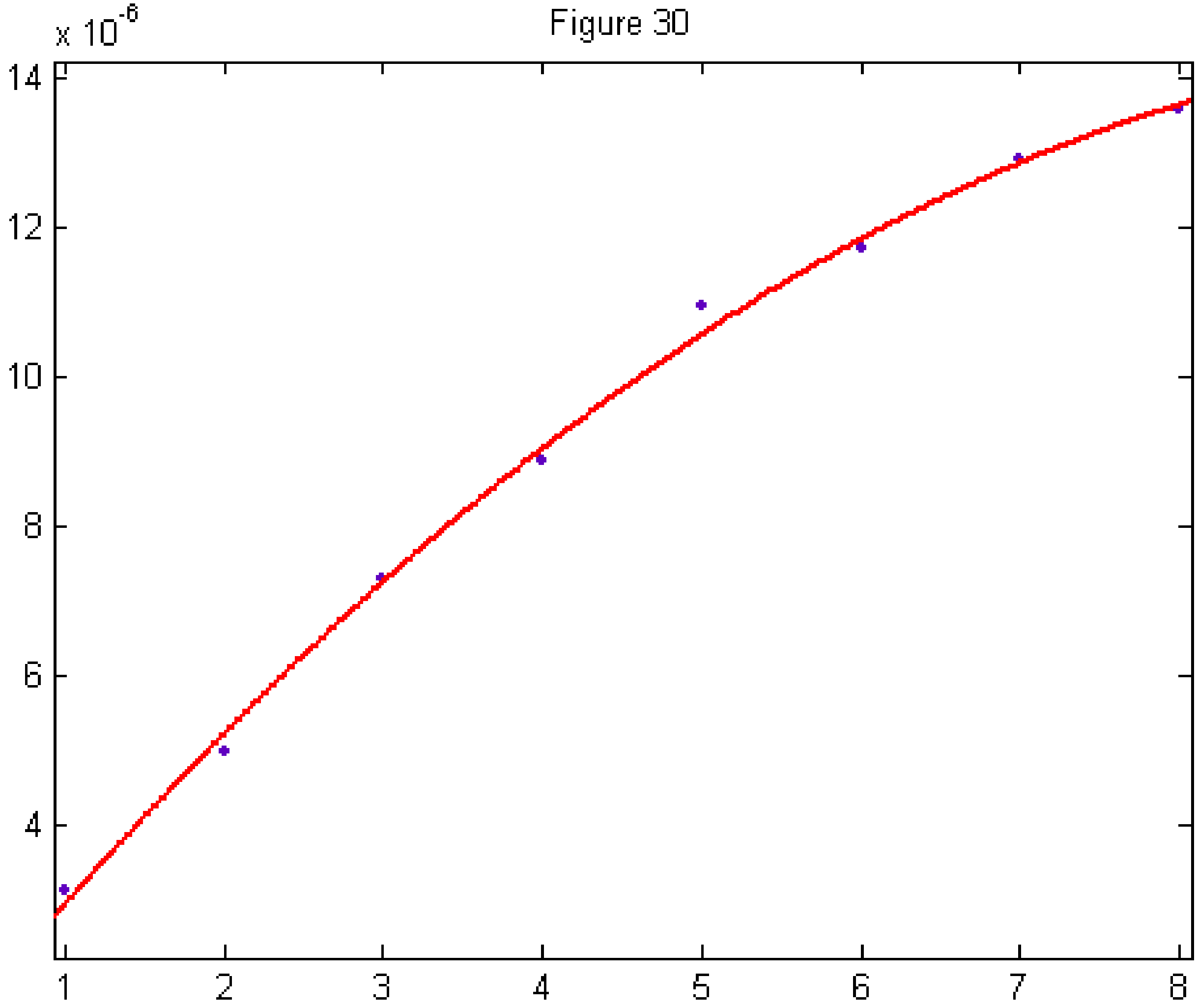
400
600
800
1000
1200
1400
1600
1800



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x 10^-6
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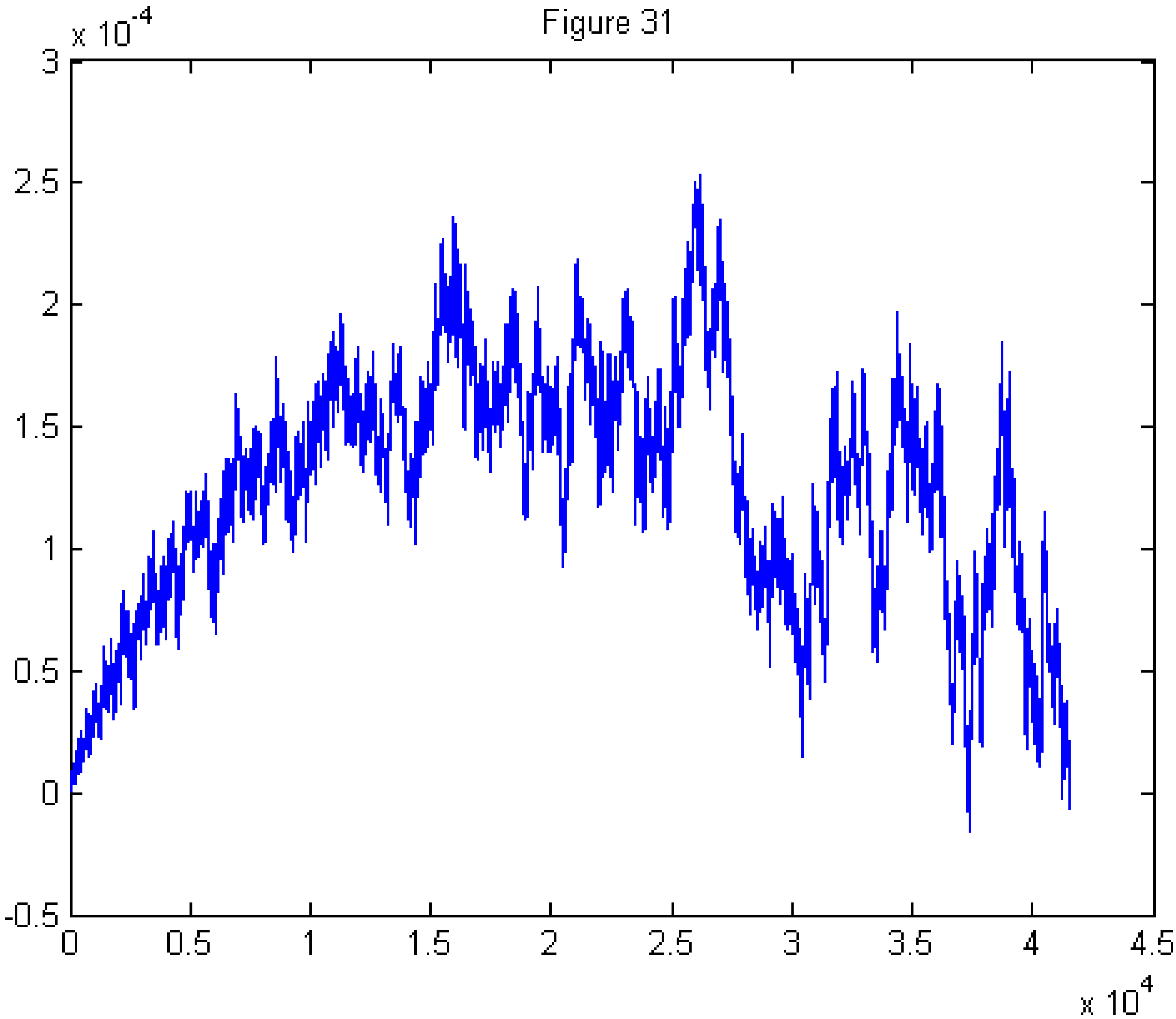
12
10
8
6
4
1
2
3
4
5
6
7
8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x 10^-4
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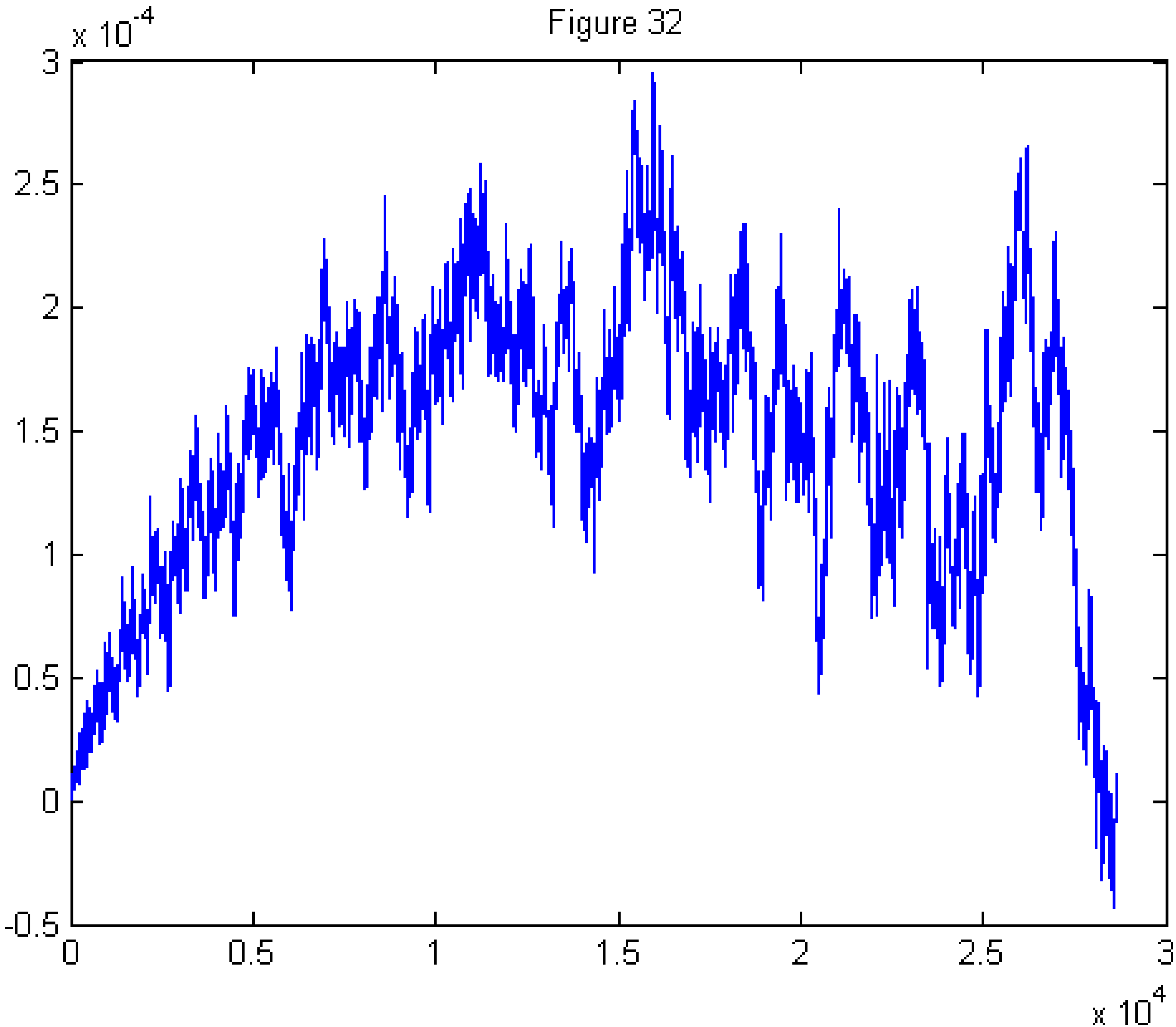
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
x 10^4



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x 10^-4
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
x 10^4

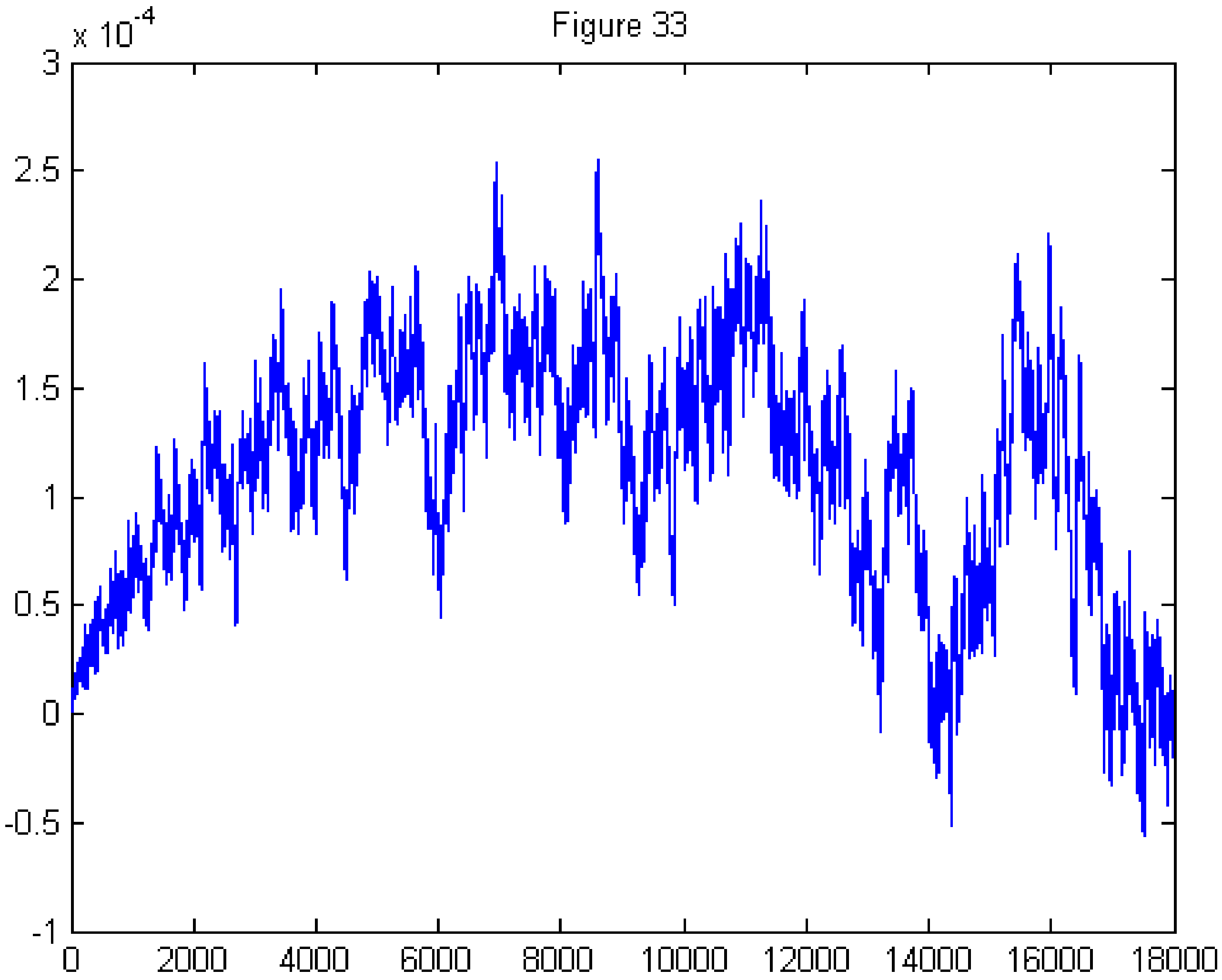
Figure 33
x 10^-4
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2.5
2
1.5
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0.5
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-0.5
-1
0
2000
4000
6000
8000
10000
12000
14000
16000
18000

Figure 34

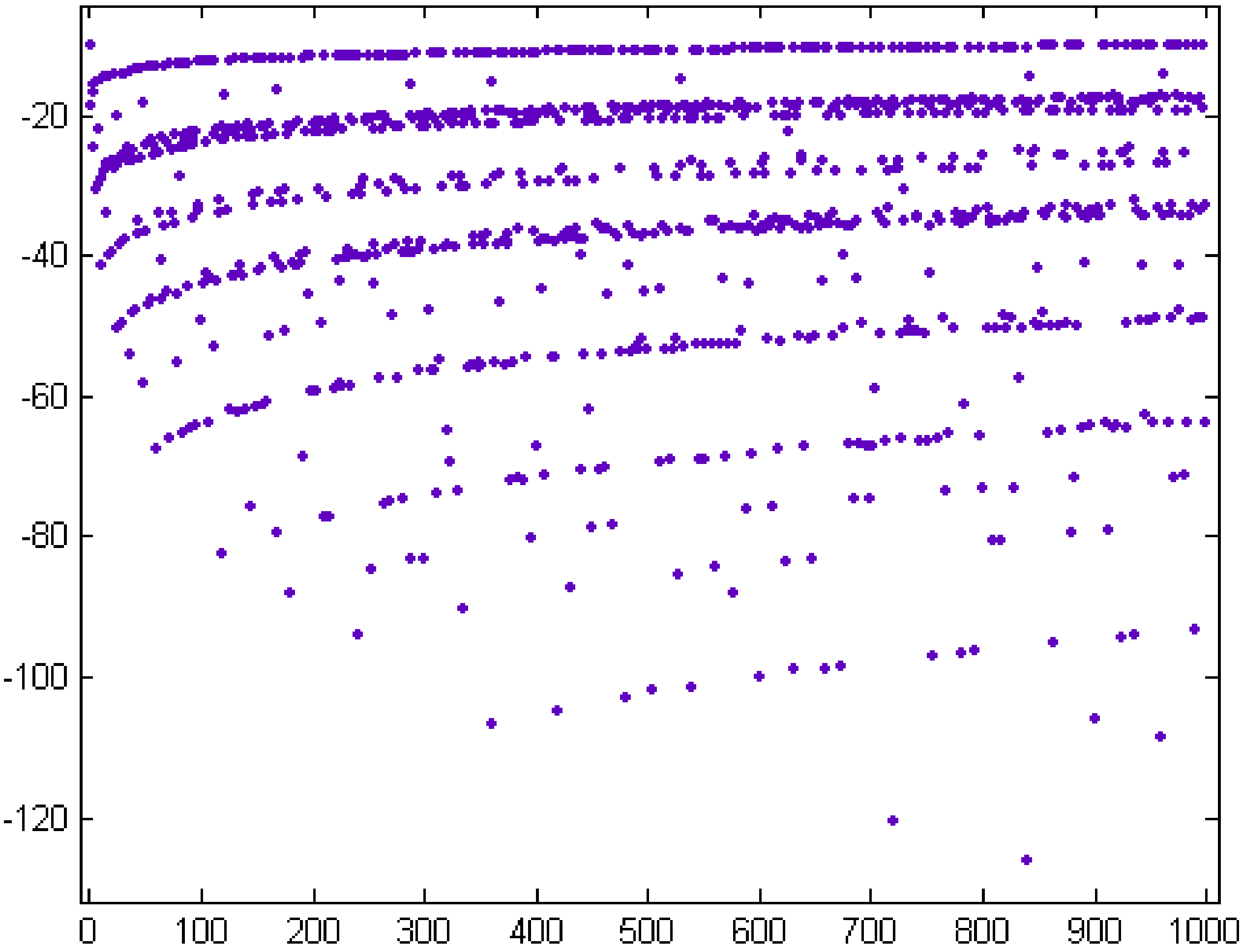


Figure 35

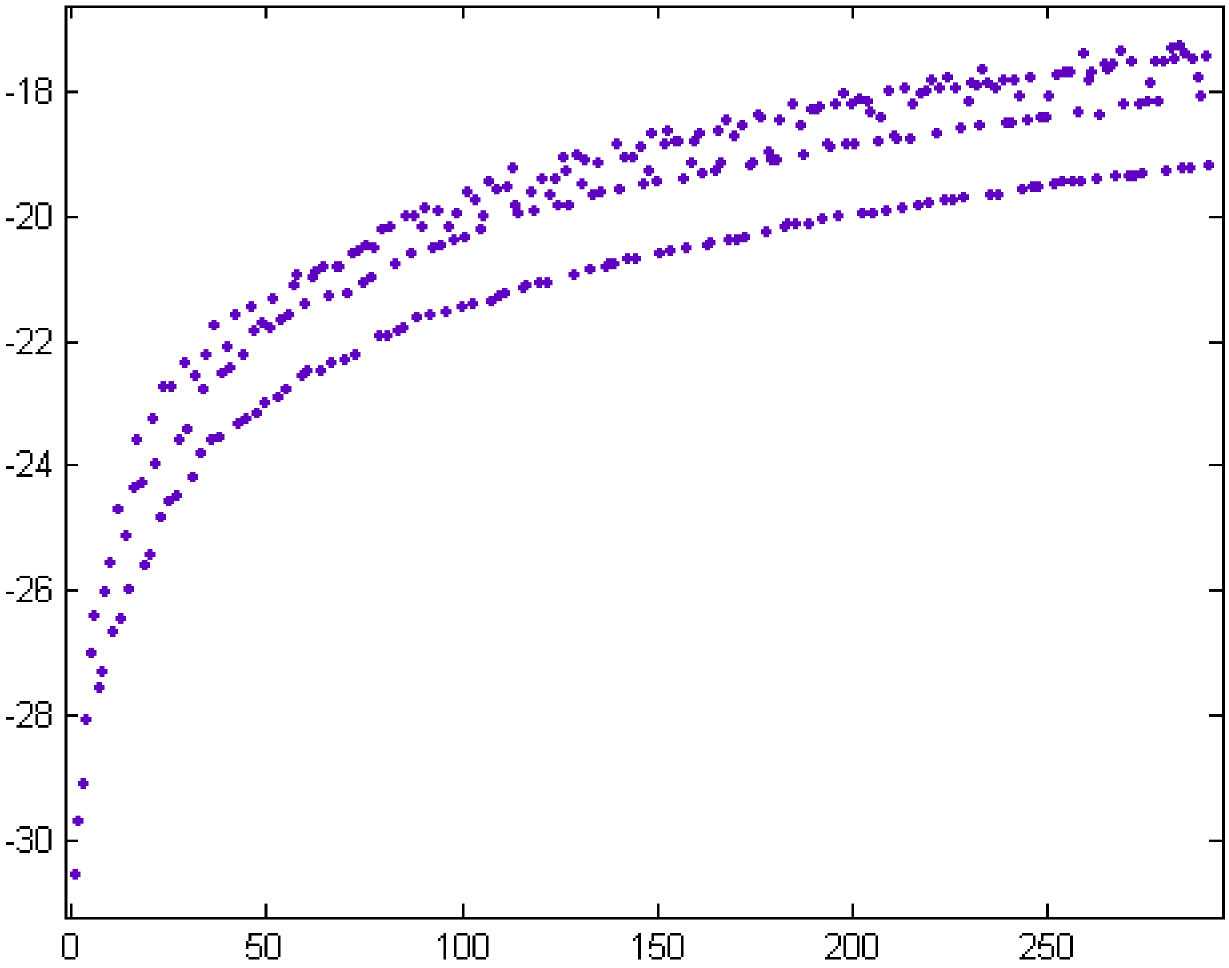
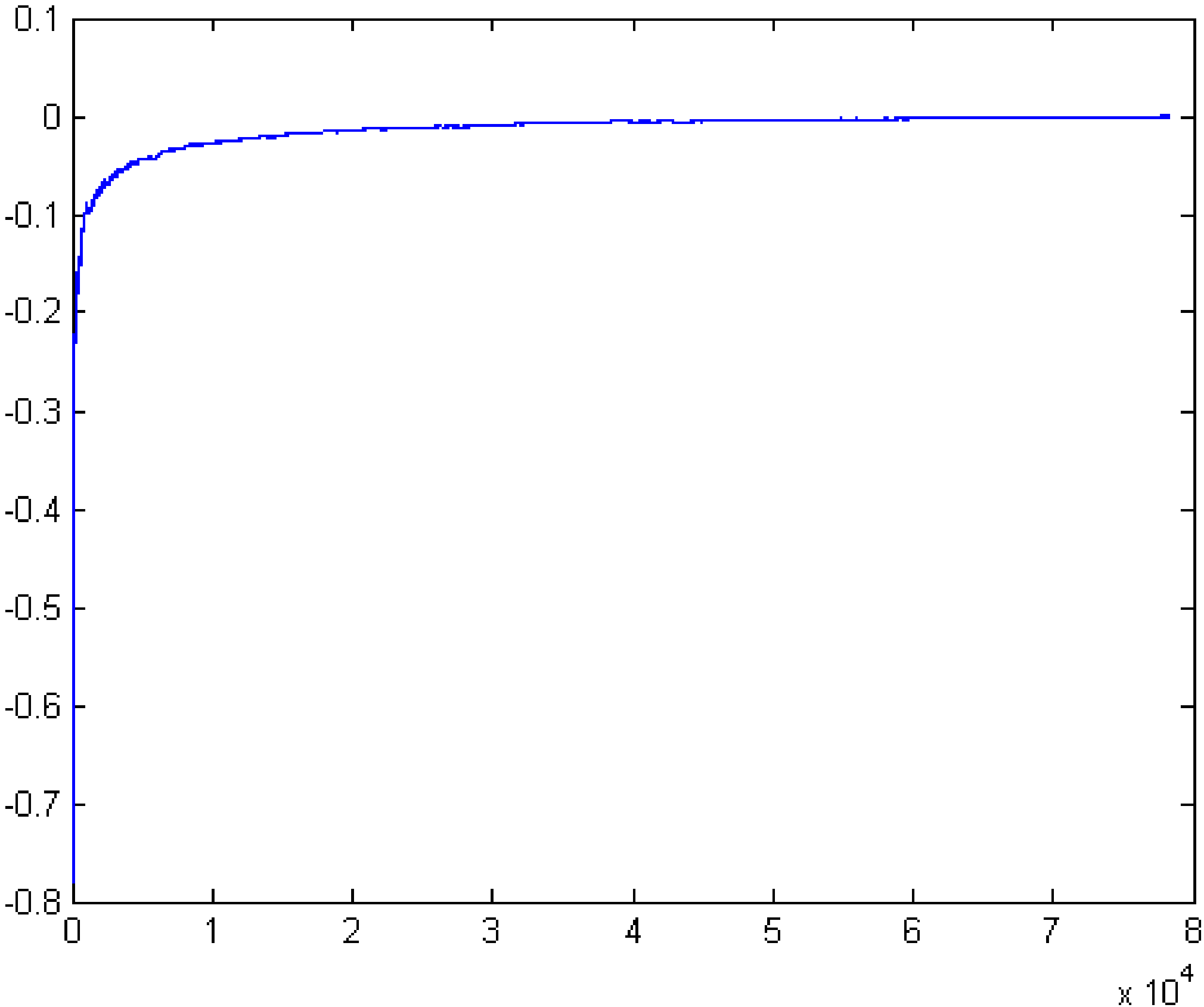
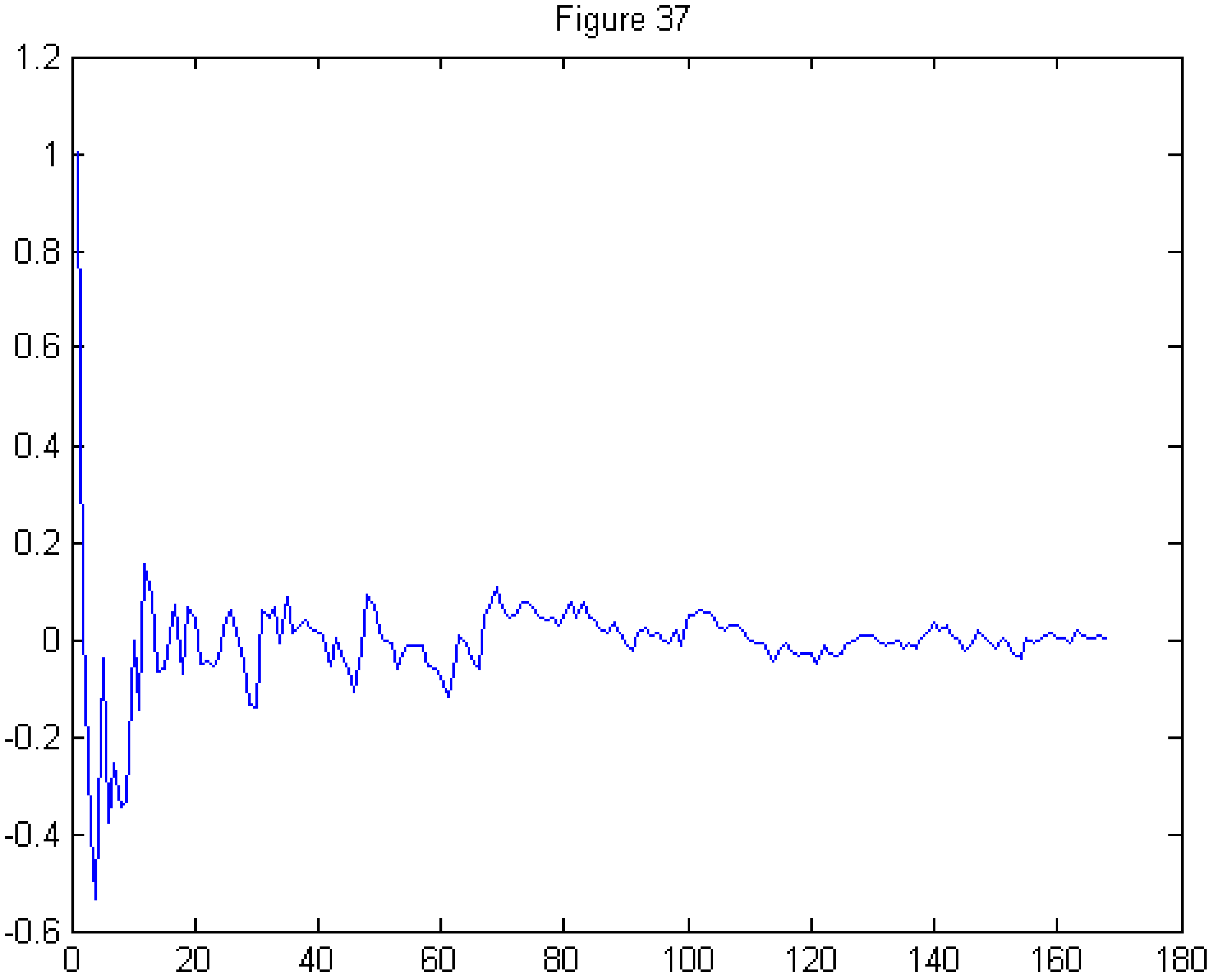


Figure 36

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1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
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100
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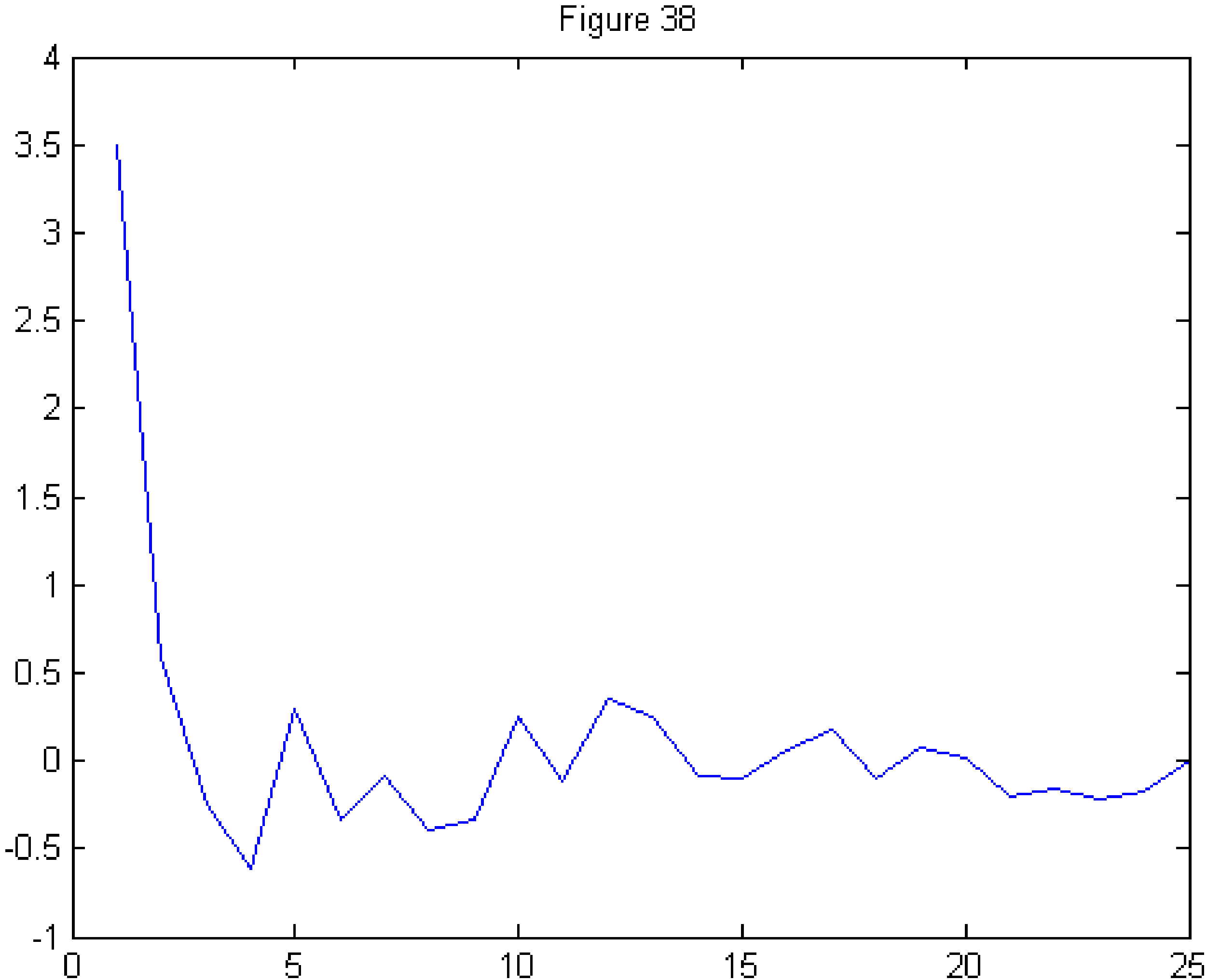
140
160
180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3.5
3
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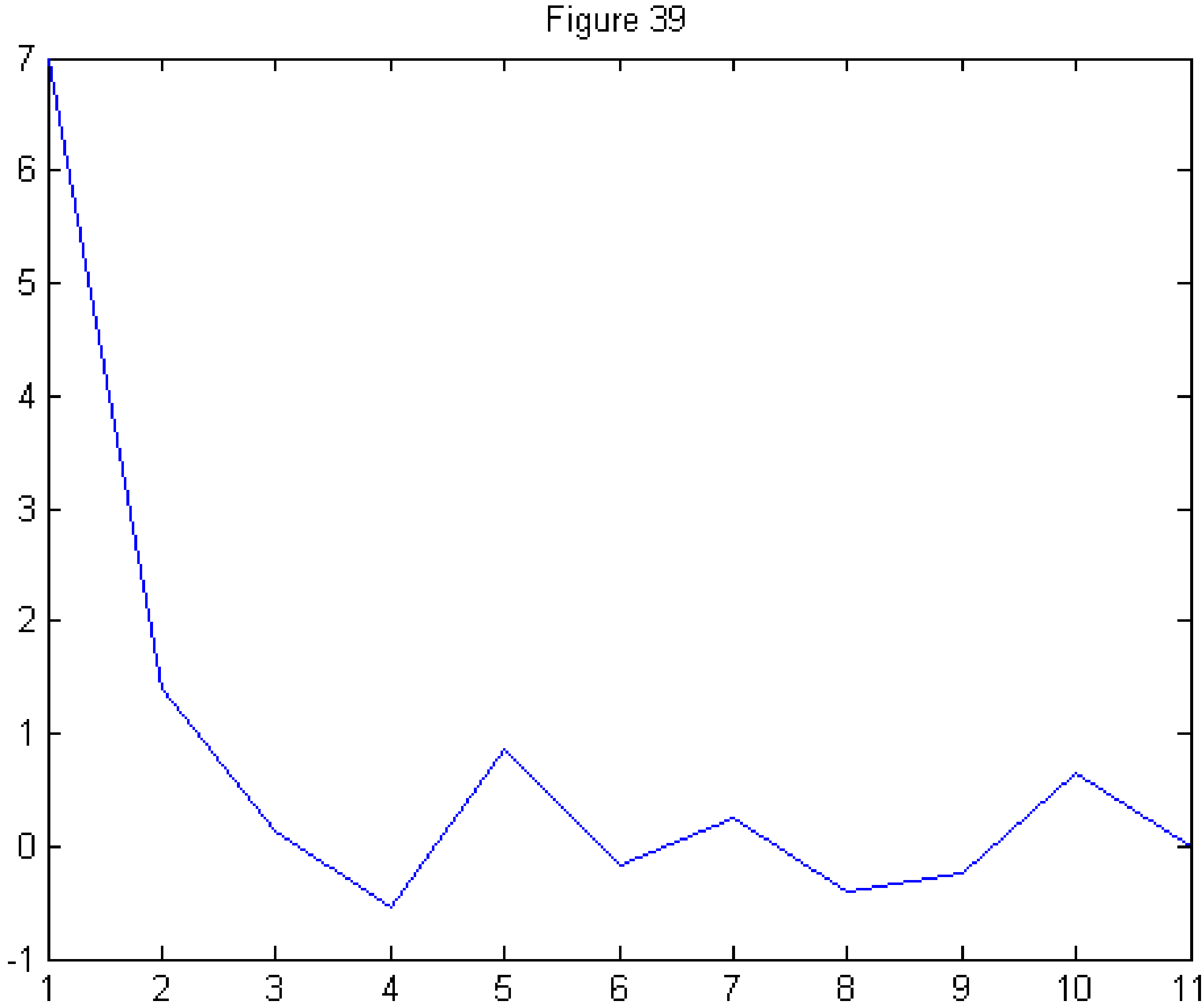
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25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6
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1
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1
2
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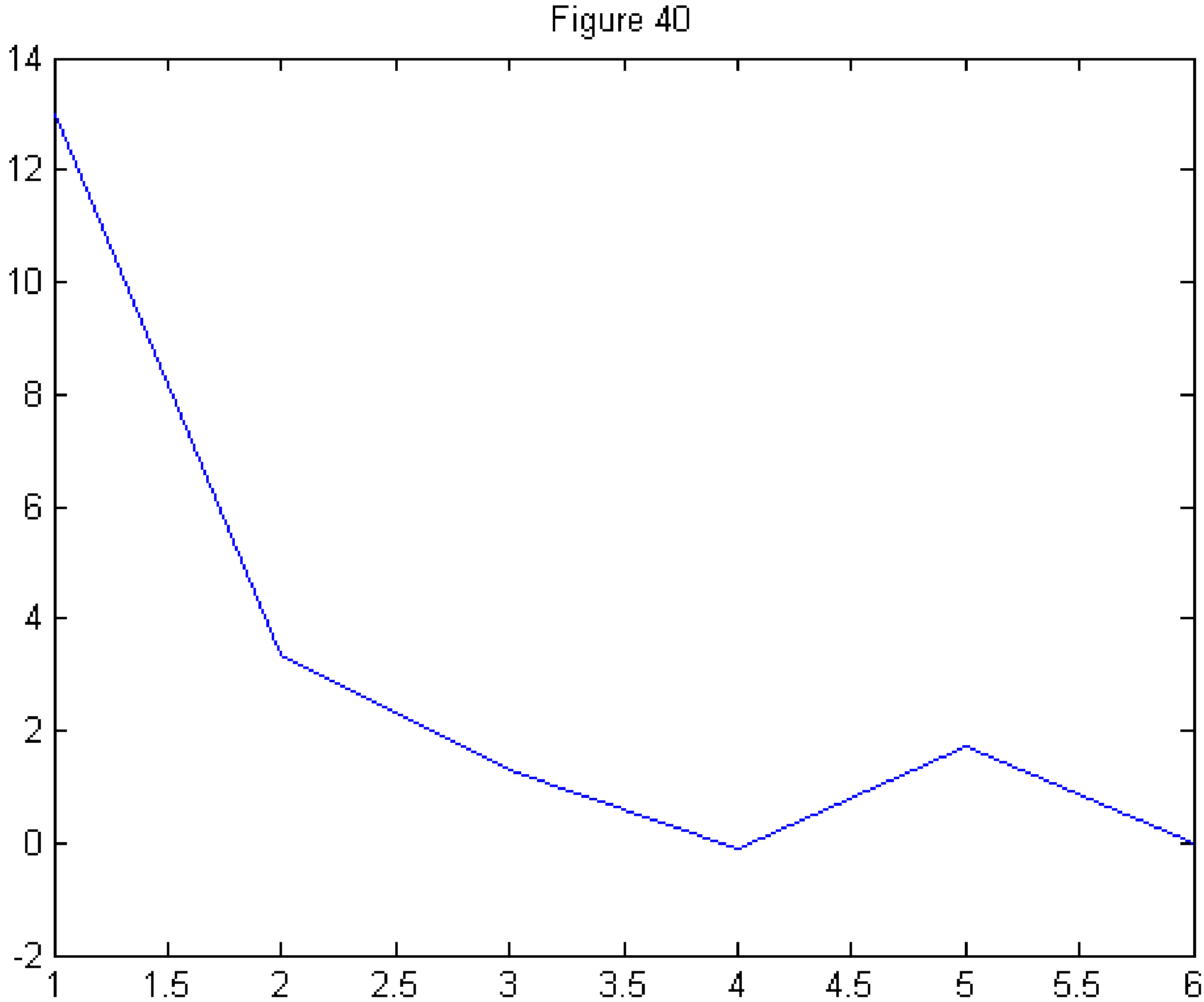
7
8
9
10
11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12
10
8
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2.5
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3.5
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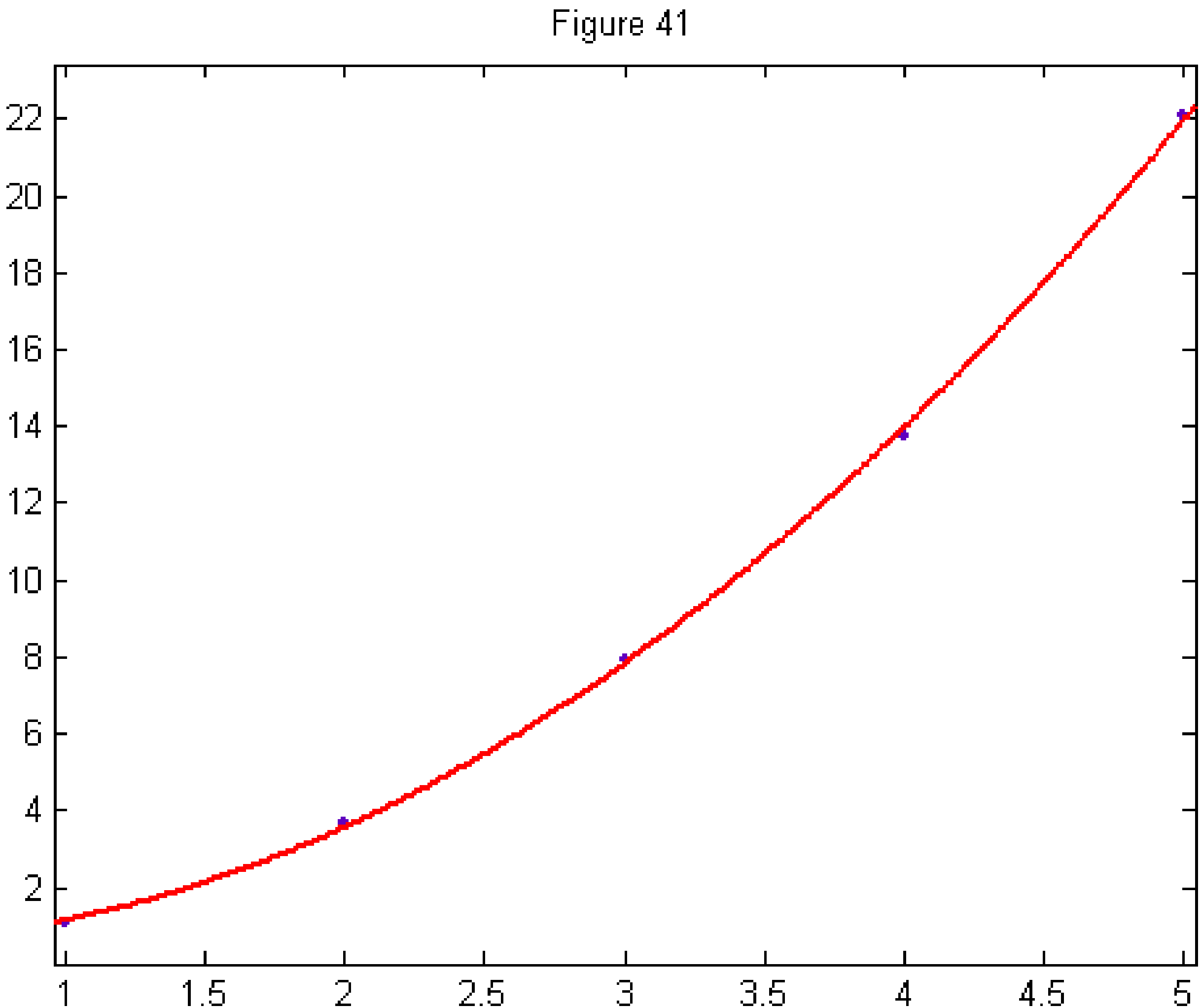
4.5
5
5.5
6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20
18
16
14
12
10
8
6
4
2
1
1.5
2
2.5
3
3.5
4
4.5
5

Figure 42

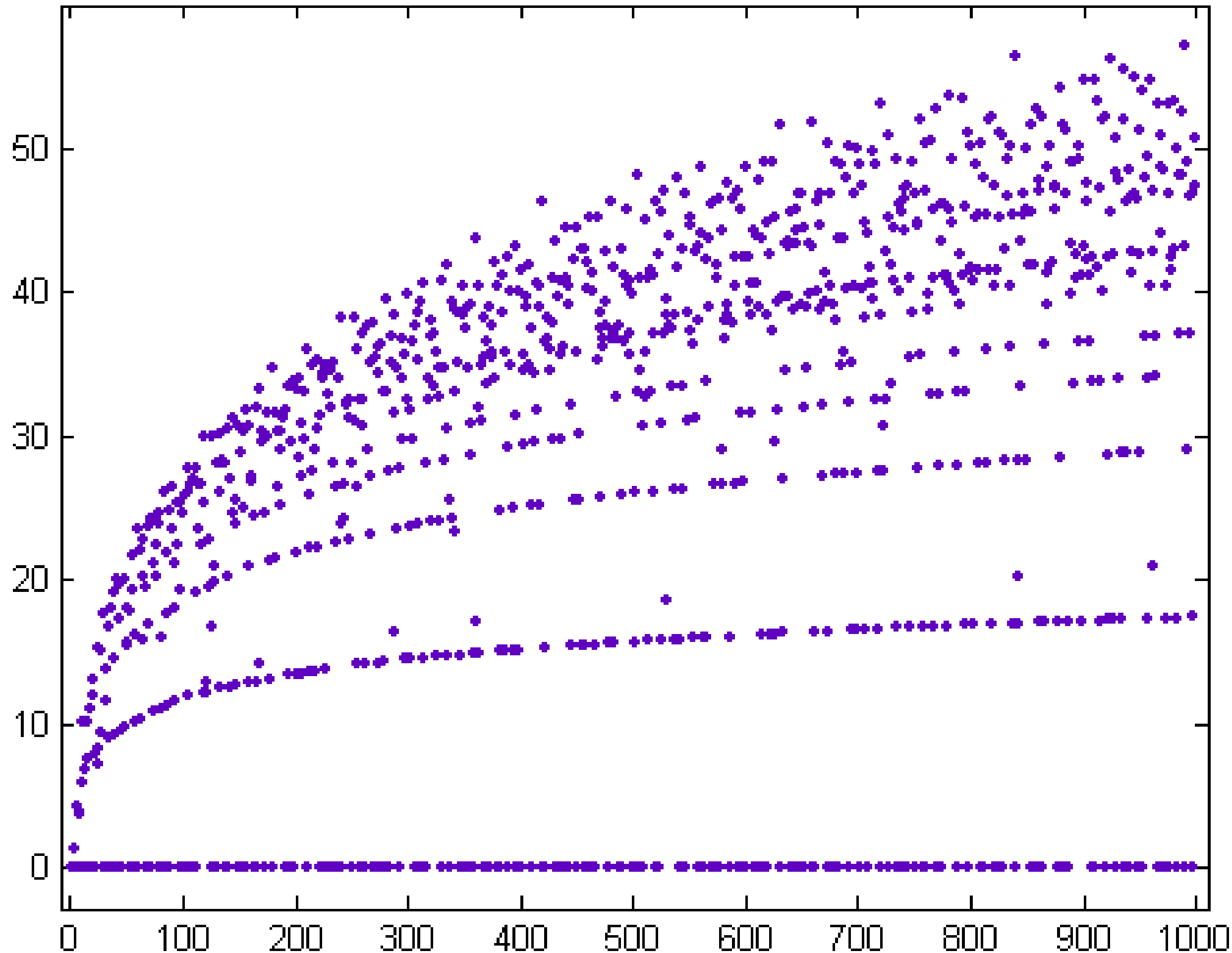


Figure 43

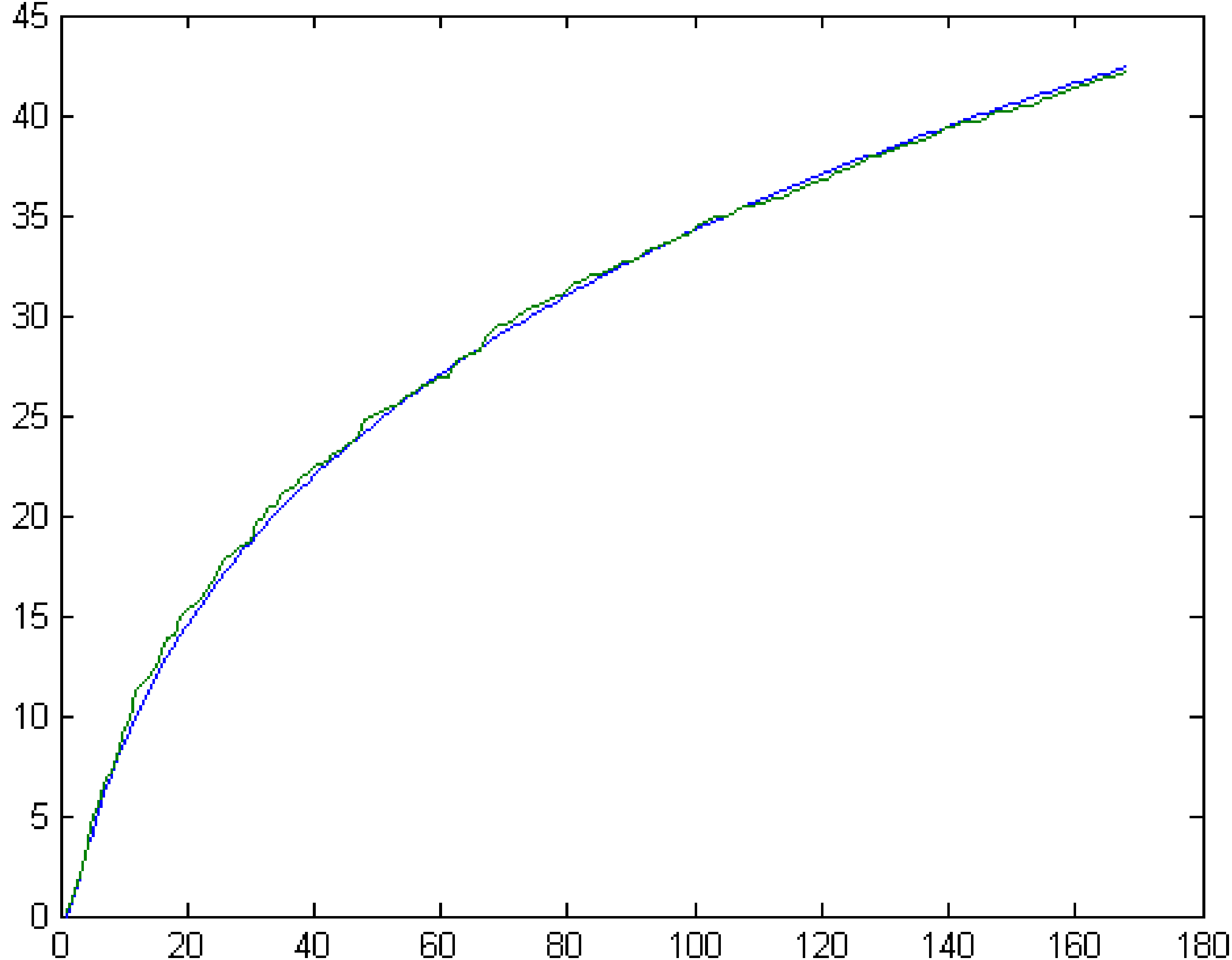


Figure 44

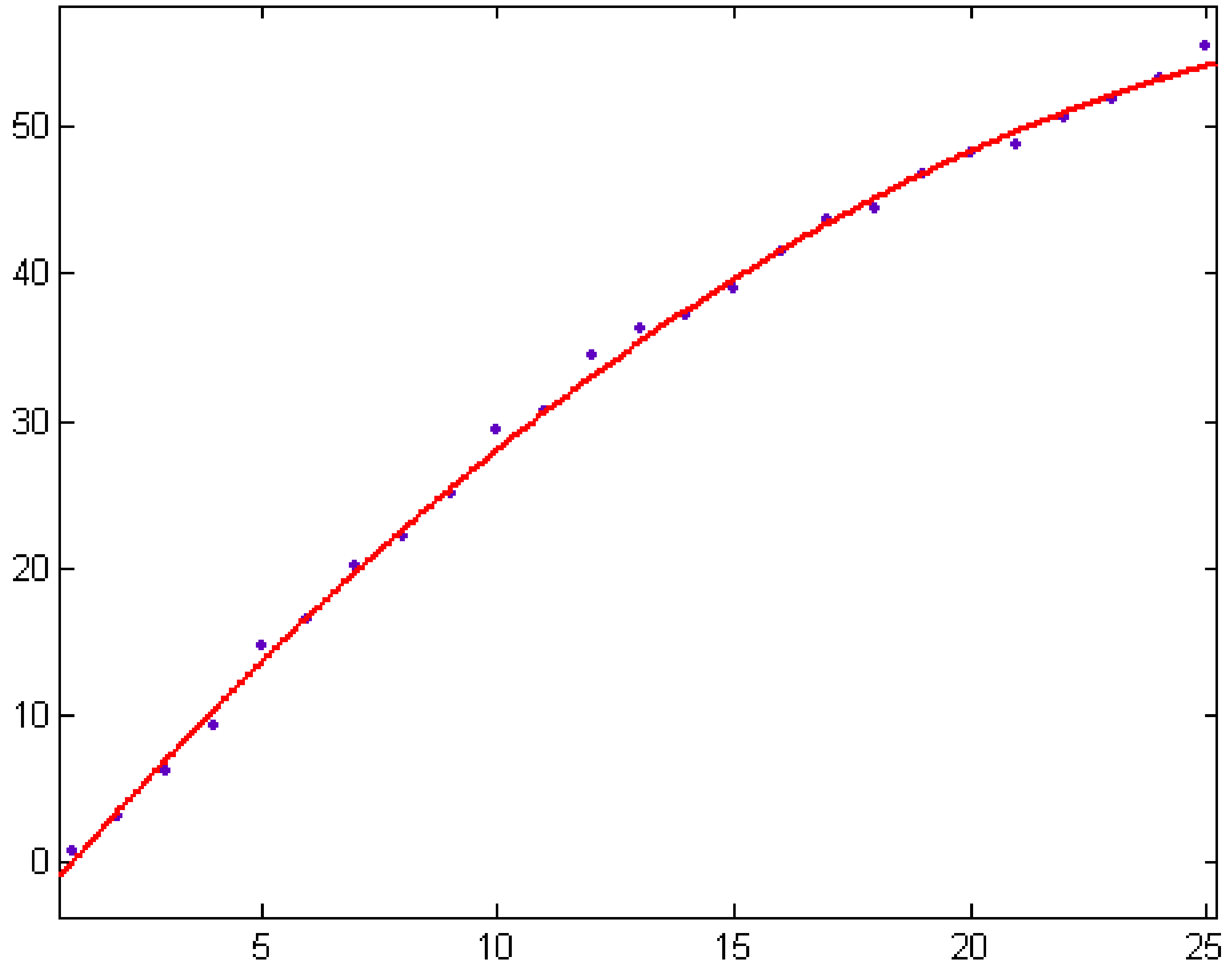


Figure 45

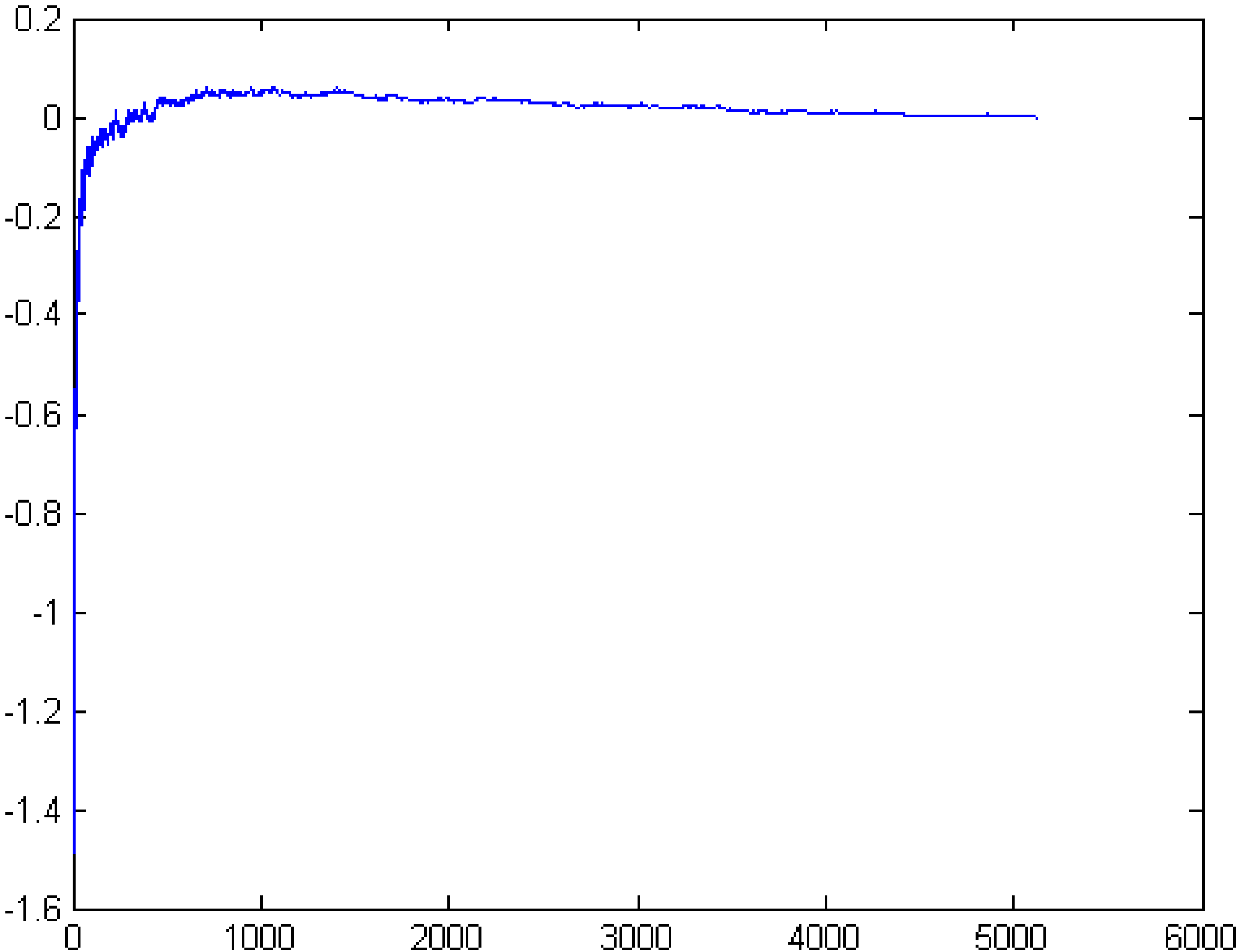


Figure 46

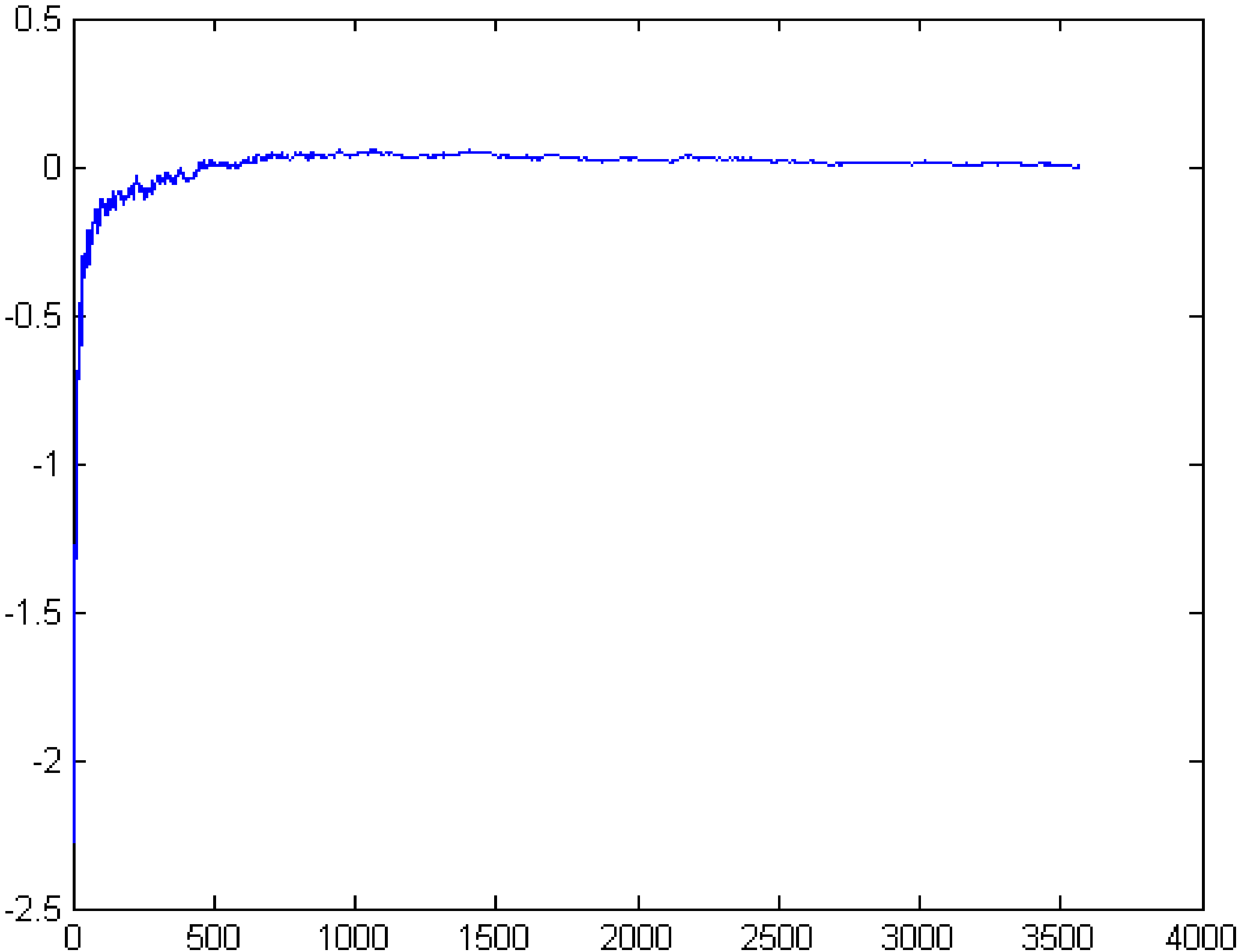
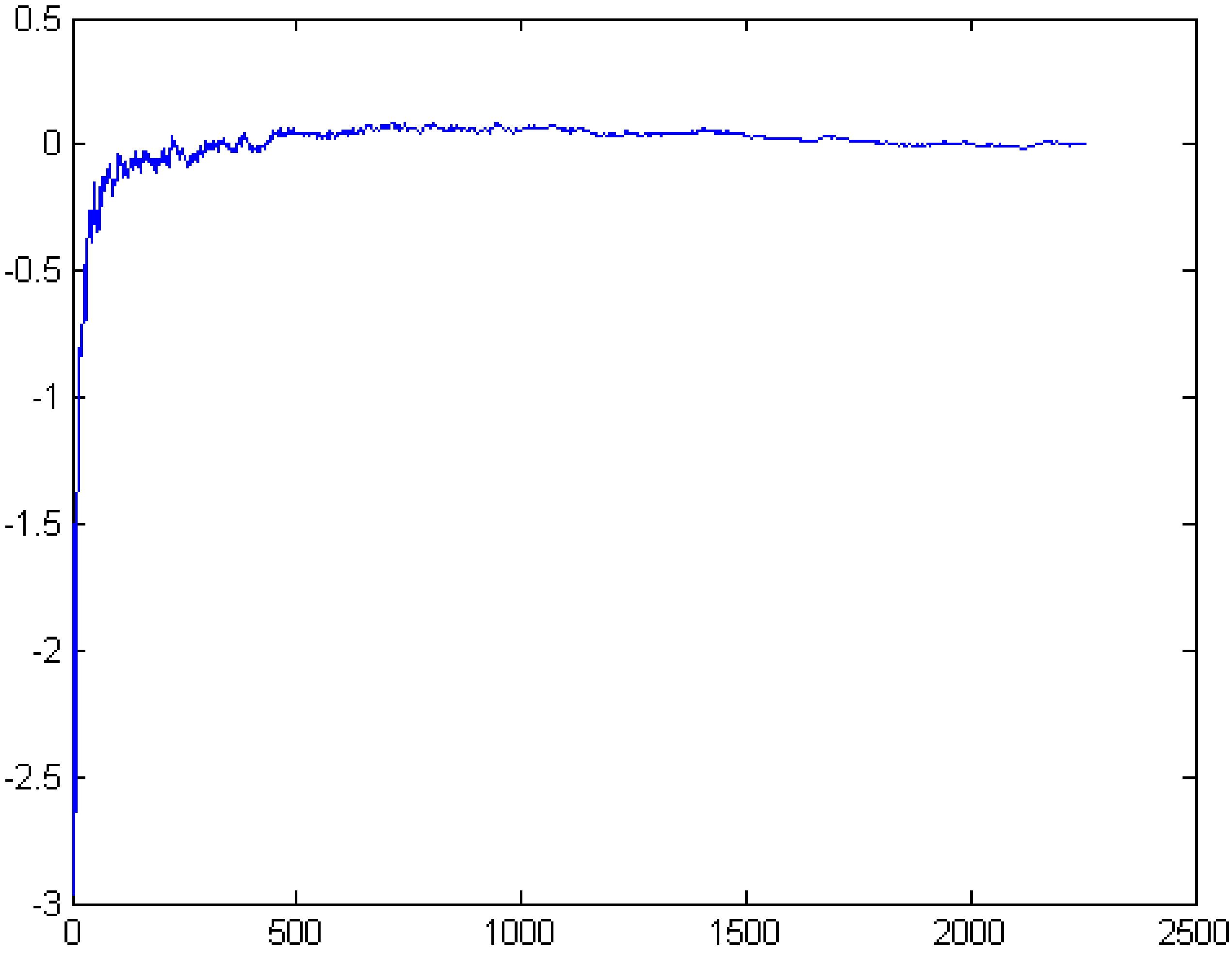


Figure 47



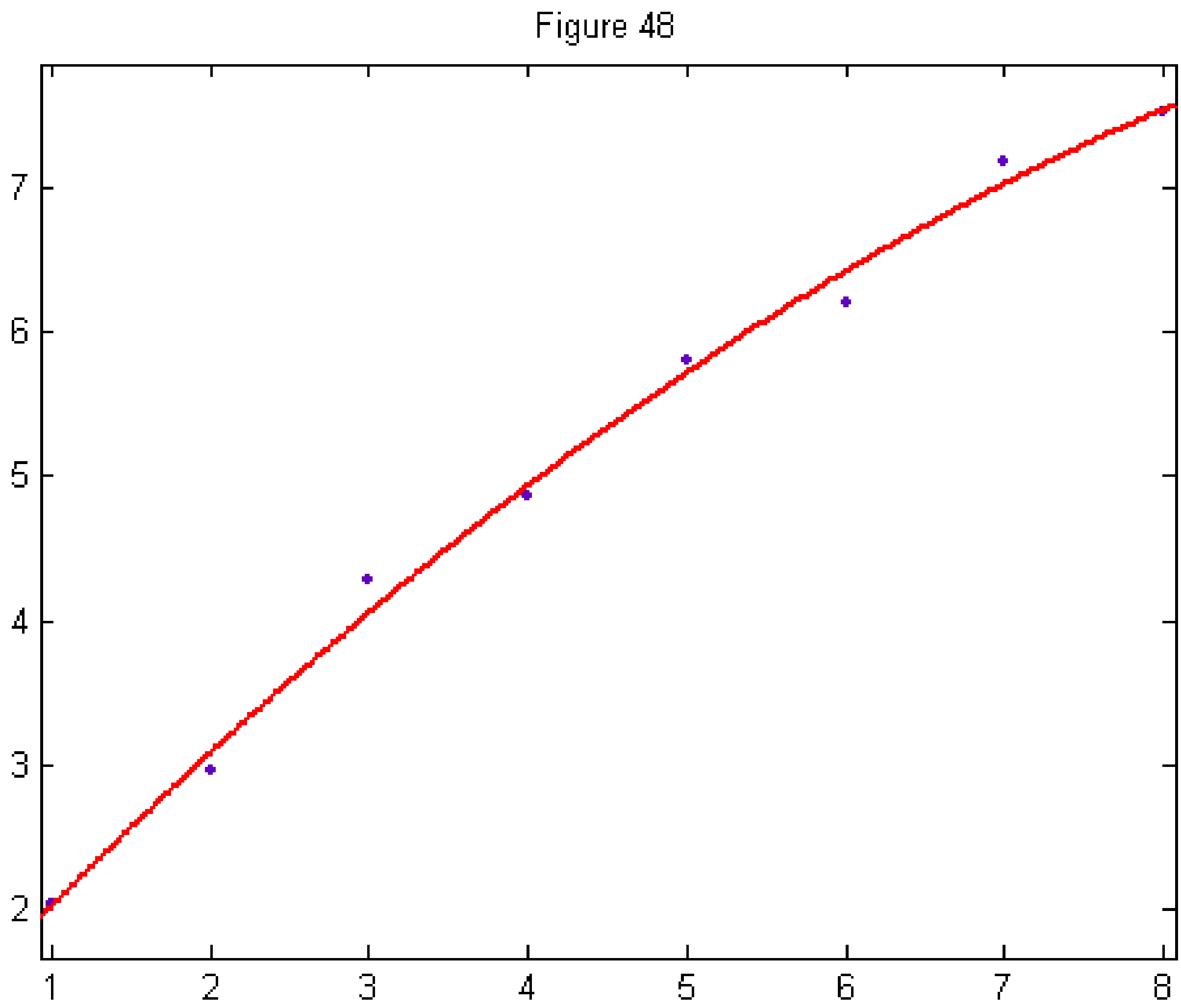


Figure 48
1
2
3
4
5
6
7
8
2
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4
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6
7