An Elementary Technique for Finding Zeta Function Zeros

Darrell Cox

February 18, 2018

Abstract

Given some zeta function zeros, the uniform distribution of the zeros is used to approximate other zeros. Certain parameters would have to be known beforehand for this technique to be of practical value. The technique to be described uses a generalized sequence of zeros - one where values have been linearly interpolated between successive zeros. Dirichlet products of this generalized sequence retain some of the properties of the original sequence (this appears to be a property of sequences with a normal distribution). Normal probability distributions and sigmoid functions (probit functions in particular) involving the zeros are also investigated. The distribution of the zeta function zeros is modeled fairly accurately.

1 Introduction

In 1903, Gram [1] published a list of the first 15 zeros of Riemann's $\zeta(s)$ function on the line Re(s)=$\frac{1}{2}$. He computed these zeros using Euler-MacLaurin summation. Gram's work was carried further by Backlund [[2], [3]] around 1912-1915. Backlund's major contribution was a method of computing, for certain values of T, the number of roots in the range $0 \leq \text{Im } s \leq T$. About 10 years later, the Riemann hypothesis was verifed up to the level $T = 300$ by Hutchinson [4], who contributed some improvements of Gram's and Backlund's methods. Hutchinson computed the first 138 zeros. Let θ_1, θ_2, θ_3, ... denote the imaginary parts of the nontrivial zeros of the Riemann zeta function (Odlyzko's [5] tables were used). Let $\kappa_1(1)$, $\kappa_1(2)$, $\kappa_1(3)$, ..., denote θ_1, θ_2, θ_3, ... and let $\kappa_n(x)$, $n = 2$, 3, 4, ..., denote these values and $n-1$ values that have been linearly interpolated between successive values.

2 Dirichlet Products Involving κ

See Figure (1) for a plot of $\sum_{i|x} \log(\kappa_1(x/i))$ for $x = 1, 2, 3, ..., 135$. (Note that all but 3 of the zeros Hutchinson computed were used.) The plot consists of 12 different slowly increasing "curves" with gaps in them. Curve #0 (numbering from the bottom) consists of a single element at $x = 1$ and having a value of $\log(\theta_1)$. Curve #1 consists of elements at $x = 2, 3, 5, ...$ (the primes) and having values of $\log(\theta_1) + \log(\theta_x)$. Curve #2 consists of elements at $x = 2^2$, 3^2, 5^2, ... Curve #3 consists of elements at $x = 2^3$, 3^3, 5^3, ... and x values that are the product of two distinct primes. Curve #4 consists of elements at $x = 2^4$, 3^4, 5^4, ... Curve #5 consists of elements at $x = 2^5$, 3^5, 5^5, ... and x values that are the product of the square of a prime and a different prime. Curve #6 consists of elements at $x = 2^6$, 3^6, 5^6, ... Curve #7 consists of elements at $x = 2^7$, 3^7, 5^7, ... , x values that are the product of the cube of a prime and a different prime, and x values that are the product of three distinct primes. Curve #8 (not shown) consists of elements at $x = 2^8$, 3^8, 5^8, ... Curve #9 consists of elements at $x = 2^9$, 3^9, 5^9, ... , x values that are the product of the fourth power of a prime and a different prime, and x values that are the product of the square of a prime and the square of a different prime (a secondary curve). The values of the secondary curve are somewhat less than those of the primary curve. Curve #10 (not shown) consists of elements at $x = 2^{10}$, 3^{10}, 5^{10}, ... Curve #11 consists of elements at $x = 2^{11}$, 3^{11}, 5^{11}, ... , x values that are the product of the fifth power of a prime and a different prime, x values that are the product of the cube of a prime and the square of a different prime, and x values that are the product of the square of a prime and two other distinct primes. Curve #12 (not shown) consists of elements at $x = 2^{12}$, 3^{12}, 5^{12}, ... Curve #13 consists of elements at $x = 2^{13}$, 3^{13}, 5^{13}, ..., x values that are the product of the sixth power of a prime and a different prime (a secondary curve), x values that are the product of the fourth power of a prime and the square of a different prime (another secondary curve), x values that are the product of the cube of a prime and the cube of a different prime, x values that are the product of the cube of a prime and two other distinct primes, and x values that are the product of four distinct primes. The x values other than those in the secondary curves constitute the primary curve. The first secondary curve has smaller values than those of the second secondary curve. Curve #14 (not shown) consists of elements at $x = 2^{14}$, 3^{14}, 5^{14}, ... Curve #15 (not shown) consists of elements at $x = 2^{15}$, 3^{15}, 5^{15}, ..., x values that are the product of the seventh power of a prime and a different prime (a secondary curve), x values that are the product of the fifth power of a prime and the square of a different prime, x values that are the product of the fourth power of a prime and the cube of a different prime (the primary curve), x values that are the product of the fourth power of a prime and two other distinct primes, and x values that are the product of the square of a prime, the square of a different prime, and another distinct prime. The x values other than those in the first secondary curve and the primary curve constitute the second secondary curve. The first secondary curve has smaller values than those of the second secondary curve. The first secondary curve coincides with the primary curve in Curve #13. Curve #16 (not shown) consists of elements at $x = 2^{16}$, 3^{16}, 5^{16}, ... Curve #17 (not shown) consists of elements at $x = 2^{17}$, 3^{17}, 5^{17}, ..., x values that are the product of the eighth power of a prime and another prime (a secondary curve), x values that are the sixth power of a prime and the square of another prime (another secondary curve), x values that are the product of the fifth power of a prime and the cube of another prime, x values that are the product of the cube of a prime, the square of another prime, and another distinct prime, and x values that are the product of the square of a prime and three other distinct primes. The x values other than those in the secondary curves constitute the primary curve. The first secondary curve has smaller values than those of the second secondary curve. The first secondary curve coincides with the primary curve in Curve #15. In general, Curve #k consists of at least elements at $x = 2^k$, 3^k, 5^k, ... The logarithm of the κ values was used to help identify the curves.

See Figure (2) for a plot of $\sum_{i|x} \kappa_1(x/i)$ for x = 1, 2, 3, ..., 135. (Note that Möbius inversion can be used to regenerate the κ values from these curves.) See Figure (3) for a plot of $\sum_{i|x} \kappa_9(x/i)$ for x = 1, 2, 3, ..., 135. (Note that only the zeros computed by Gram were used.) This plot has almost the same pattern as the plot in Figure (2), but the magnitudes of the points are smaller. Let y_1, y_2, y_3, ..., y_{32} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x = 2, 3, 5, ..., 131 and let z_1, z_2, z_3, ..., z_{32} equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at x = 2, 3, 5, ..., 131. See Figure (4) for a plot of y and z. For a linear least-squares fit of y, R-square=0.998 and for a linear least-squares fit of z, R-square=0.9972. To compare the curves, they would have to be first normalized (reduced in value by subtracting the first element of the array from all the elements) and then the normalized z values could be scaled up by multiplying by the ratio of the last elements of the normalized arrays. Note that these are the aforementioned parameters needed to predict the y values using the z values. Let y_1, y_2, y_3, ..., y_5 equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at $x = 2^2$, 3^2, 5^2, ..., 11^2 and let z_1, z_2, z_3, ..., z_5 equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at $x = 2^2$, 3^2, 5^2, ..., 11^2. See Figure (5) for a plot of y and z. For a quadratic least-squares fit of y, R-square=0.9915 and for a quadratic least-squares fit of z, R-square=0.9927. (For the corresponding 65 y values obtained when 100000 zeros are used, a quadratic least-squares fit gives R-square=0.9988.) Let y_1, y_2, y_3, ..., y_{43} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes and let z_1, z_2, z_3, ..., z_{43} equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes. See Figure (6) for a plot of y and z. See Figure (7) for a plot of the normalized and scaled y and z values. See Figure (8) for a plot of the difference between the normalized and scaled y and z values divided by the normalized y values (where the first element of the resulting array has been set to 0). The relative error is usually less than 5%. (Let y_1, y_2, y_3, ..., y_{14} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the cube of a prime when $x \leq 100000$ and 100000 zeros are used. For a cubic least-squares fit of y, R-square=0.9919.) Let y_1, y_2, y_3, ..., y_{19} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime and let z_1, z_2, z_3, ..., z_{19} equal the values of $\sum_{i|x} \kappa_9(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime. See Figure (9) for a plot of the normalized and scaled y and z values. See Figure (10) for a plot of the relative errors (the errors are usually less than 9%).

Let y_1, y_2, y_3, ..., y_{194} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime and let z_1, z_2, z_3, ..., z_{194} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime where $x \leq 2000$ (2000 zeta function zeros are used). See Figure (11) for a plot of the normalized and scaled y and z values. A factor of 1.769148 was used to scale the normalized z values. See Figure (12) for a plot of the relative errors (the errors are less than 7%). Let y_1, y_2, y_3, ..., y_{194} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime and let z_1, z_2, z_3, ..., z_{194} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and a different prime where $x \leq 2000$ and 2000 elements from a standard uniform distribution are used in place of the zeros. See Figure (13) for a plot of the normalized and scaled y and z values. A factor of 1.993304 was used to scale the normalized z values. See Figure (14) for a plot of the relative errors (the errors are usually less than 13%). Let y_1, y_2, y_3, ..., y_{568} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes and let z_1, z_2, z_3, ..., z_{568} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes where $x \leq 2000$ and 2000 zeta function zeros are used. See Figure (15) for a plot of the normalized and scaled y and z values. A factor of 1.773339 was used to scale the normalized z values. See Figure (16) for a plot of the relative errors (the errors are less than 9%).

Let y_1, y_2, y_3, ..., y_{1229} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are prime and let z_1, z_2, z_3, ..., z_{1229} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are prime where $x \leq 10000$ and 10000 zeta function zeros are used. See Figure (17) for a plot of the normalized and scaled y and z values. A factor of 1.81505 was used to scale the normalized z values. See Figure (18) for a plot of the relative errors. As the sample size increases, the zeta function zeros appear to become more uniformly distributed and the scaling factor appears to approach 2. For $x = 100, 200, 300, ..., 100000$, the scaling factors slowly increase from 1.701628 to 1.853194. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (19) for a plot of $10.42(f(n) - 1.701628)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (20) for a plot of the difference between the two curves. Let y_1, y_2, y_3, ..., y_{32} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the cube of a prime and x values that are the product of two distinct primes and let z_1, z_2, z_3, ..., z_{32} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at

x values that are the cube of a prime and x values that are the product of two distinct primes where $x \leq 100$ and 100 zeta function zeros are used. See Figure (21) for a plot of the normalized and scaled y and z values. A factor of 1.703358 was used to scale the normalized z values. For $x = 100, 200, 300, ..., 100000$, the scaling factors slowly increase from 1.703358 to 1.849745. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (22) for a plot of $9.8(f(n) - 1.703358)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (23) for a plot of the difference between the two curves. Let y_1, y_2, y_3, ..., y_{16} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another prime and let z_1, z_2, z_3, ..., z_{16} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the fifth power of a prime and x values that are the product of the square of a prime and another prime where $x \leq 100$ and 100 zeta function zeros are used. See Figure (24) for a plot of the normalized and scaled y and z values. A factor of 1.687551 was used to scale the normalized z values. For $x = 100, 200, 300, ..., 100000$, the scaling factors slowly increase from 1.687551 to 1.847807. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (25) for a plot of $9.65(f(n) - 1.687551)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (26) for a plot of the difference between the two curves. The scaling factors can be smoothed out by taking larger samples. For $x = 1000, 2000, 3000, ..., 10000$, the scaling factors slowly increase from 1.748444 to 1.803921. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 11$. See Figure (27) for a plot of $22.0(f(n) - 1.748444)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 11$. Let y_1, y_2, y_3, ..., y_{10} equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the seventh power of a prime, x values that are the product of the cube of a prime and another prime, and x values that are the product of three distinct primes and let z_1, z_2, z_3, ..., z_{10} equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the seventh power of a prime, x values that are the product of the cube of a prime and another prime, and x values that are the product of three distinct primes where $x \leq 100$ and 100 zeta function zeros are used. See Figure (28) for a plot of the normalized and scaled y and z values. A factor of 1.692376 was used to scale the normalized z values. For $x = 100, 200, 300, ..., 100000$, the scaling factors slowly increase from 1.692376 to 1.843177. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (29) for a plot of $9.2(f(n) - 1.692376)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (30) for a plot of the difference between the two curves. Let y_1, y_2, y_3, ..., y_4 equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the ninth power of a prime, x values that are the product of the fourth power of a prime and another prime, and x values that are the product of the square of a prime and the square of another prime and let z_1, z_2, z_3, ..., z_4 equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the ninth power of a prime, x values that are the product of the fourth power of a prime and another prime, and x values that are the product of the square of a prime and the square of a different prime where $x \leq 100$ and 100 zeta function zeros are used. See Figure (31) for a plot of the normalized and scaled y and z values. A factor of 1.691521 was used to scale the normalized z values. For $x = 100$, 200, 300, ..., 100000, the scaling factors slowly increase from 1.691521 to 1.842967. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (32) for a plot of $9.35(f(n) - 1.691521)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (33) for a plot of the difference between the two curves. Let y_1, y_2, y_3, ..., y_5 equal the values of $\sum_{i|x} \kappa_1(x/i) - \theta_1$ at x values that are the eleventh power of a prime, x values that are the product of the fifth power of a prime and another prime, x values that are the product of the cube of a prime and the square of another prime, and x values that are the product of the square of a prime and two other distinct primes and let z_1, z_2, z_3, ..., z_5 equal the values of $\sum_{i|x} \kappa_2(x/i) - \theta_1$ at x values that are the eleventh power of a prime, x values that are the product of the fifth power of a prime and another prime, x values that are the product of the cube of a prime and the square of another prime, and x values that are the product of the square of a prime and two other distinct primes where $x \leq 100$ and 100 zeta function zeros are used. See Figure (34) for a plot of the normalized and scaled y and z values. A factor of 1.686757 was used to scale the normalized z values. For $x = 100$, 200, 300, ..., 100000, the scaling factors slowly increase from 1.686757 to 1.842156. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (35) for a plot of $9.0(f(n) - 1.686757)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to become fairly constant. See Figure (36) for a plot of the difference between the two curves.

Odlyzko [6] investigated the spacing of the zeta function zeros. Let $\kappa_1'(1)$, $\kappa_1'(2)$, $\kappa_1'(3)$, ..., denote $\theta_1 \log(\theta_1)$, $\theta_2 \log(\theta_2)$, $\theta_3 \log(\theta_3)$, ... and let $\kappa_n'(x)$, $n = 2, 3, 4, ...$, denote these values and $n - 1$ values that have been linearly interpolated between successive values. Let y_1, y_2, y_3, ..., y_{1229} equal the values of $\sum_{i|x} \kappa_1'(x/i) - \theta_1 \log(\theta_1)$ at x values that are prime and let z_1, z_2, z_3, ..., z_{1229} equal the values of $\sum_{i|x} \kappa_2'(x/i) - \theta_1 \log(\theta_1)$ at x values that are prime where $x \leq 10000$ and 10000 zeta function zeros are used. See Figure (37) for a plot of the normalized and scaled y and z values (note that they are no longer linear). A factor of 1.939215 was used to scale the normalized z values. This is more consistent with the scaling factor found for the standard uniform distribution. See Figure (38) for a plot of the relative errors. For $x = 100$, 200, 300, ..., 100000, the scaling factors slowly increase from 1.844432 to 1.960727. Denote these values by $f(n)$, $n = 2, 3, 4, ..., 1001$. See Figure (39) for a plot of $21.0(f(n) - 1.844432)$ and $\log(\log(n)) - \log(\log(2))$ for $n = 2, 3, 4, ..., 1001$. The difference between the two curves appears to slowly increase no matter what factor (21.0 in this instance) is used. See Figure (40) for a plot of the difference between the two curves.

3 Normal Probability Distributions Involving the Zeta Function Zeros

Let y_i equal $|\theta_{i+1} - \theta_i|$, $i = 1, 2, 3, ..., 199999$ and let z_j, $j = 1, 2, 3, ...,$ 199757 equal y_i that are less than or equal to 2.0. Let w_i equal $\lfloor 200.0 z_i \rfloor$ for $i = 1, 2, 3, ..., 199757$. See Figure (41) for a histogram (with 399 bins) of w. (Thresholds less than 2.0 cut off the right-hand tail of the distribution.) See Figure (42) for a plot of the sorted z values (the plot resembles a sigmoid curve). The quantiles of the distribution at .025, .25, .50, .75, and .975 are .2141, .4866, .6660, .8721, and 1.35 respectively. The mean of the distribution is 0.6954 with a 95% confidence interval of (0.6941, 0.6967) and the standard deviation is 0.2912 with a 95% confidence interval of (0.2903, 0.292). See Figure (43) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. (Sigmoid functions are used in artificial neural networks as an activation function, mapping a value of $(-\infty, \infty)$ to (0, 1). Applying the sigmoid function $\frac{1}{1+e^{-z}}$ to a normal distribution with mean μ and standard deviation σ gives an expected value of $\Phi(\frac{\lambda\mu}{\sqrt{(1+\lambda^2\sigma^2)}})$ (approximately) where Φ is the cumulative distribution function and $\lambda = \pi/8$. In this case, the expected value is 0.0726. The expected value can be computed more accurately using MacLaurin approximation.) Let y_i equal $|\theta_{i+1} - \theta_i|$, $i = 1, 2, 3, ..., 199999$ and let z_j, $j = 1, 2, 3, ..., 242$ equal y_i that are greater than 2.0. Let w_i equal $\lfloor 3.0 z_i \rfloor$ for $i = 1, 2, 3, ..., 242$. See Figure (44) for a histogram (with 9 bins) of w. See Figure (45) for a plot of the sorted z values. The mean of the distribution is 2.3936 with a 95% confidence interval of (2.3220, 2.4652) and the standard deviation is 0.5653 with a 95% confidence interval of (0.5191, 0.6207). See Figure (46) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. Let y_i equal $|\theta_{i+1} - \theta_i|$, $i = 1, 2, 3, ...,$ 1999999 and let z_j, $j = 1, 2, 3, ..., 1999757$ equal y_i that are less than or equal to 2.0. Let w_i equal $\lfloor 200.0 z_i \rfloor$ for $i = 1, 2, 3, ..., 1999757$. See Figure (47) for a histogram (with 400 bins) of w. See Figure (48) for a plot of the sorted z values. The quantiles of the distribution at .025, .25, .50, .75, and .975 are .1702, .3943, .5434, .7113, and 1.092 respectively. The mean of the distribution is 0.5657 with a 95% confidence interval of (0.5654, 0.5661) and the standard deviation is 0.2280 with a 95% confidence interval of (0.2377, 0.2382). See Figure (49) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. The value of the activation function is 0.0654. (Using Maclaurin approximation, the values of the activation function are 0.0370, 0.0625, 0.0697, and 0.0857 for zero sample sizes of 10000, 100000, 200000, and 2000000 respectively.) No new instances where the absolute values of the differences were greater than 2.0 were found.

Let u_x equal $\sum_{i|x} \kappa_1(x/i)$ for $x = 1, 2, 3, ..., 200000$ and let v_i equal $u_i/\log(u_i)$ for $i = 1, 2, 3, ... 200000$. Let w_i equal $|v_{i+1} - v_i|$, $i = 1, 2, 3, ..., 199999$ and let y_j, $j = 1, 2, 3, ..., 197801$, equal w_i values that are less than or equal to 2.0. Let

z_i equal $\lfloor 200.0 y_i \rfloor$ for $i = 1, 2, 3, ..., 197801$. See Figure (50) for a histogram (with 401 bins) of z. See Figure (51) for a plot of the sorted y values. The mean of the distribution is 0.1452 with a 95% confidence interval of (0.1443, 0.1461) and the standard deviation is 0.2072 with a 95% confidence interval of (0.2066, 0.2079). See Figure (52) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. Let z_i, $i = 1, 2, ..., 197801$ equal the v_i values (the first in each comparison) corresponding to the w_i values. See Figure (53) for a plot of the sorted z values (the plot resembles a sigmoid curve). The mean of the distribution is 10672 with a 95% confidence interval of (10643, 10700) and the standard deviation is 6500.6 with a 95% confidence interval of (6480.5, 6521.0). See Figure 54 for the corresponding normal probability plot. The linearity of the curve indicates that the data came from a normal probability distribution. The value of the activation function is 0.0504. Let y_j, $j = 1, 2, 3, ..., 2198$, equal w_i values that are greater than 2.0. Let z_i equal $\lfloor 0.5 y_i \rfloor$ for $i = 1, 2, 3, ..., 2198$. See Figure (55) for a histogram (with 1198 bins) of z. See Figure (56) for a plot of the sorted y values. The mean of the distribution is 11.7201 with a 95% confidence interval of (8.6810, 14.7593) and the standard deviation is 72.6566 with a 95% confidence interval of (70.5706, 74.8707). See Figure (57) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. Let z_i, $i = 1, 2, 3, ..., 2198$ equal the v_i values (the first in each comparison) corresponding to the w_i values. See Figure (58) for a plot of the sorted z values (the plot resembles a sigmoid curve). The mean of the distribution is 32294 with a 95% confidence interval of (23080, 325081) and the standard deviation is 5116 with a 95% confidence interval of (4969.1, 5271.9). See Figure (59) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. The value of the activation function is $1.3851 \cdot 10^{-10}$.

The scaled Dirichlet products of the zeros then behave similarly to the zeros. The repulsion effect is much more pronounced due to number-theoretic reasons. For the Lehmer [7] pair at 6709 and 6710, $\theta_{6709} = 7005.06266$ and $\theta_{6710} = 7005.10056$. 6709 is prime and $6710 = 2 \cdot 5 \cdot 11 \cdot 61$ so the corresponding Dirichlet products are in Curve #1 and Curve #13 respectively. Curve #9 in particular rarely has adjacent points due to the scarcity of solutions of the Diophantine equations $p^9 - r^4 s = 1$, $p^9 - r^2 s^2 = 1$, $p^4 q - r^2 s^2 = 1$ and $p^4 q - r^4 s = 1$ where p, q, r, and s are primes. More generally, it's more likely that the points in the increasing curve will be at about the same height if the difference in x values is small.

Let θ'_i denote the above sorted and scaled Dirichlet products for $i = 1, 2, 3, ..., 10000$. These values can in turn be used to generate other sorted and scaled Dirichlet products. Eventually, all the absolute values of differences of successive "zeros" appear to become less than 2.0. Let w_i equal $|\theta'_{i+1} - \theta'_i|$, $i = 1, 2, 3, ..., 9999$ after 3 iterations of this process. Let y_j, $j = 1, 2, 3, ..., 9981$, equal the w_i values that are less than or equal to 2.0. Let z_i, $i = 1, 2, ..., 9981$ equal the θ_i' values (the first in each comparison) corresponding to the y_i values. See Figure (60) for a plot of the sorted z values. The mean of the distribution is 84.6263 with a 95% confidence interval of (83.6517, 85.6009) and the standard deviation is 49.6718 with a 95% confidence interval of (48.9923, 50.3707). See Figure (61) for the corresponding normal distribution plot. The linearity of the plot indicates that the data came from a normal probability distribution. The value of the activation function is 0.0475.

Let u_x equal $\sum_{i|x} \kappa_4(x/i)$ for $x = 1, 2, 3, ..., 100000$ and let v_i equal $u_i/\log(u_i)$ for $i = 1, 2, 3, ... 100000$. Let w_i equal $|v_{i+1} - v_i|$, $i = 1, 2, 3, ..., 99999$ and let y_j, $j = 1, 2, 3, ..., 99612$, equal w_i that are less than or equal to 2.0. Let z_i, $i = 1, 2, ..., 99612$ equal the θ_i' values (the first in each comparison) corresponding to the y_i values. See Figure (62) for a plot of the sorted z values. The mean of the distribution is 2087.7 with a 95% confidence interval of (2079.9, 2905.6) and the standard deviation is 1263.9 with a 95% confidence interval of (1258.4, 1269.5). See Figure (63) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution. The value of the activation function is 0.0494.

Let θ_i' denote the above sorted and scaled Dirichlet products for $i = 1, 2, 3, ...,$ 100000. These values can in turn be used to generate other sorted and scaled Dirichlet products. Let w_i equal $|v_{i+1} - v_i|$, $i = 1, 2, 3, ..., 99999$ after 3 iterations of this process and let y_j, $j = 1, 2, 3, ..., 99970$, equal w_i that are less than or equal to 2.0. Let z_i, $i = 1, 2, ..., 99970$ equal the θ_i' values (the first in each comparison) corresponding to the y_i values. See Figure (64) for a plot of the sorted z values. The mean of the distribution is 139.4849 with a 95% confidence interval of (138.9594, 140.0105) and the standard deviation is 84.7758 with a 95% confidence interval of (84.4059, 85.1491). See Figure (65) for the corresponding normal probability plot. The linearity of the plot indicates that the data came from a normal probability distribution.

Let θ_i' denote sorted and scaled Dirichlet products for $i = 1, 2, 3, ..., 10000$ (where no interpolation between successive zeros is done). Let w_i equal $|\theta_{i+1}' - \theta_i'|$, $i = 1, 2, 3, ..., 9999$ and let y_j, $j = 1, 2, 3, ..., 9762$, equal w_i that are less than or equal to 2.0. Let z_i, $i = 1, 2, 3, ..., 9762$ equal the θ_i' values (the first in each comparison) corresponding to the y_i values. Let n denote the number of times sorted and scaled Dirichlet products are generated. For $n = 1, 2, 3, ..., 20$, the number of elements in the z array are 9762, 9952, 9981, 9994, 9998, 9998, 9998, 9998, 9998, 9999, 9999, 9999, 9999, 9999, 9999, 9999, 9999, 9999, 9999, and 9999 respectively. See Figures (66) and (67) for the means and standard deviations of the corresponding normal probability distributions. After $n = 13$, the distributions reach equilibrium with means of about 28.6 and standard deviations of about 488.4. Let θ_i' denote sorted and scaled Dirichlet products for $i = 1, 2, 3, ..., 100000$. Let w_i equal $|\theta_{i+1}' - \theta_i'|$, $i = 1, 2, 3, ..., 99999$ and let y_j, $j = 1, 2, 3, ..., 98698$, equal w_i that are less than or equal to 2.0. Let z_i, $i = 1,$

2, 3, ..., 98698 equal the θ'_i values (the first in each comparison) corresponding to the y_i values. Let n denote the number of times sorted and scaled Dirichlet products are generated. For $n = 1, 2, 3, ..., 20$, the number of elements in the z array are 98698, 99755, 99942, 99975, 99992, 99995, 99997, 99998, 99998, 99998, 99998, 99998, 99998, 99998, 99998, 99998, 99998, 99998, 99998, and 99998 respectively. See Figures (68) and (69) for the means and standard deviations of the corresponding normal probability distributions. After $n = 12$, the distributions almost reach equilibrium with means of about 306.5 and standard deviations of about 5465. The consecutive z values with a difference greater than 2.0 are 84.006780 and 87.646047. The effect of using limited precision floating point arithmetic (especially in computing the logarithms) is unknown.

Let u_x equal $\sum_{i|x} \kappa_1(\lfloor x/i \rfloor)$ for $x = 1, 2, 3, ..., 10009$ and let v_i equal $u_i / \log(u_i)$ for $i = 1, 2, 3, ...$ 10009. Let w_i equal $|v_{i+1} - v_i|$, $i = 1, 2, 3, ..., 10009$ and let y_j, $j = 1, 2, 3, ..., 9770$, equal w_i values that are less than or equal to 2.0. The mean of the y distribution is 0.2457 with a 95% confidence interval of (0.2399, 0.2514) and the standard deviation is 0.2906 with a 95% confidence interval of (0.2866, 0.2947). Let t_i equal $|\theta_{i+1} - \theta_i|$, $i = 1, 2, 3, ..., 9999$ and let y'_j, $j = 1, 2, 3, ..., 9970$ equal t_i values that are less than or equal to 2.0. The mean of the y' distribution is 0.9530 with a 95% confidence interval of (0.9456, 0.9605) and the standard deviation is 0.3736 with a 95% confidence interval of (0.3684, 0.3789). See Figure (70) for a plot of y, y', and $y + y'$. The mean of the $y + y'$ distribution is 1.1987 with a 95% confidence interval of (1.1859, 1.2115) and the standard deviation is 0.6455 with a 95% confidence interval of (0.6366, 0.6547). See Figure (71) for the corresponding normal probability plot of $y + y'$. The linearity of the plot indicates that the data came from a normal probability distribution. Note that the two curves (y and y') are not independent since they must end at about the same point. Qualitatively, the curves should remain the same for larger numbers of zeros - the inner portions of the curves should approximate straight lines.

4 Probit Functions

Let t_i equal $\theta_{i+1} - \theta_i$, $i = 1, 2, 3, ..., 999$. The mean of the distribution is 1.4067 with a 95% confidence interval of (1.3659, 1.4475) and the standard deviation is 0.6569 with a 95% confidence interval of (.6293, .6871). See Figure (72) for a plot of the sorted t_i values, $\Phi^{-1}(p)$ (the probit function), and $F^{-1}(p) = \mu + \sigma\Phi^{-1}(p)$ (where $\mu = 1.4067$ and $\sigma = 0.6569$) for $p = 0.0, 0.001, 0.002, ..., 0.999$. The maximum and minimum t values are 6.887315 and 0.161501 respectively. The maximum and minimum Φ^{-1} values are 3.215562 and 0.002507 respectively. This appears to be the maximum t value for all larger samples of the zeros. Of course, this is also the maximum Φ^{-1} value. The probit function can be computed using an ordinary differential equation. Let t_i equal $\theta_{i+1} - \theta_i$, $i = 1, 2, 3, ..., 100000$ and let y_j, $j = 1, 2, 3, ..., 99757$ equal t_i values that are less than or equal to 2.0. The mean of the y distribution is 0.7451 with a 95% confidence interval of (0.7432, 0.7470) and the standard deviation is 0.3103 with a 95% confidence interval of (0.3090, 0.3117). See Figure (73) for a plot of the sorted y values and the corresponding F^{-1} values. Let t_i equal $\theta_{i+1} - \theta_i$, $i = 1, 2, 3, ..., 200000$ and let y_j, $j = 1, 2, 3, ..., 199757$ equal t_i values that are less than or equal to 2.0. The mean of the y distribution is 0.6954 with a 95% confidence interval of (0.6941, 0.6967) and the standard deviation is 0.2912 with a 95% confidence interval of (0.2903, 0.292). See Figure (74) for a plot of the sorted y values and the corresponding F^{-1} values. Let t_i equal $\theta_{i+1} - \theta_i$, $i = 1, 2, 3, ..., 2000000$ and let y_j, $j = 1, 2, 3, ..., 1999757$ equal t_i values that are less than or equal to 2.0. The mean of the y distribution is 0.5657 with a 95% confidence interval of (0.5654, 0.566) and the standard deviation is 0.2380 with a 95% confidence interval of (0.2377, 0.2382). See Figure (75) for a plot of the sorted y values and the corresponding F^{-1} values. These plots indicate that the distribution of the zeros is modeled fairly accurately and can be modeled more accurately with a common "correction" function. For zero sample sizes of 100000, 200000, 300000, ..., 2000000, the means of the above distributions are .7451, .6954, .6689, .6511, .6379, .6275, .6189, .6116, .6053, .5998, .5949, .5905, .5865, .5825, .5794, .5763, .5734, .5707, .5681 and .5657 respectively. See Figure (76) for a plot of these means. For zero sample sizes of 100000, 200000, 300000, ..., 2000000, the standard deviations of the above distributions are .3103, .2912, .2806, .2734, .2680, .2637, .2602, .2572, .2546, .2523, .2502, .2484, .2467, .2451, .2437, .2424, .2412, .2401, .2390, and .2380 respectively. See Figure (77) for a plot of these standard deviations. The normalized and scaled reciprocals of the means and standard deviations closely approximate the logarithm function. For zero sample sizes of 100000, 200000, 300000, ..., 2000000, the smallest y values of the above distributions are .0147, .0147, .0147, .0131, .0057, .0057, .0057, .0057, .0057, .0057, .0057, .0030, .0030, .0030, .0030, .0030, .0030, .0030, .0030, and .0030 respectively. See Figure (78) for a plot of these smallest values. There doesn't appear to be any lower bounds of these values. See Figure (79) for a plot of the differences between the curves in Figure (73). See Figure (80) for a plot of the differences between the curves in Figure (75). For differences less than about .25, the differences in Figure (80) can be approximated by interpolating between the differences in Figure (79).

References

[1] Gram, J.-P. Sur les Zéros de la Fonction $\zeta(s)$ de Riemann. *Acta Math.* 27, 289-304 (1903)

[2] Backlund, R., Über die Nullstellen der Riemannschen Zetafunktion, *Acta Math.* 41, 345-375 (1918)

[3] Backlund, R., Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion. *Ofversigt Finska Vetensk. Soc.* 61, No. 9 (1918-1919)

[4] Hutchinson, J. I. On the roots of the Riemann zeta-function. *Trans. Amer. Math. Soc.* 27, 49-60 (1925)

[5] Odlyzko, A. M., "www.dtc.umn.edu/~odlyZko/zeta_tables/index.html"

[6] Odlyzko, A. M., On the distribution of spacings between zeros of the zeta function, *Math. Comp.* 48, 273-308 (1987)

[7] Lehmer, D. H., On the roots of the Riemann zeta-function, *Acta Mathematica* 95: 291-298 (1956)

Figure 1

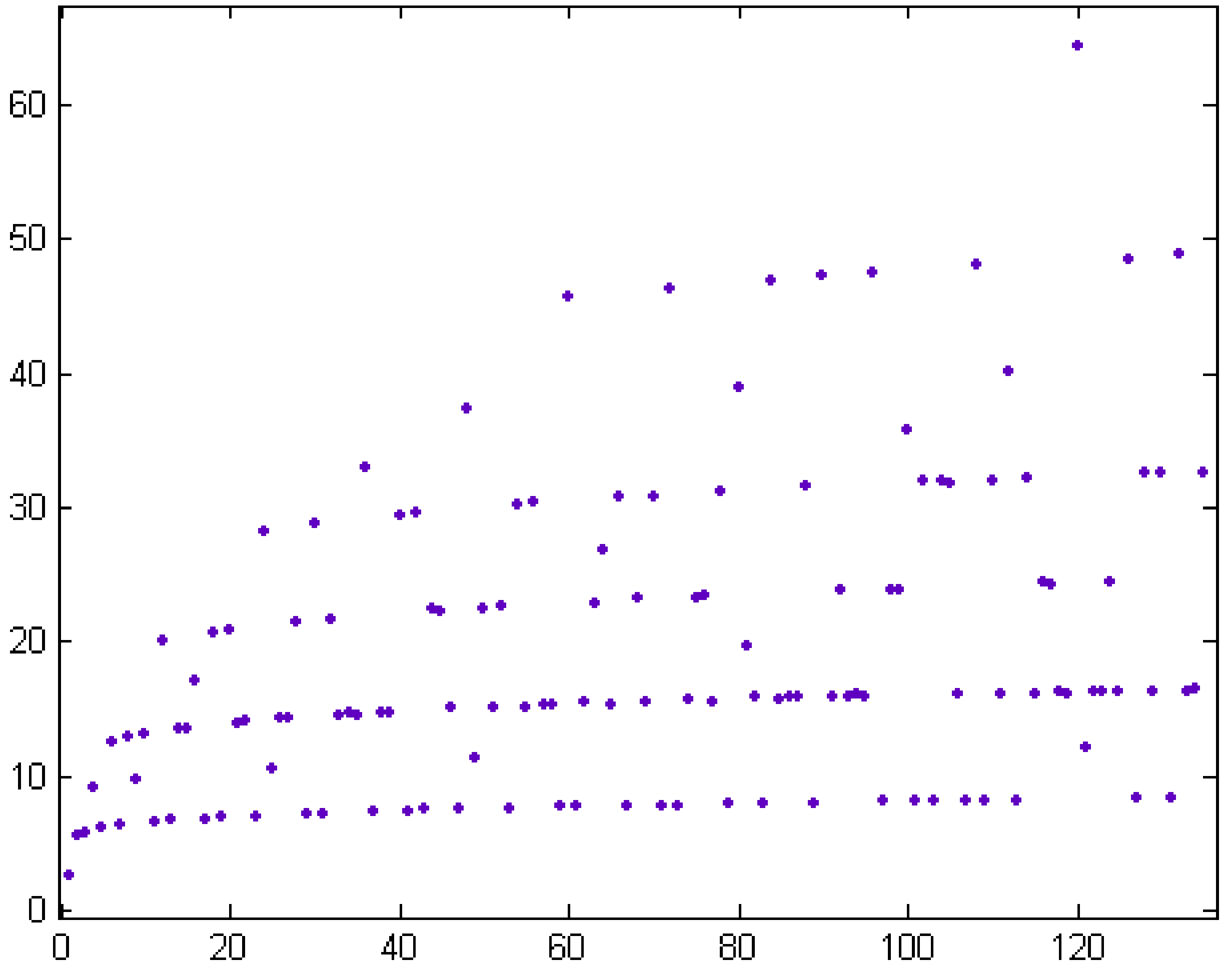
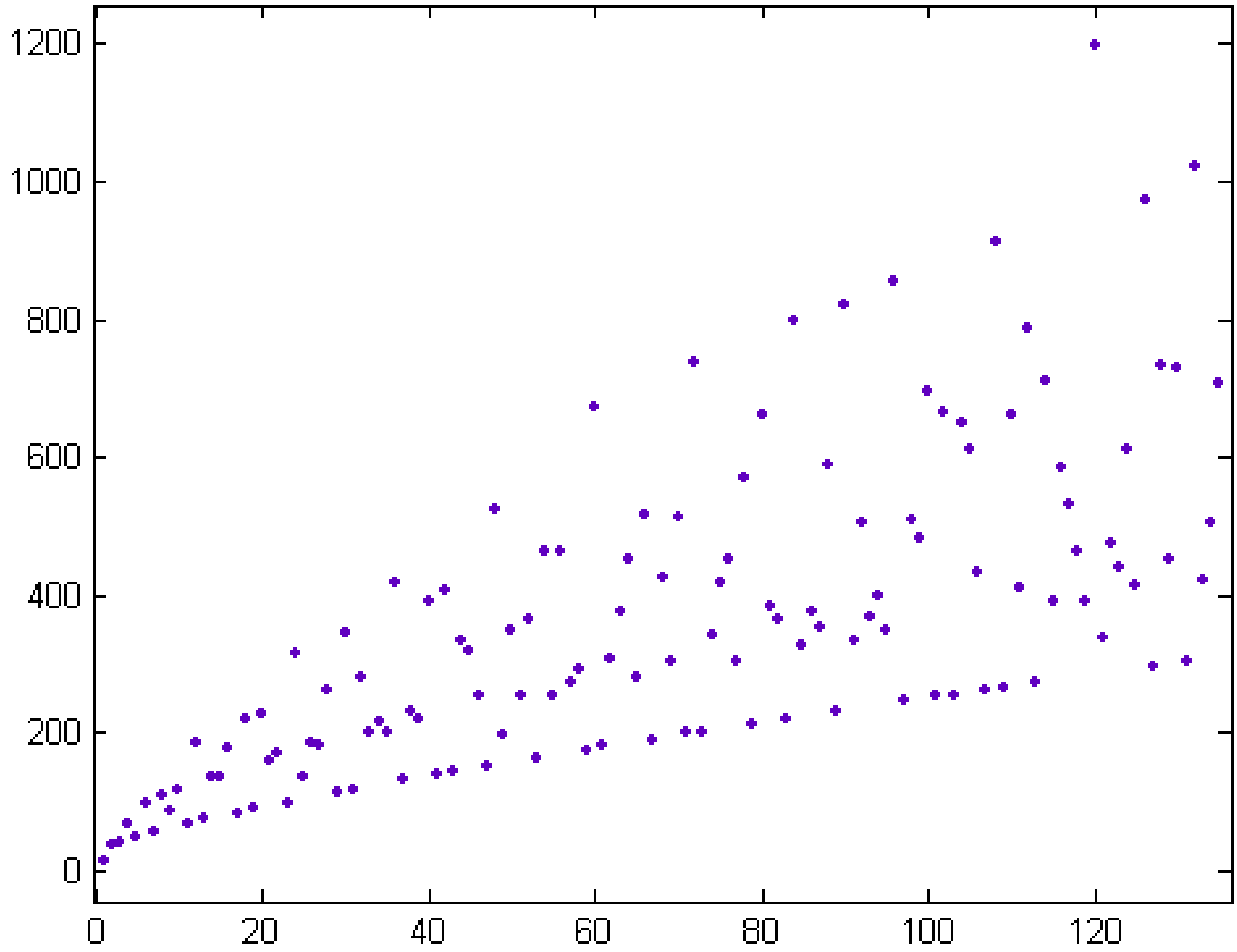


Figure 2



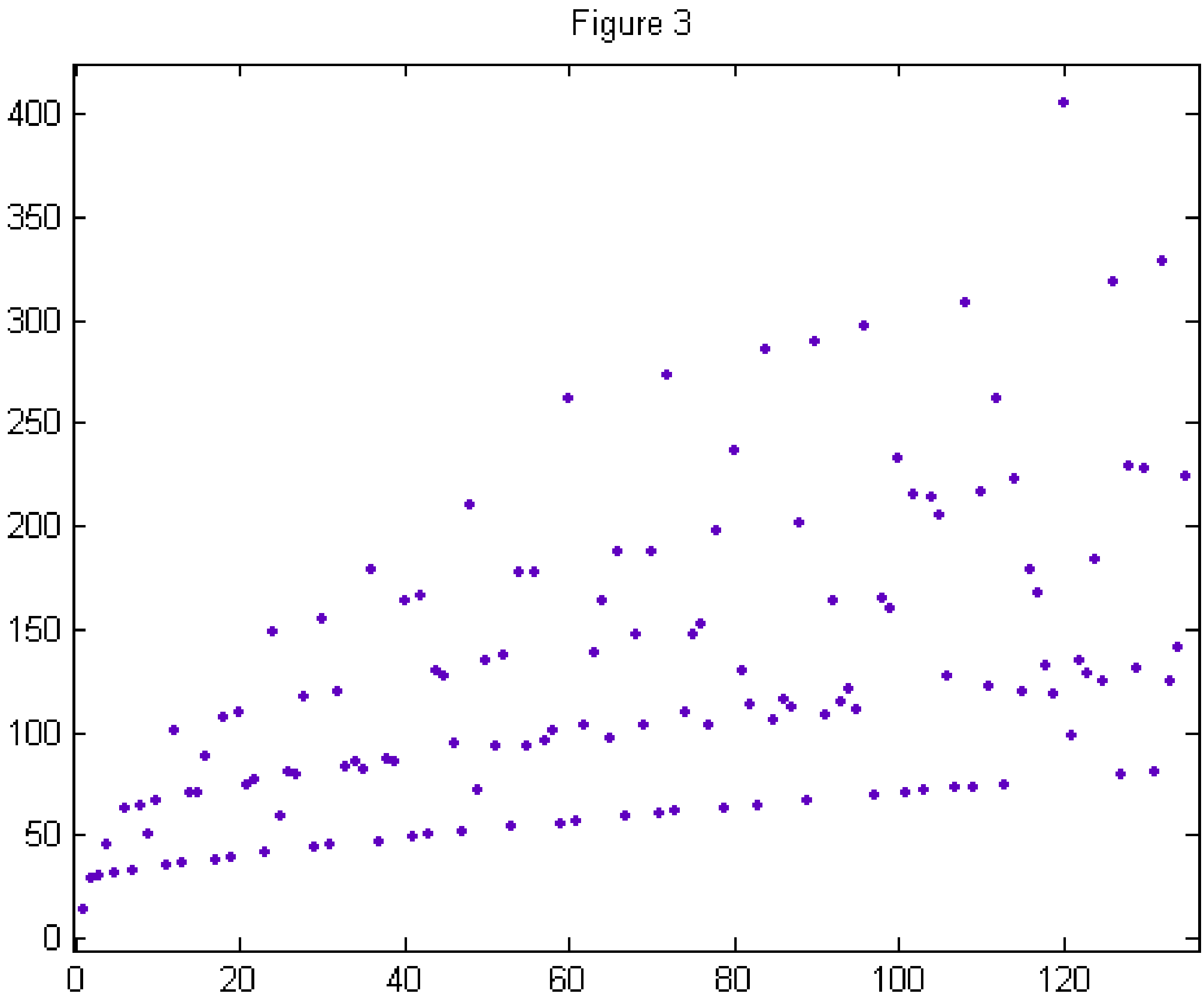


Figure 3
0
50
100
150
200
250
300
350
400
0
20
40
60
80
100
120

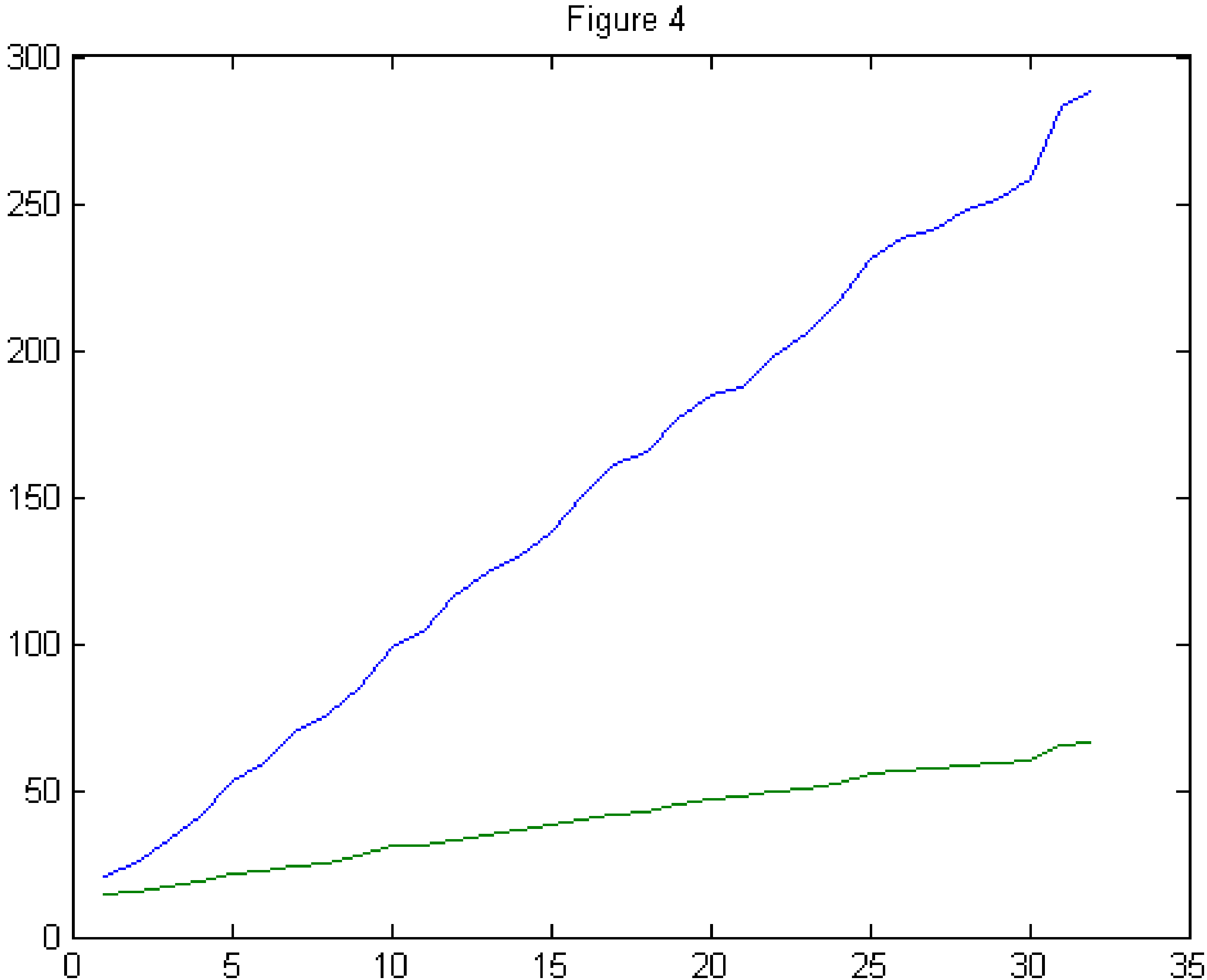


Figure 4
300
250
200
150
100
50
0
0
5
10
15
20
25
30
35

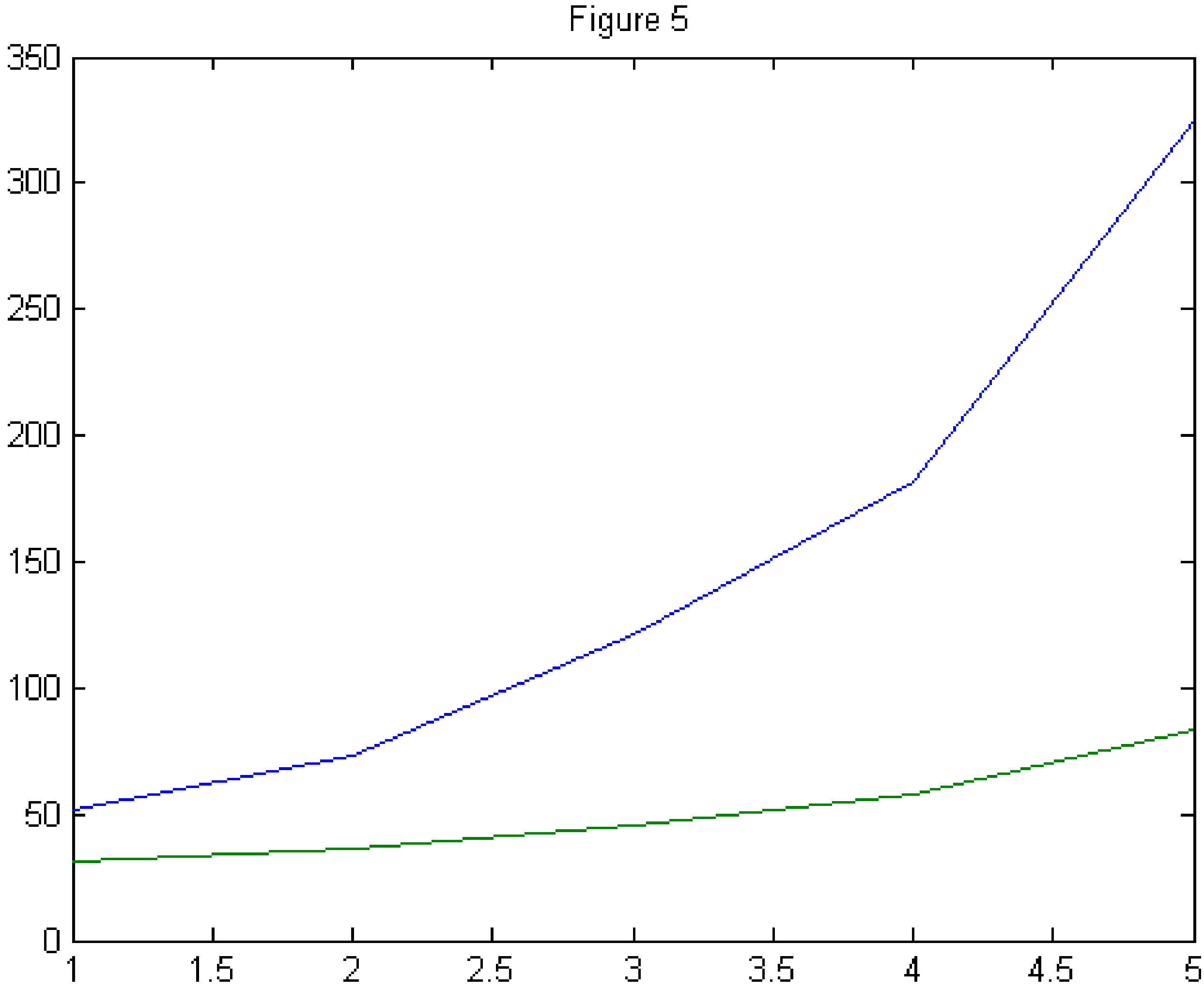


Figure 5
350
300
250
200
150
100
50
0
1
1.5
2
2.5
3
3.5
4
4.5
5

Figure 6

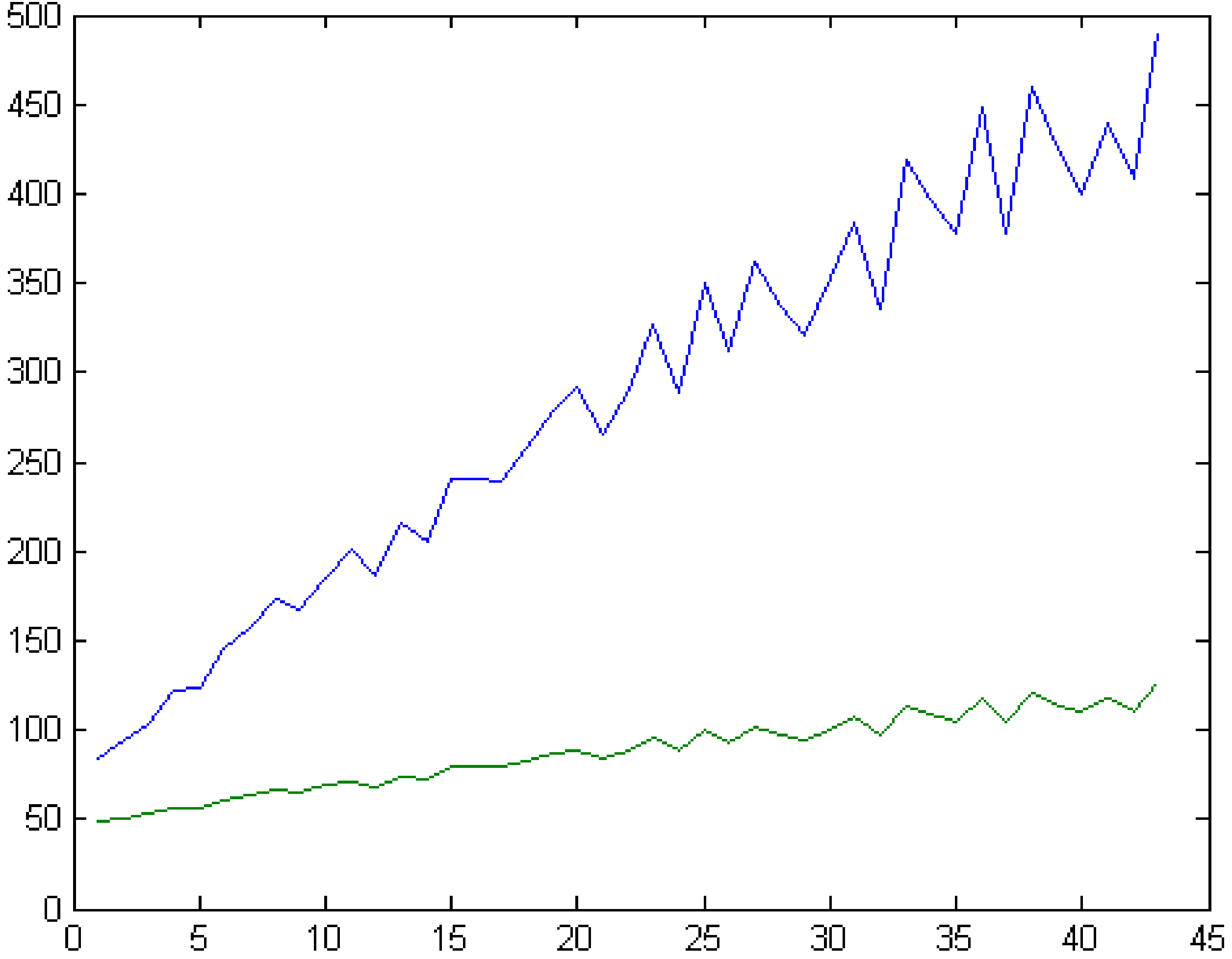
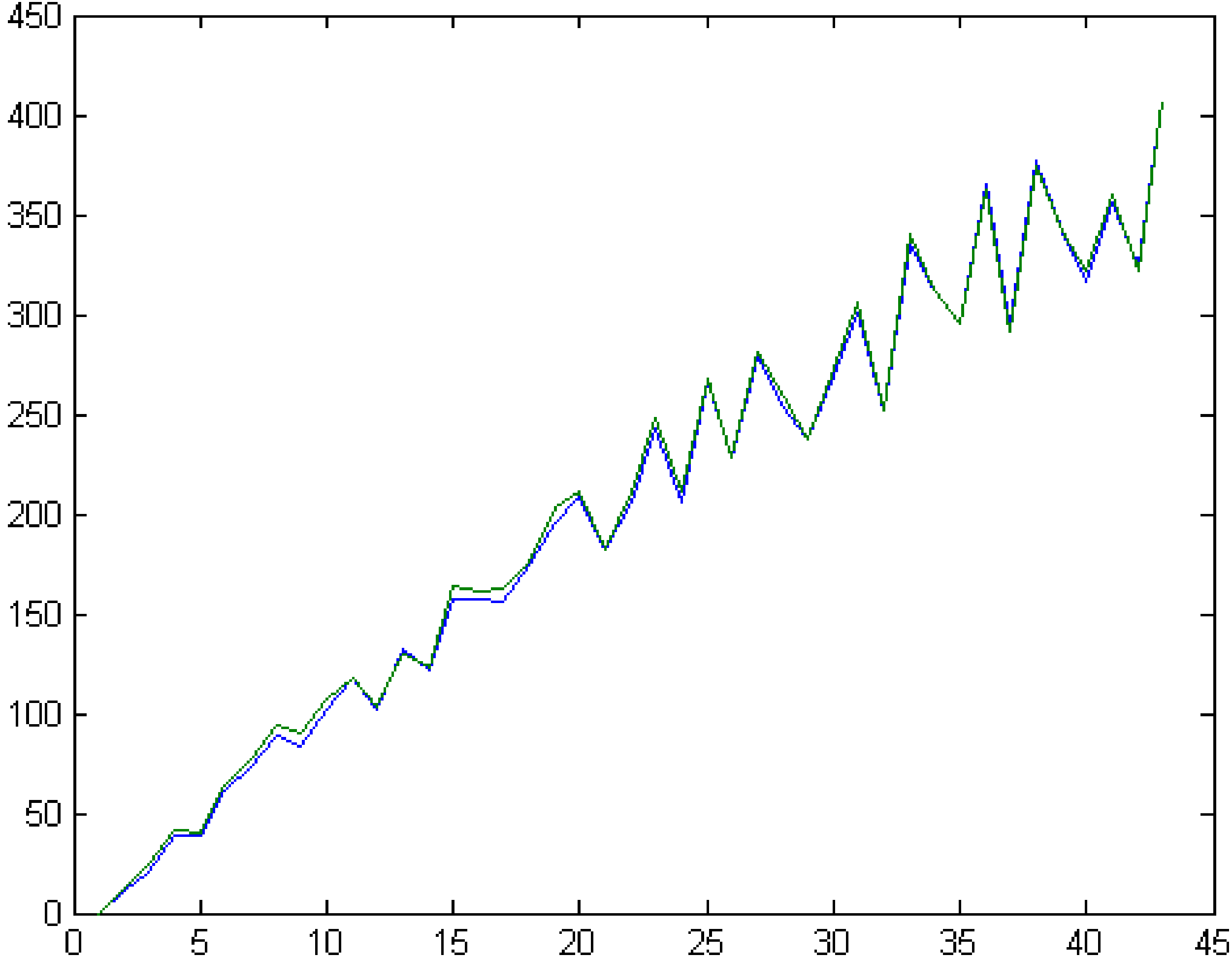


Figure 7



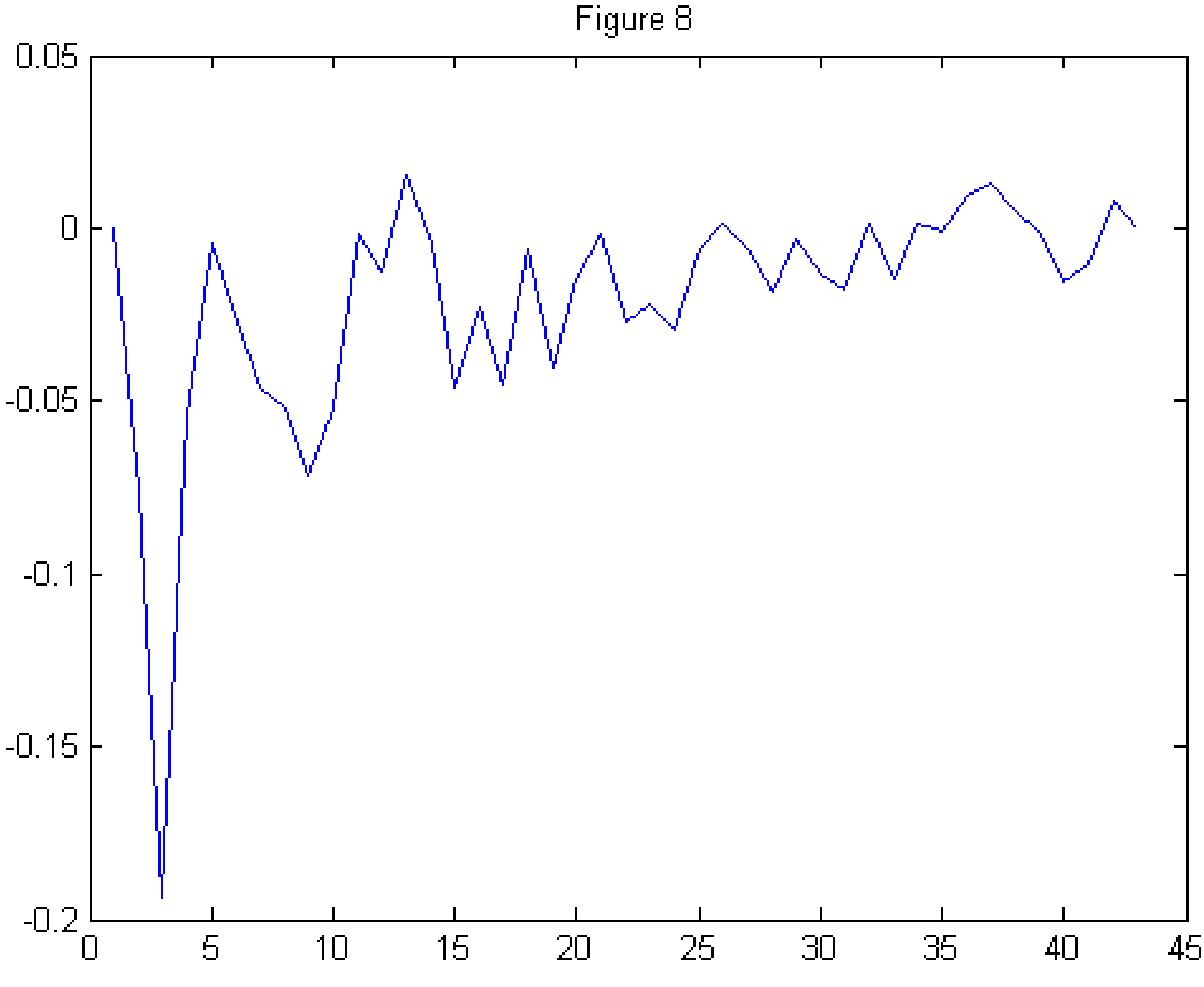


Figure 8
0.05
0
-0.05
-0.1
-0.15
-0.2
0
5
10
15
20
25
30
35
40
45

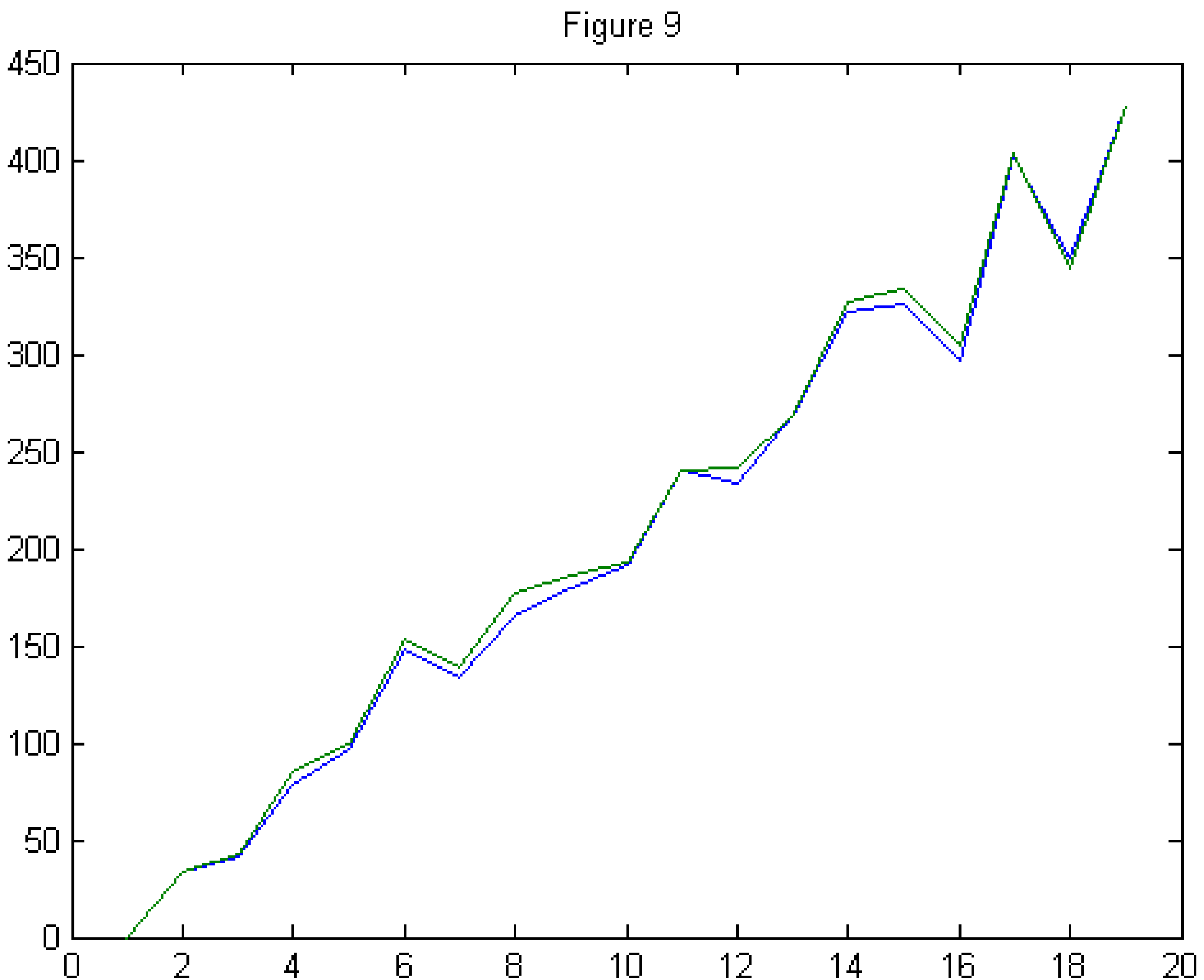


Figure 9
450
400
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
14
16
18
20

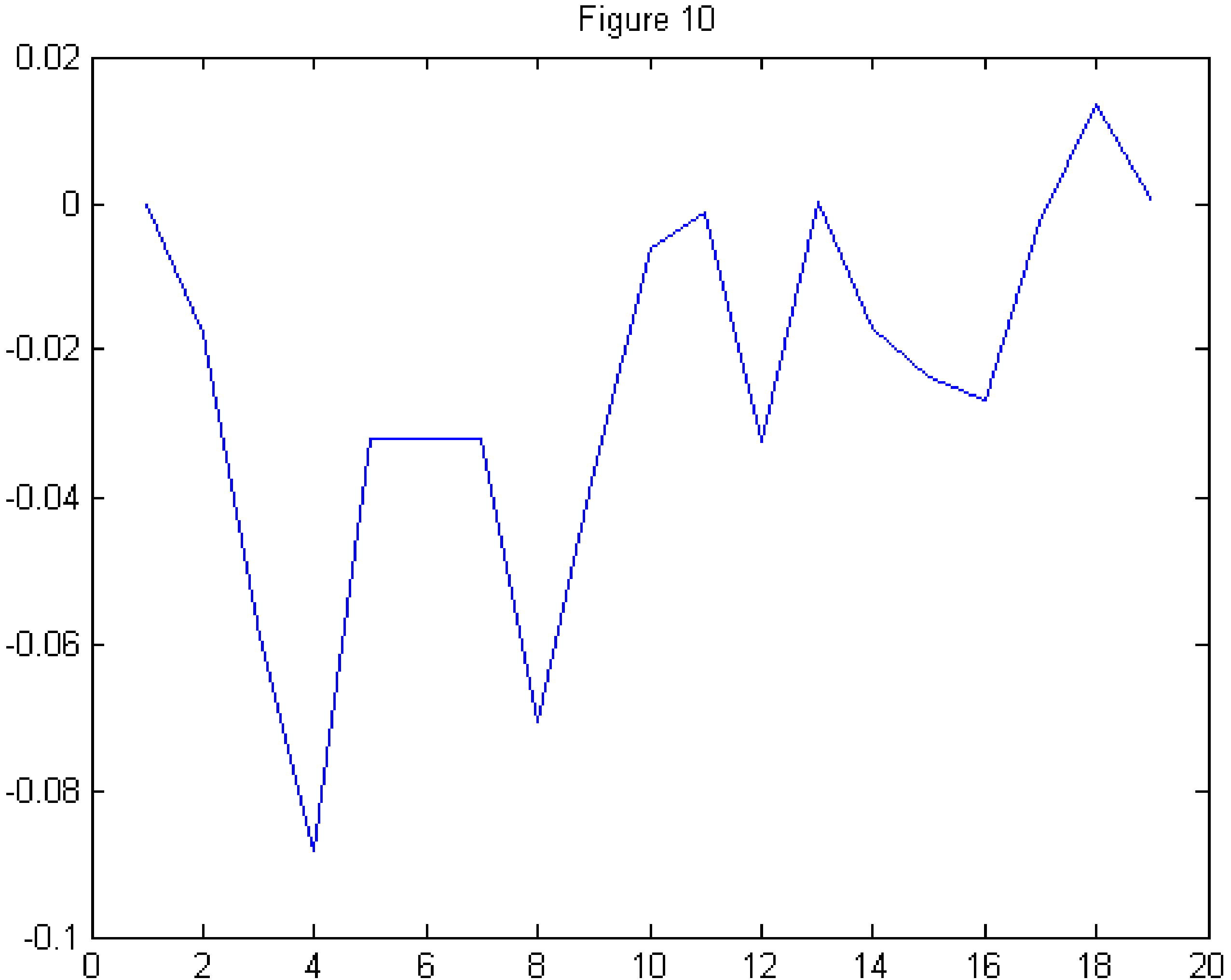


Figure 10
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
2
4
6
8
10
12
14
16
18
20

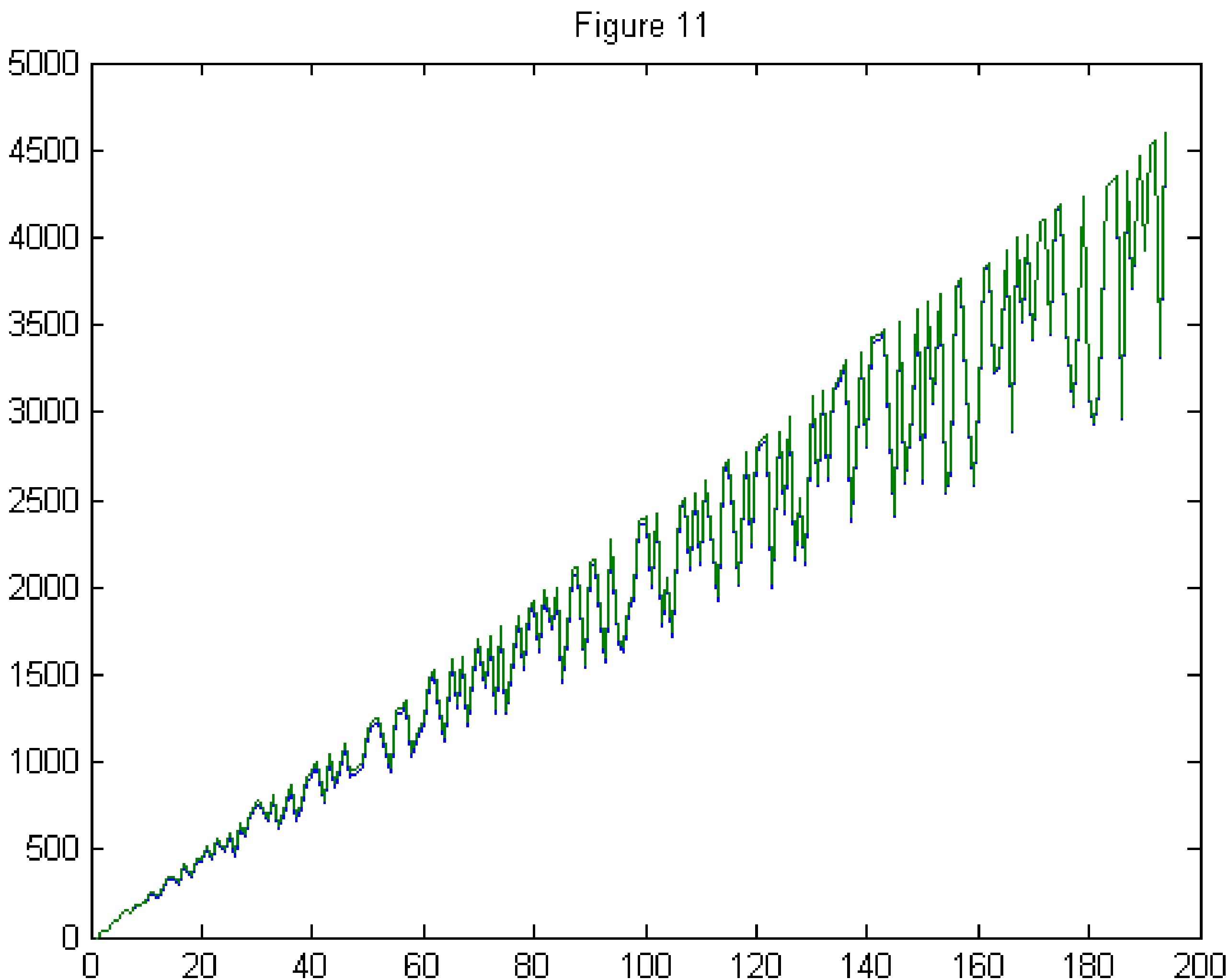
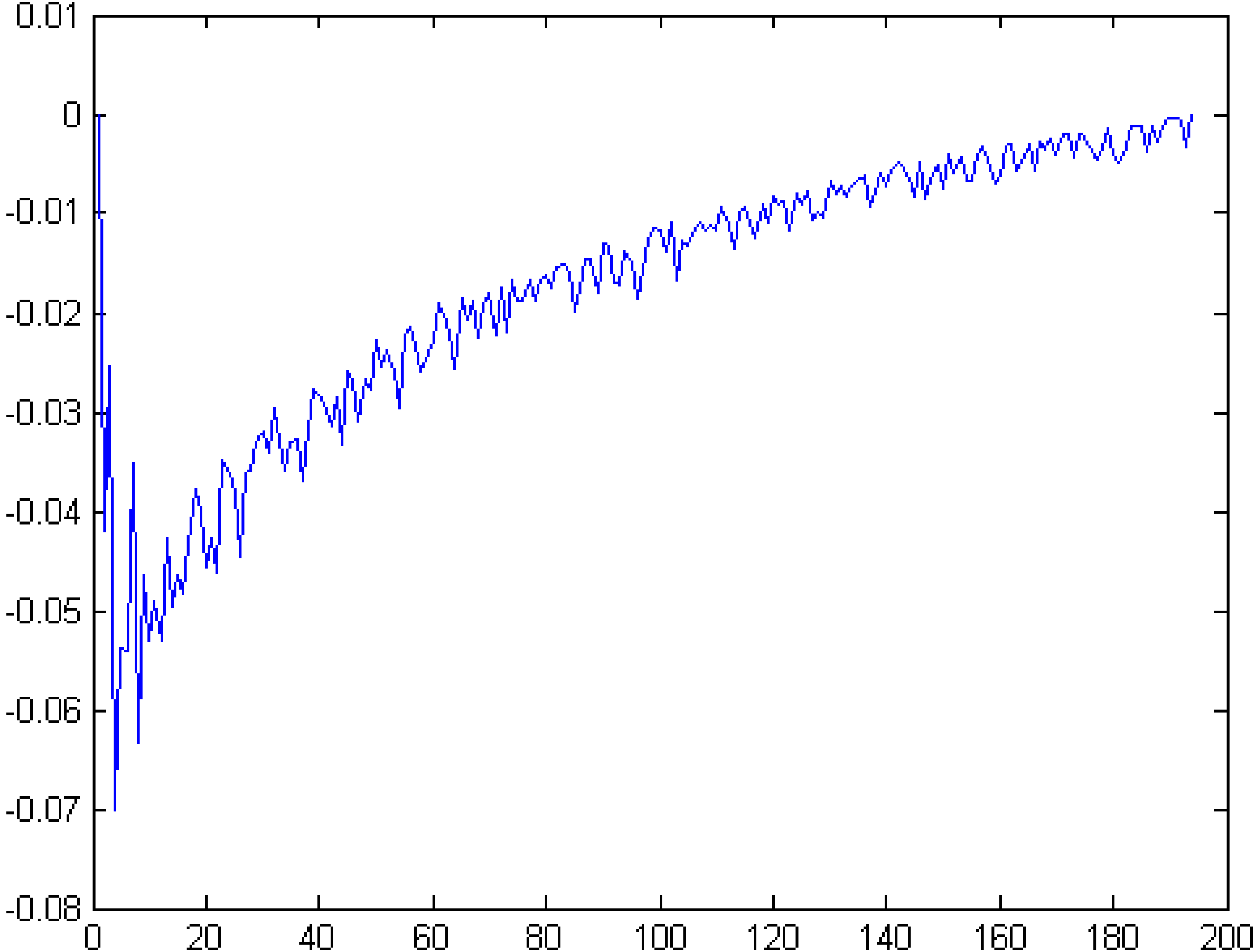


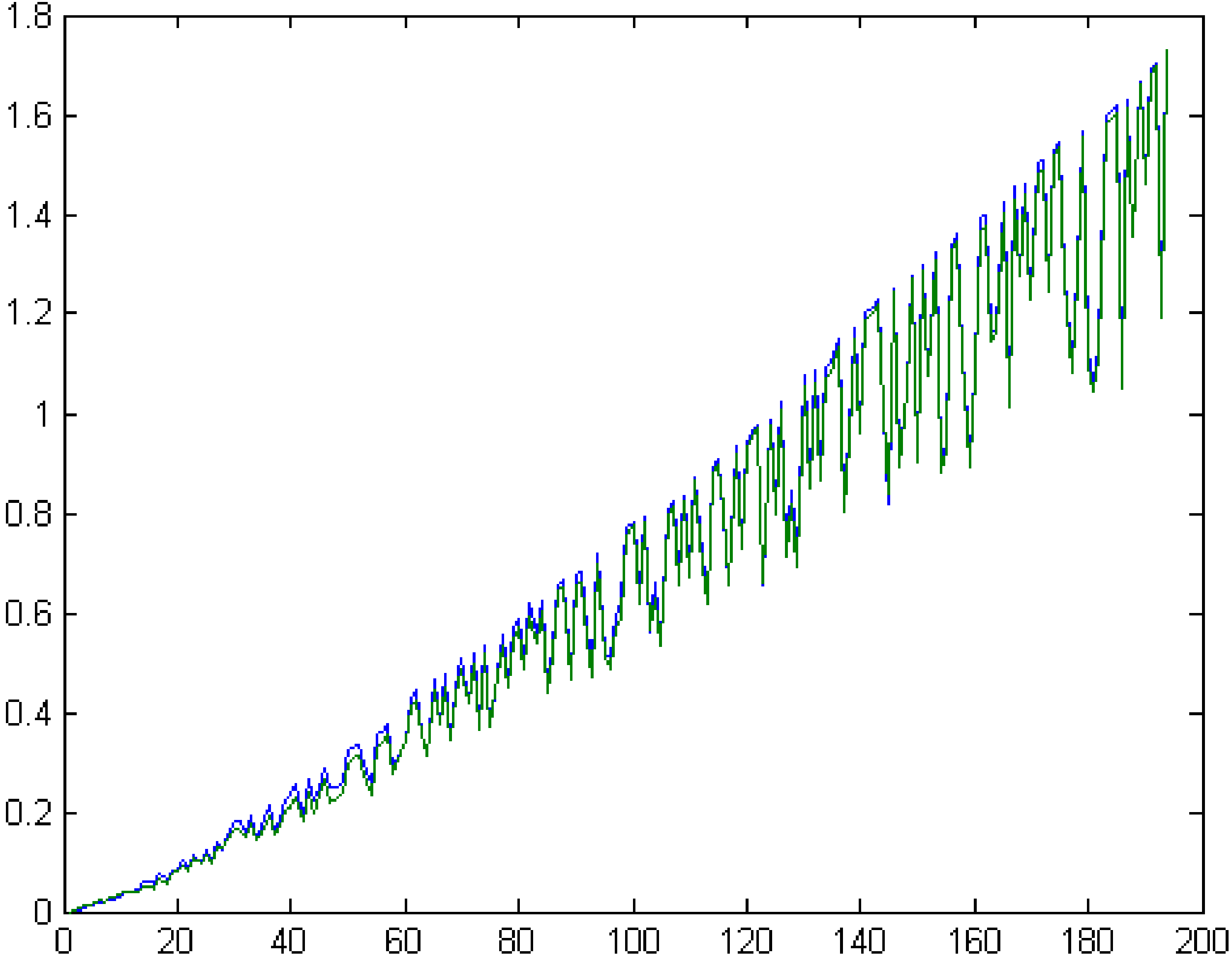
Figure 11
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
0
20
40
60
80
100
120
140
160
180
200

Figure 12



0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
0
20
40
60
80
100
120
140
160
180
200

Figure 13



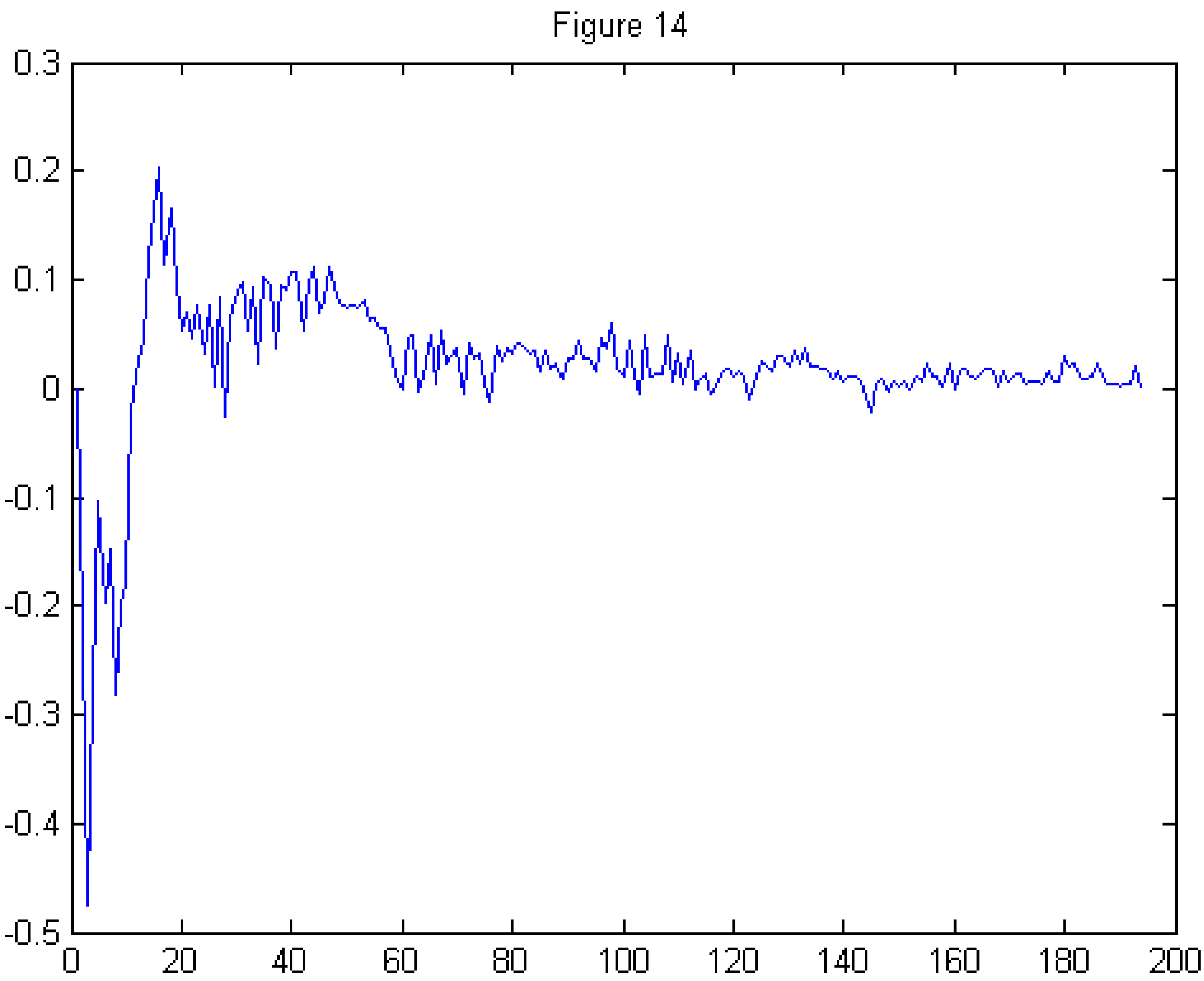


Figure 14
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
20
40
60
80
100
120
140
160
180
200

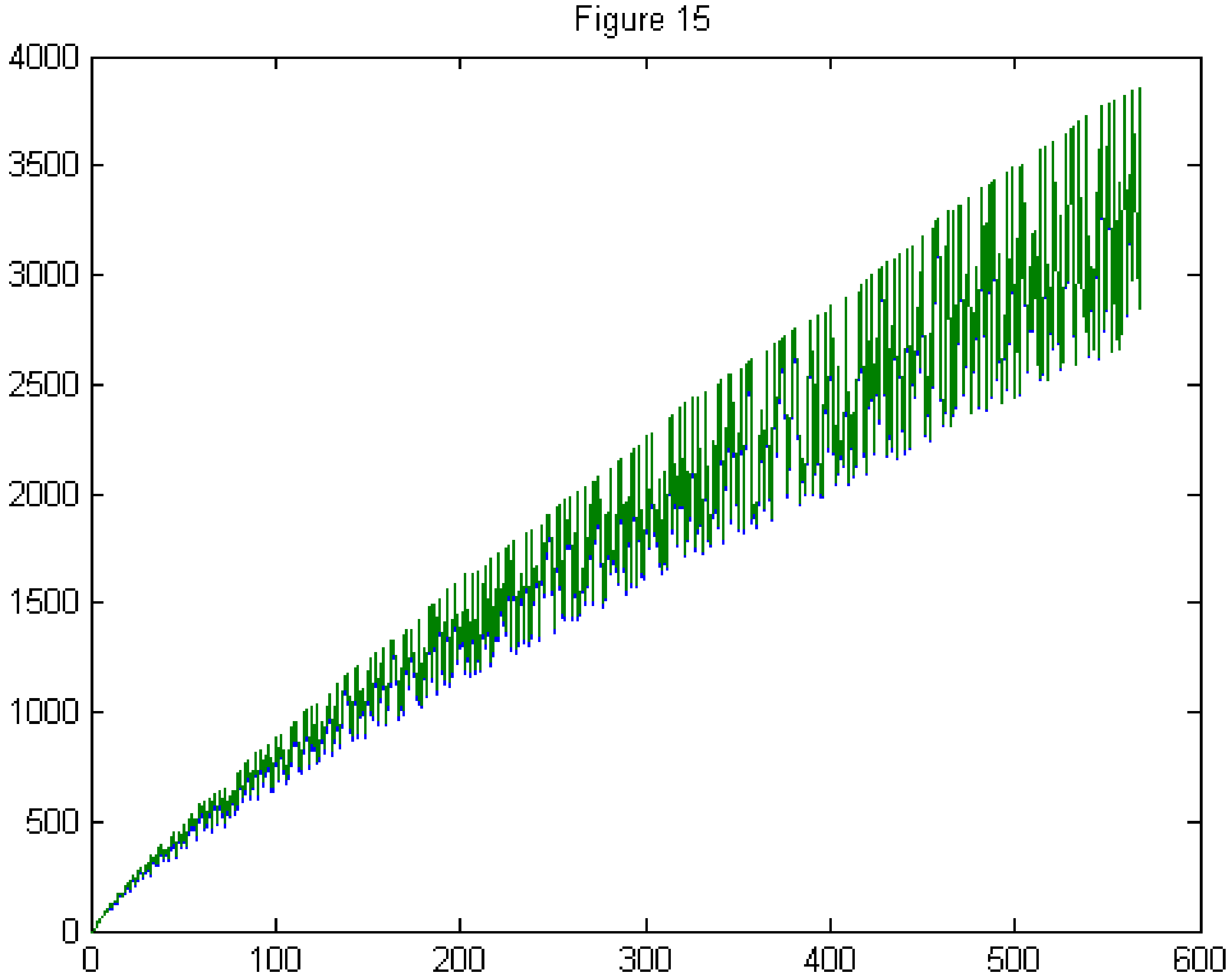


Figure 15
4000
3500
3000
2500
2000
1500
1000
500
0
0
100
200
300
400
500
600

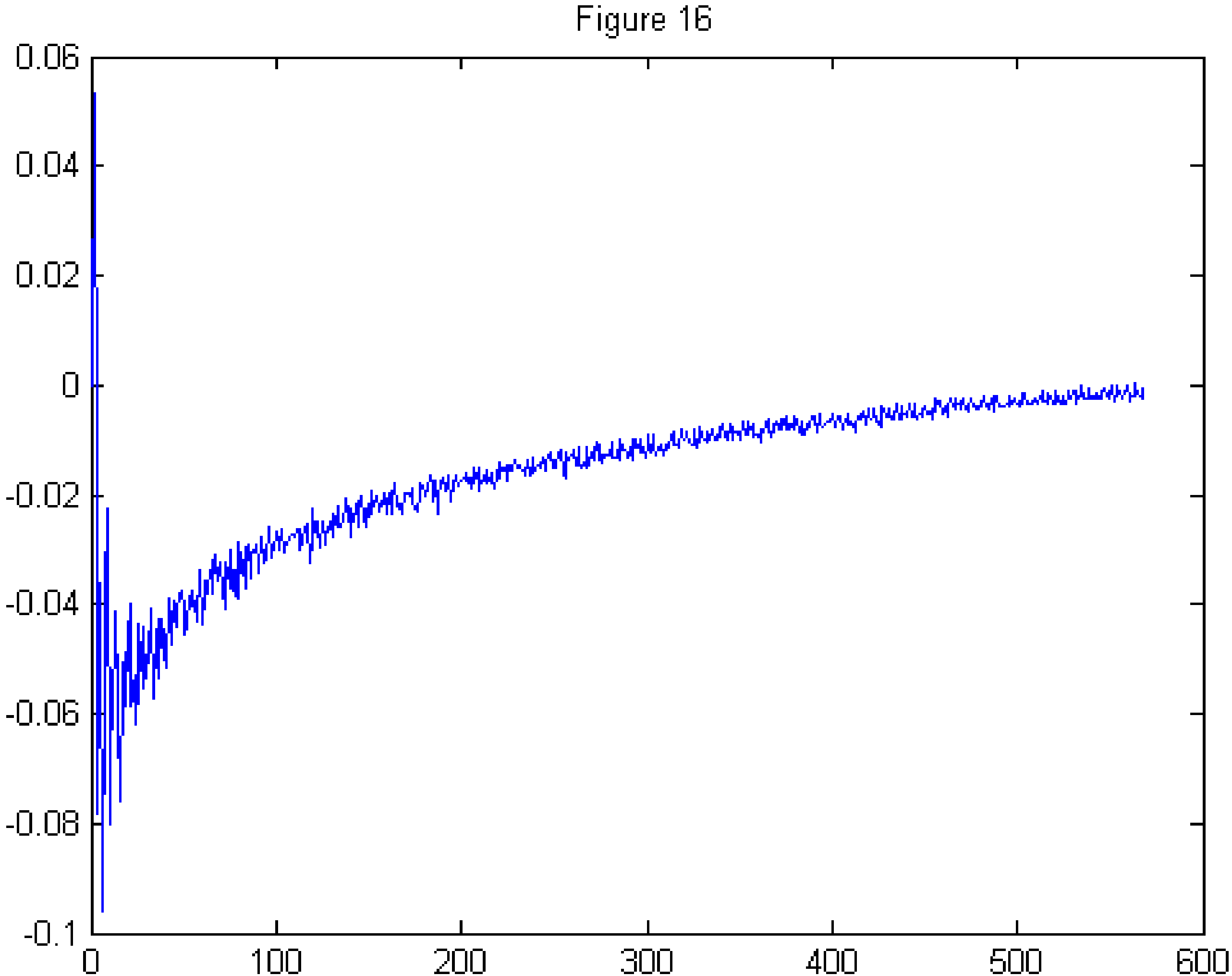


Figure 16
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
100
200
300
400
500
600

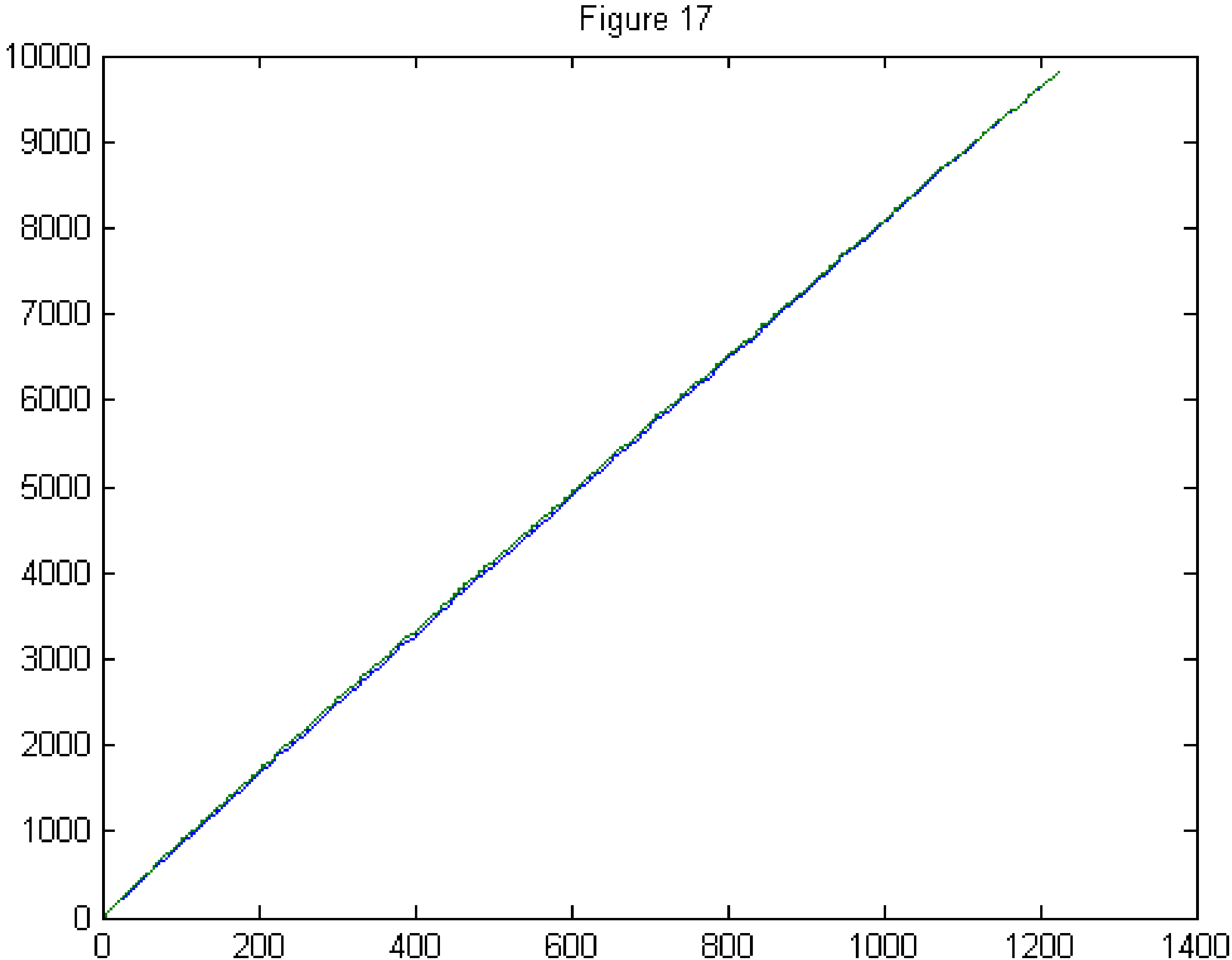


Figure 17
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
200
400
600
800
1000
1200
1400

Figure 18

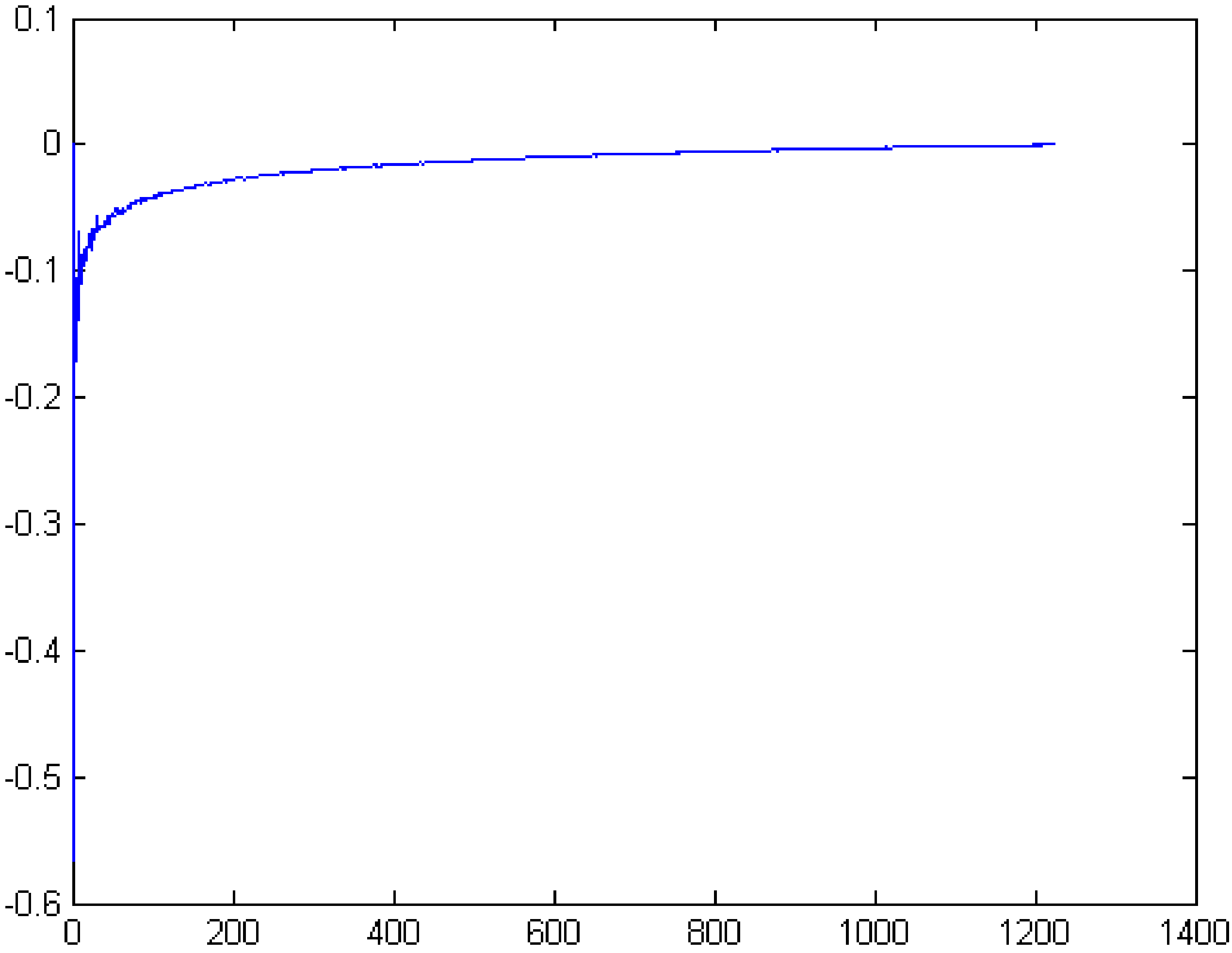
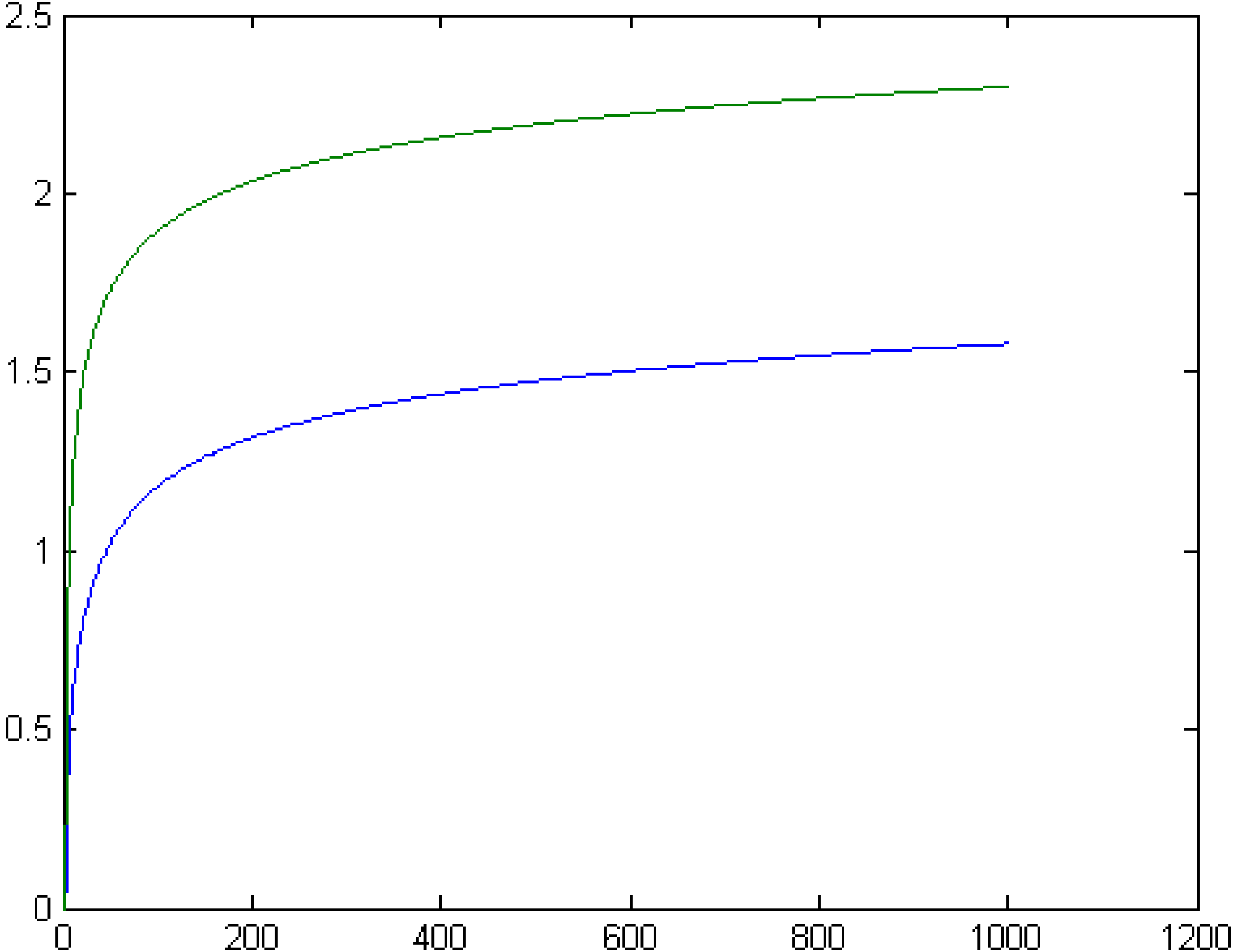


Figure 19



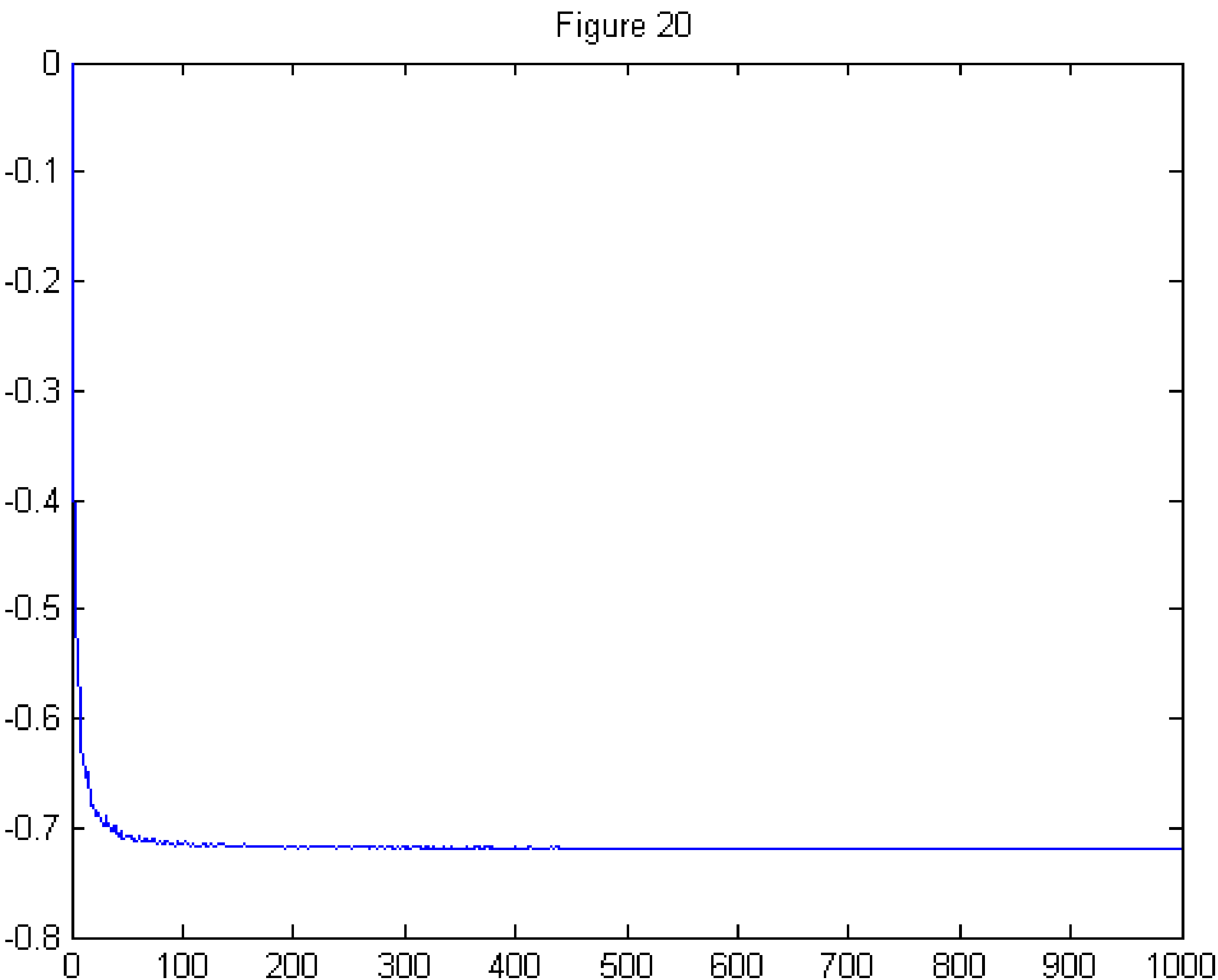


Figure 20
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0
100
200
300
400
500
600
700
800
900
1000

Figure 21

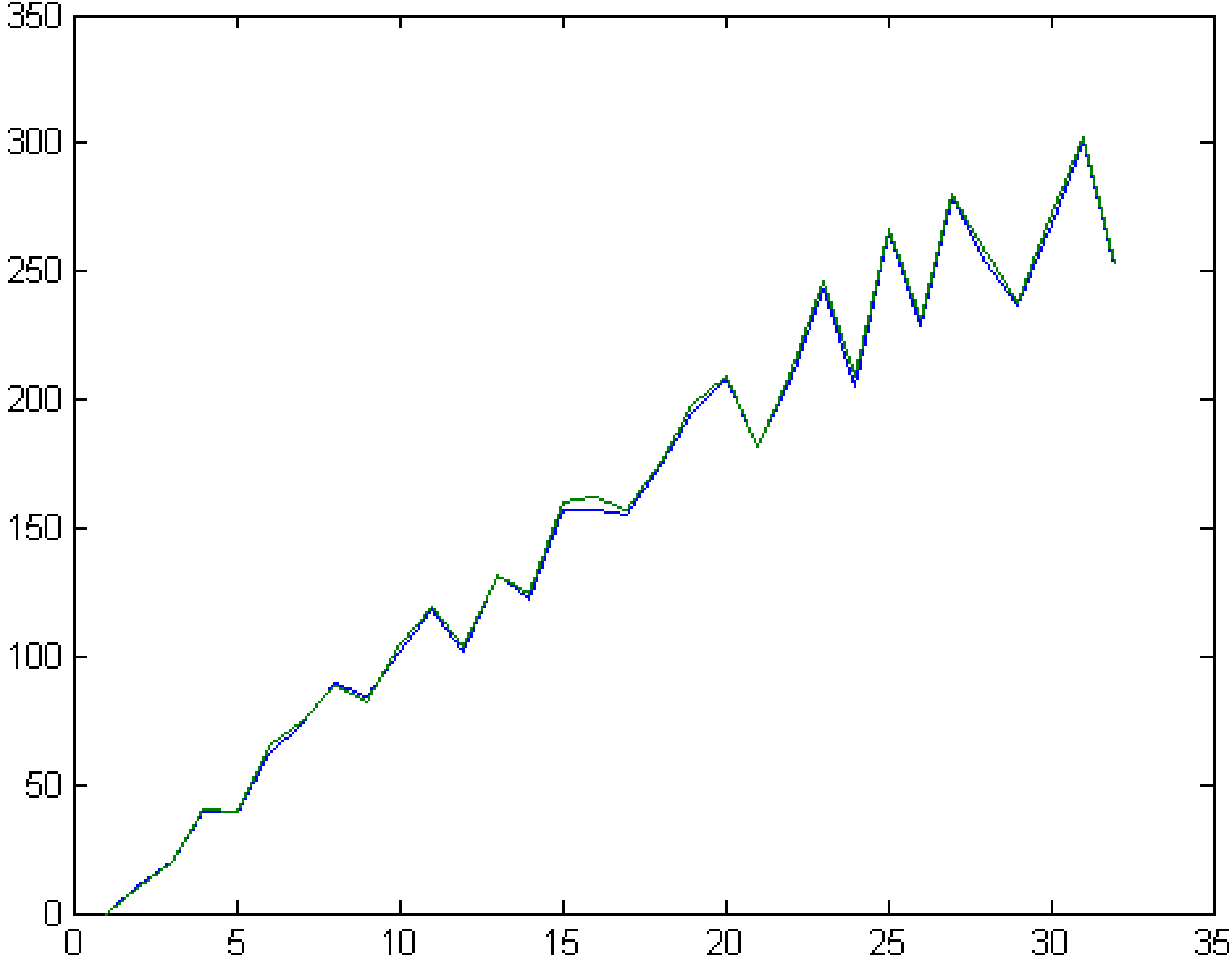
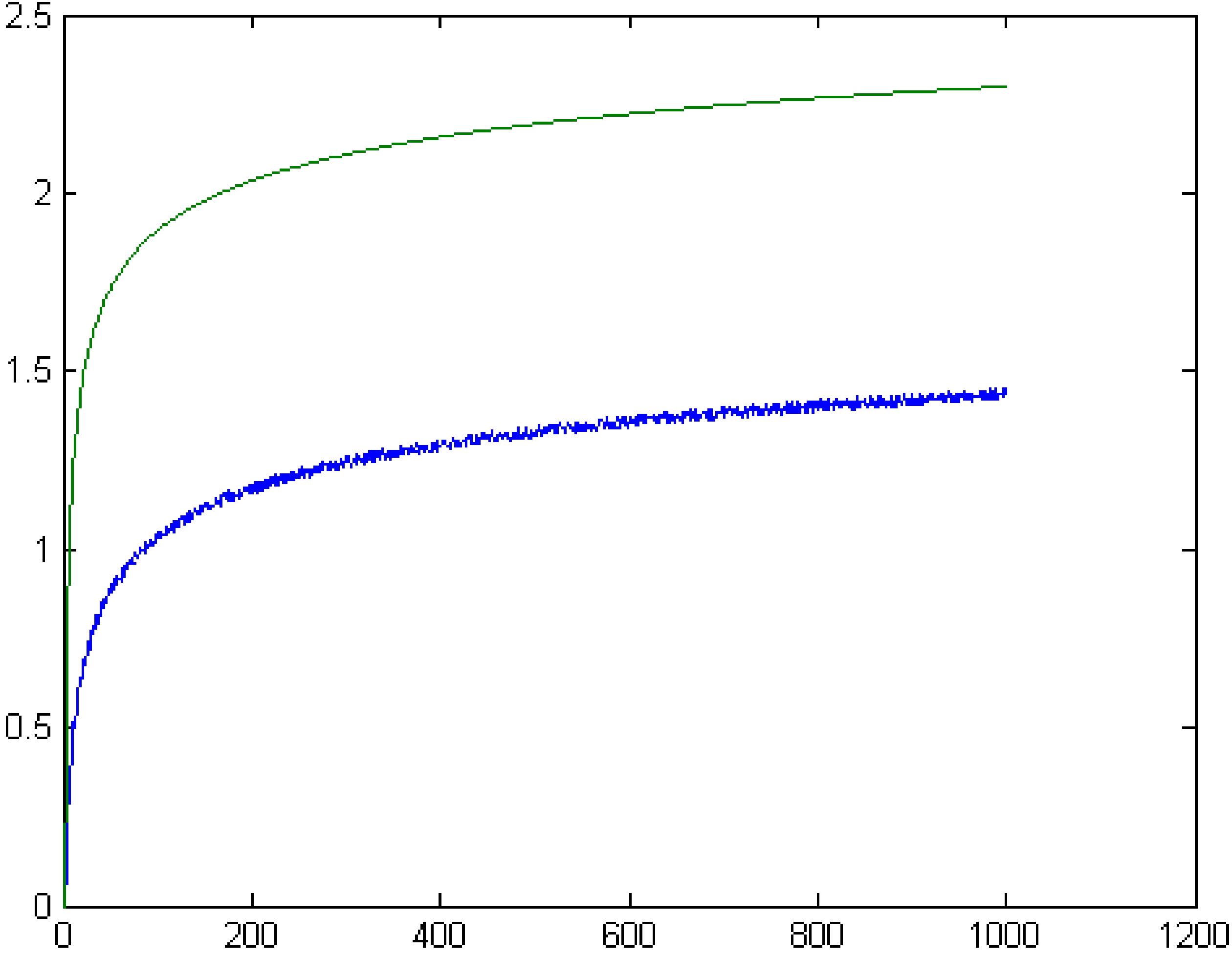


Figure 22



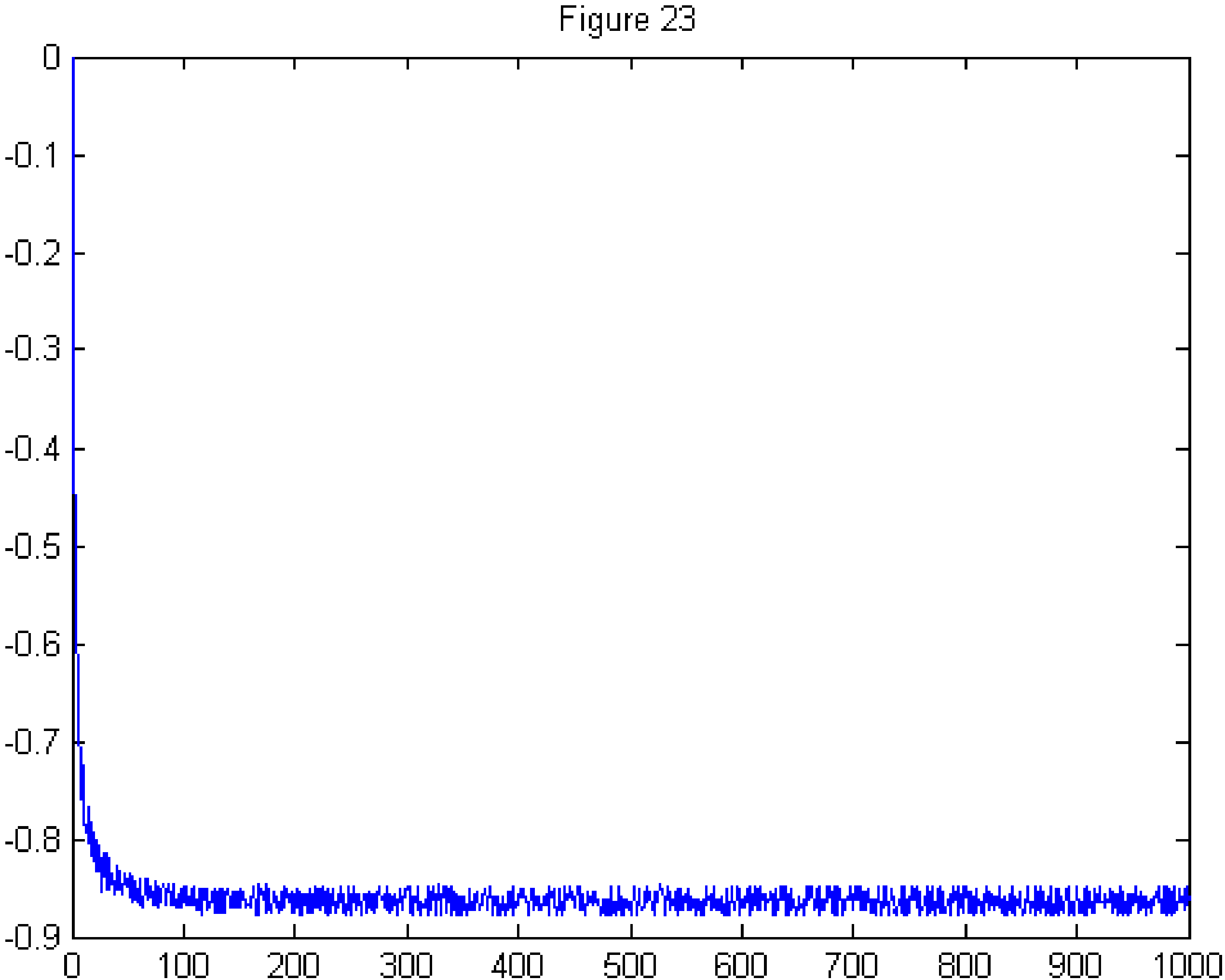


Figure 23
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
100
200
300
400
500
600
700
800
900
1000

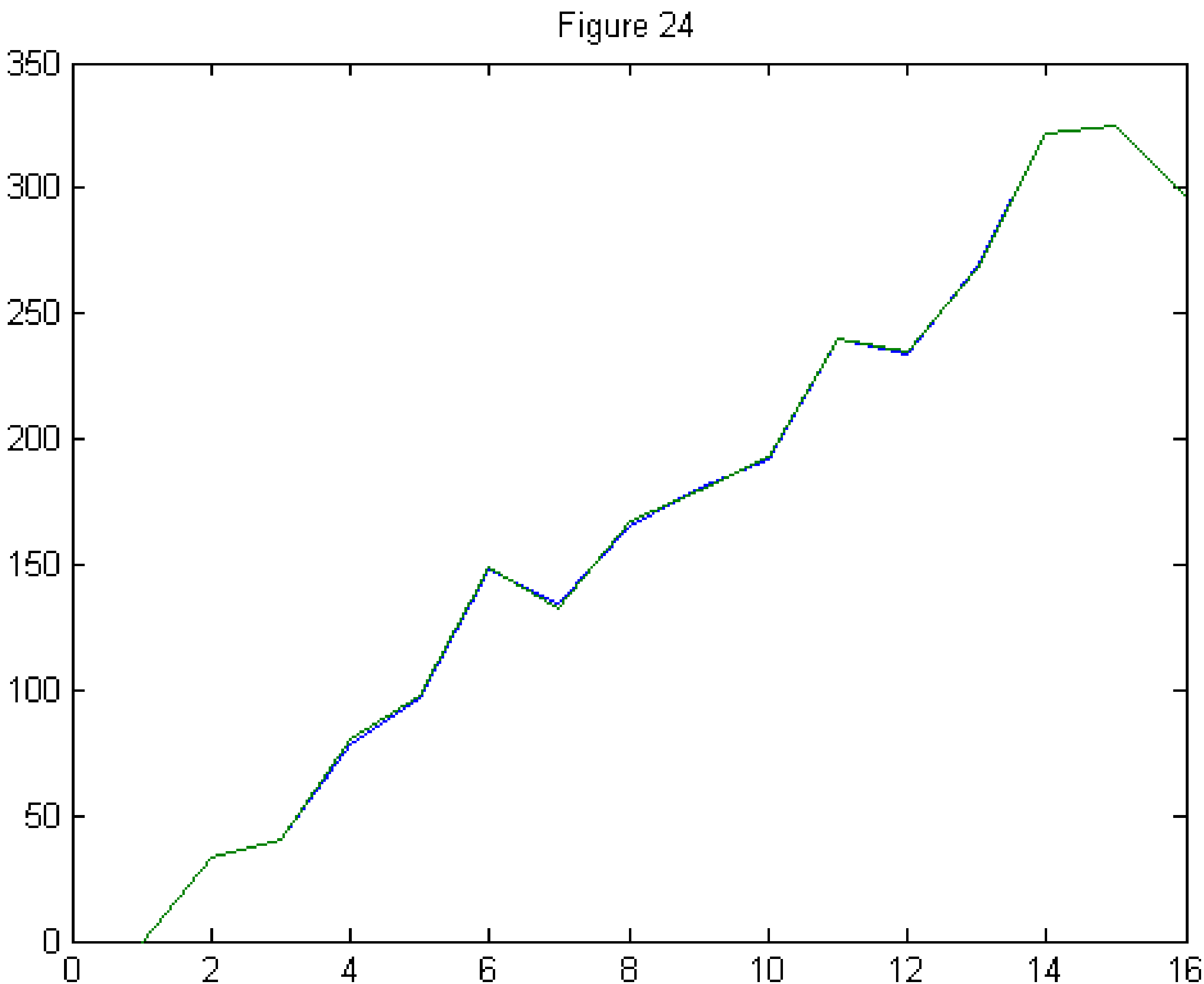
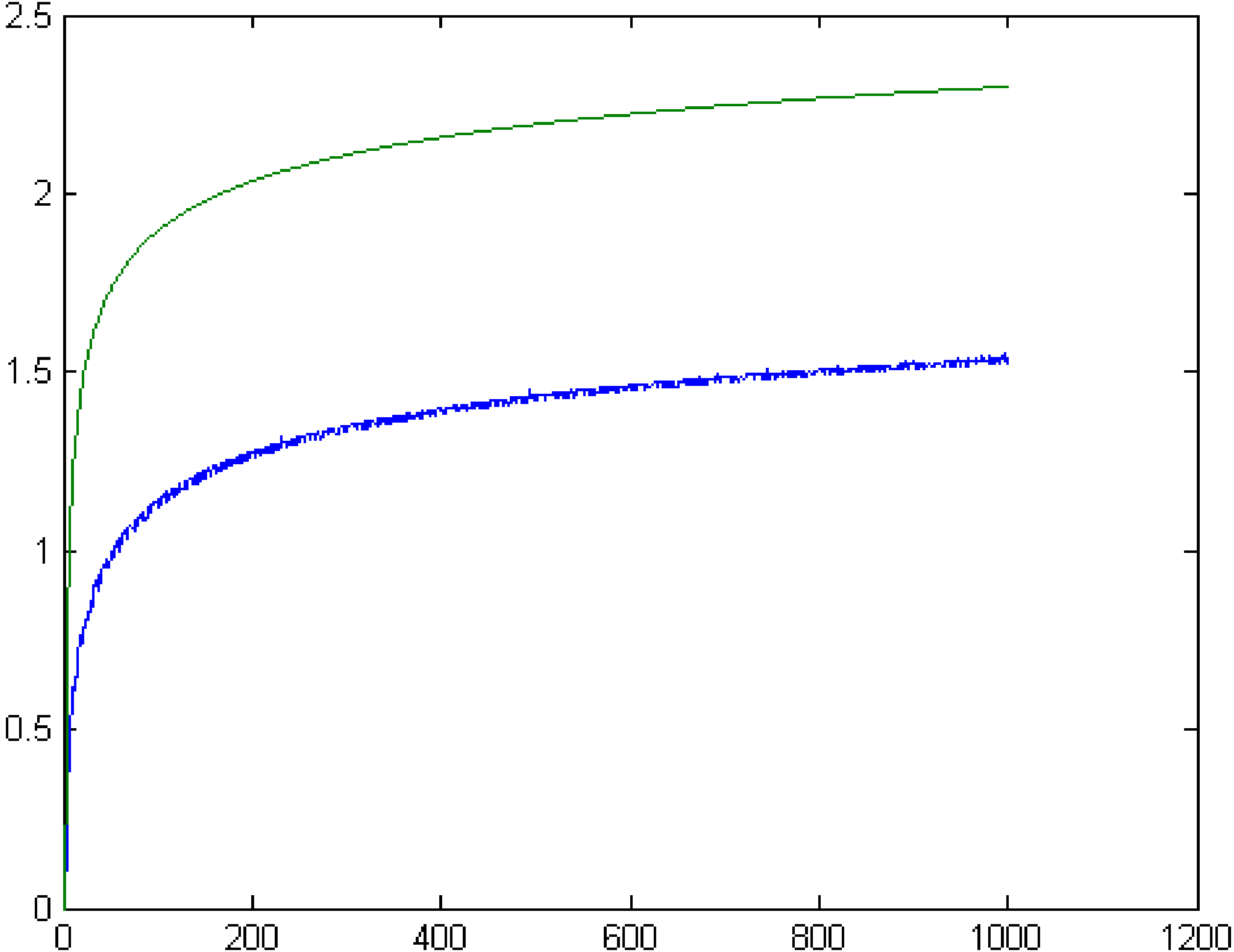


Figure 24
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
14
16

Figure 25



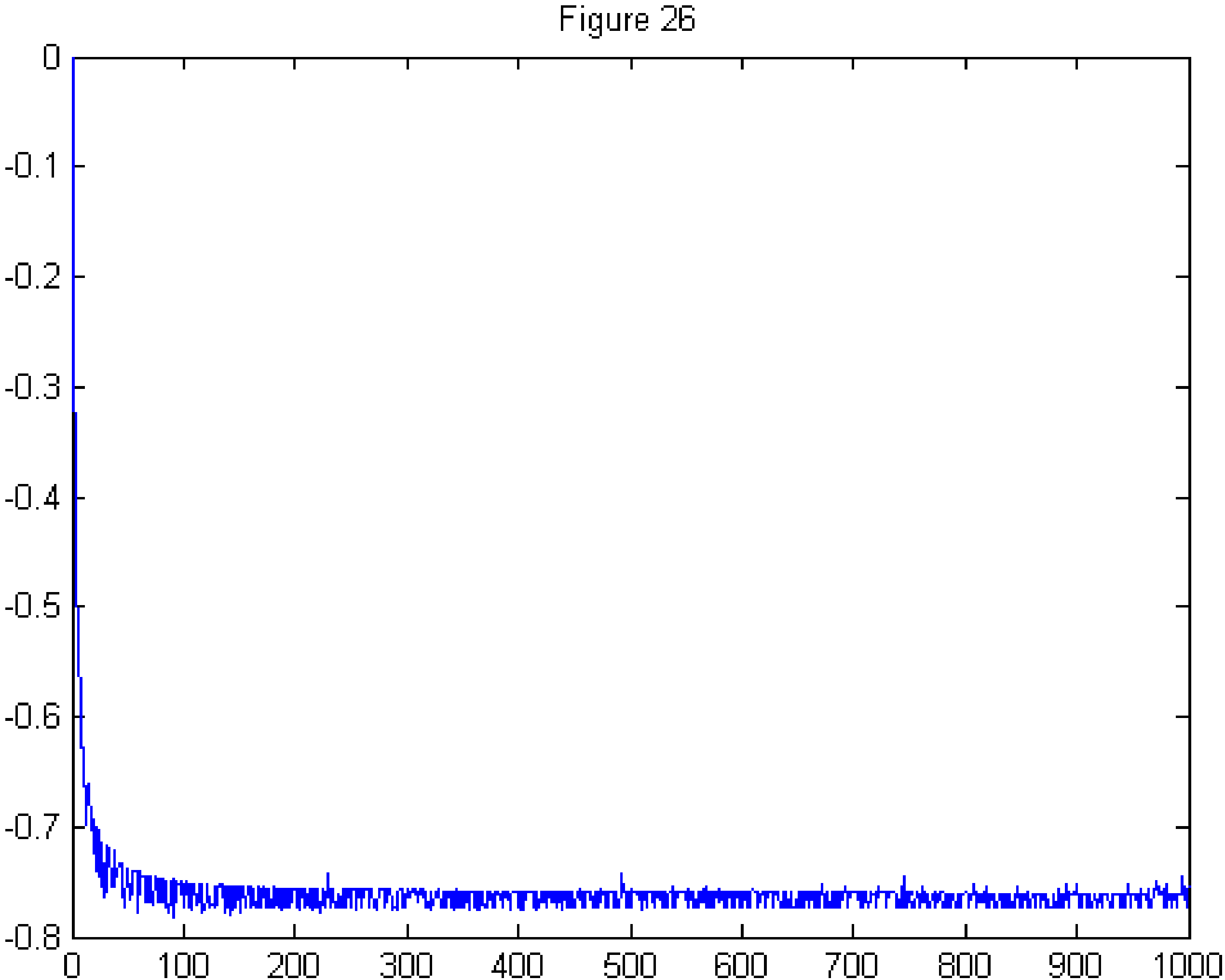


Figure 26
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0
100
200
300
400
500
600
700
800
900
1000

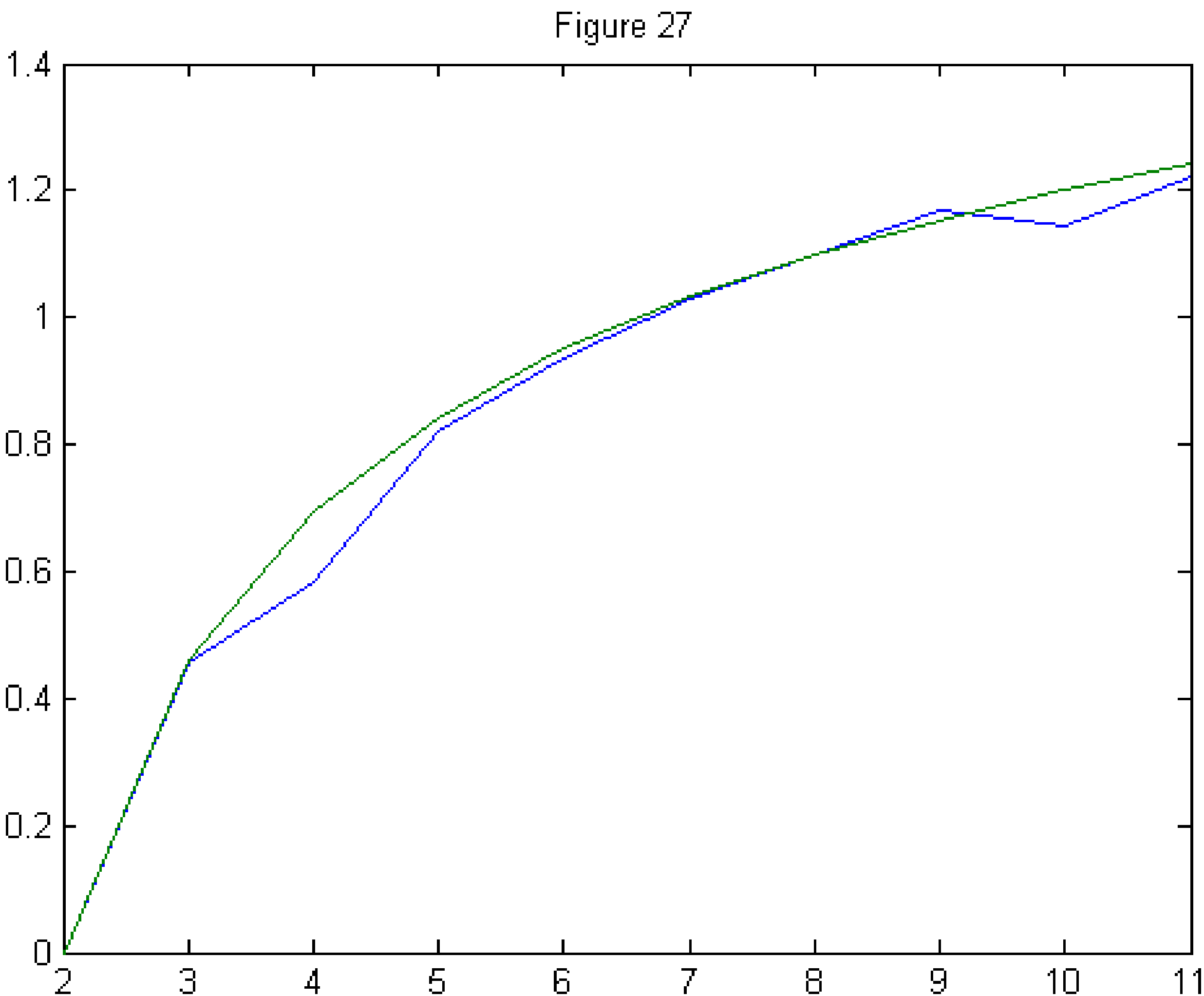


Figure 27
1.4
1.2
1
0.8
0.6
0.4
0.2
0
2
3
4
5
6
7
8
9
10
11

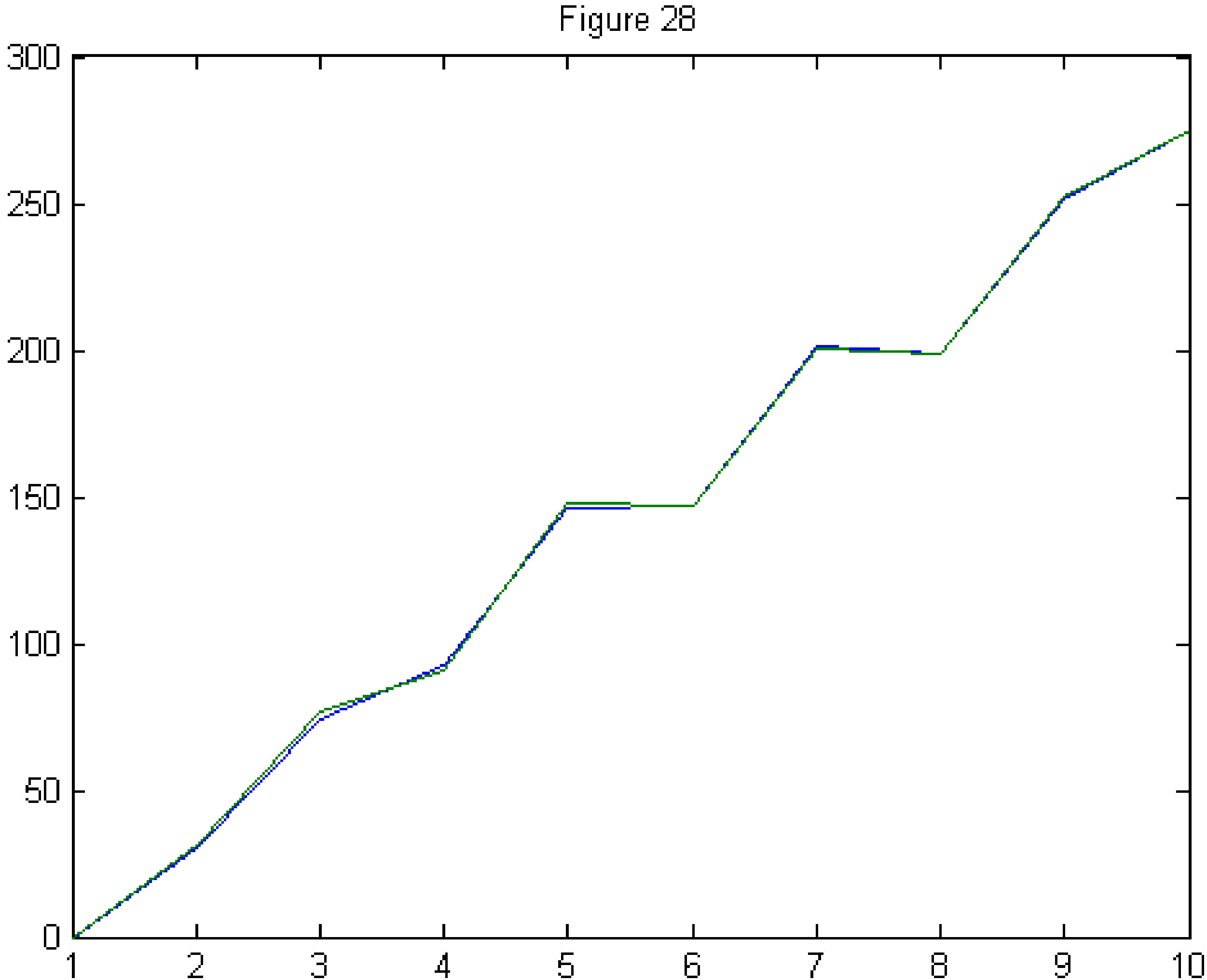


Figure 28
300
250
200
150
100
50
0
1
2
3
4
5
6
7
8
9
10

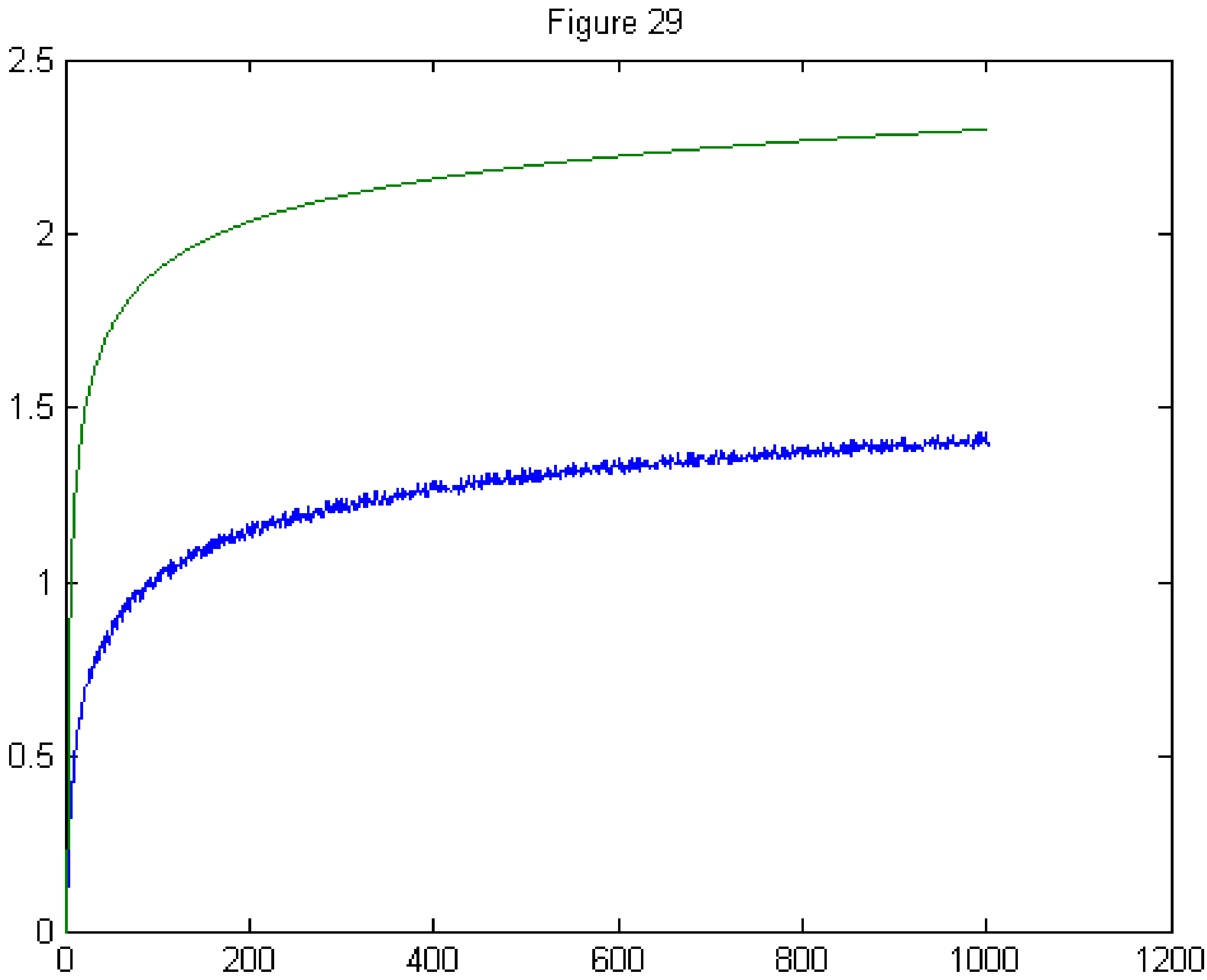


Figure 29
2.5
2
1.5
1
0.5
0
0
200
400
600
800
1000
1200

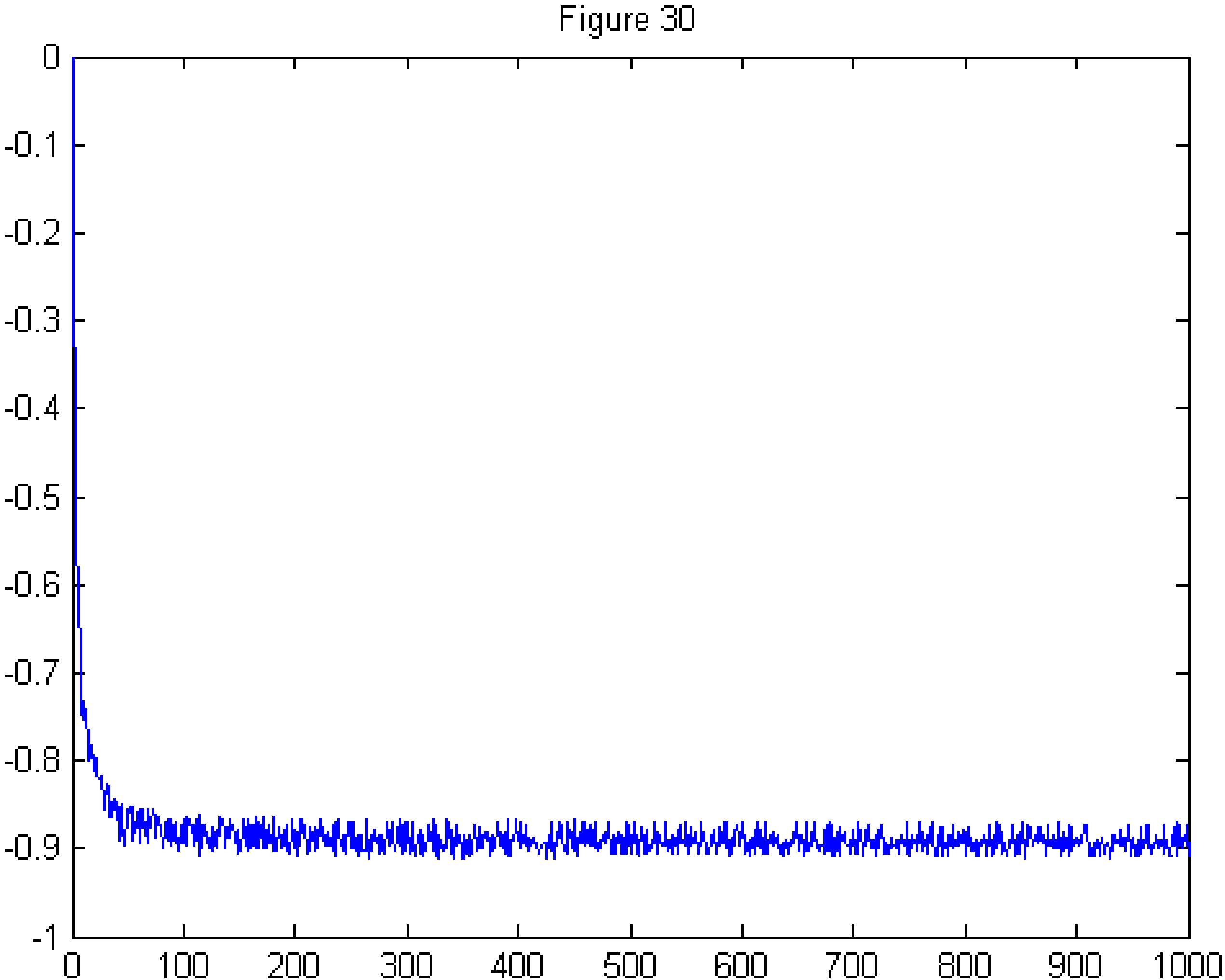


Figure 30
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
0
100
200
300
400
500
600
700
800
900
1000

Figure 31

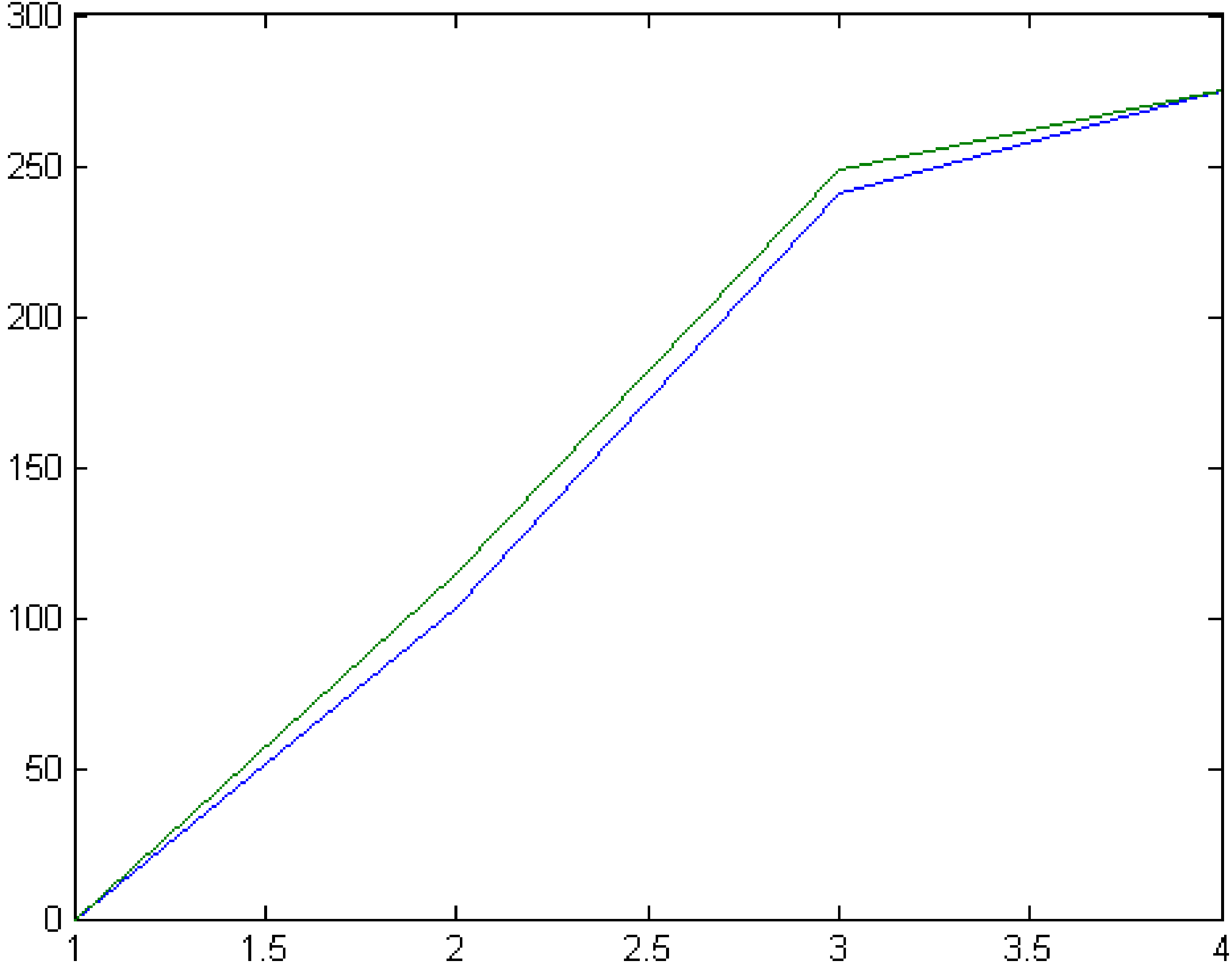
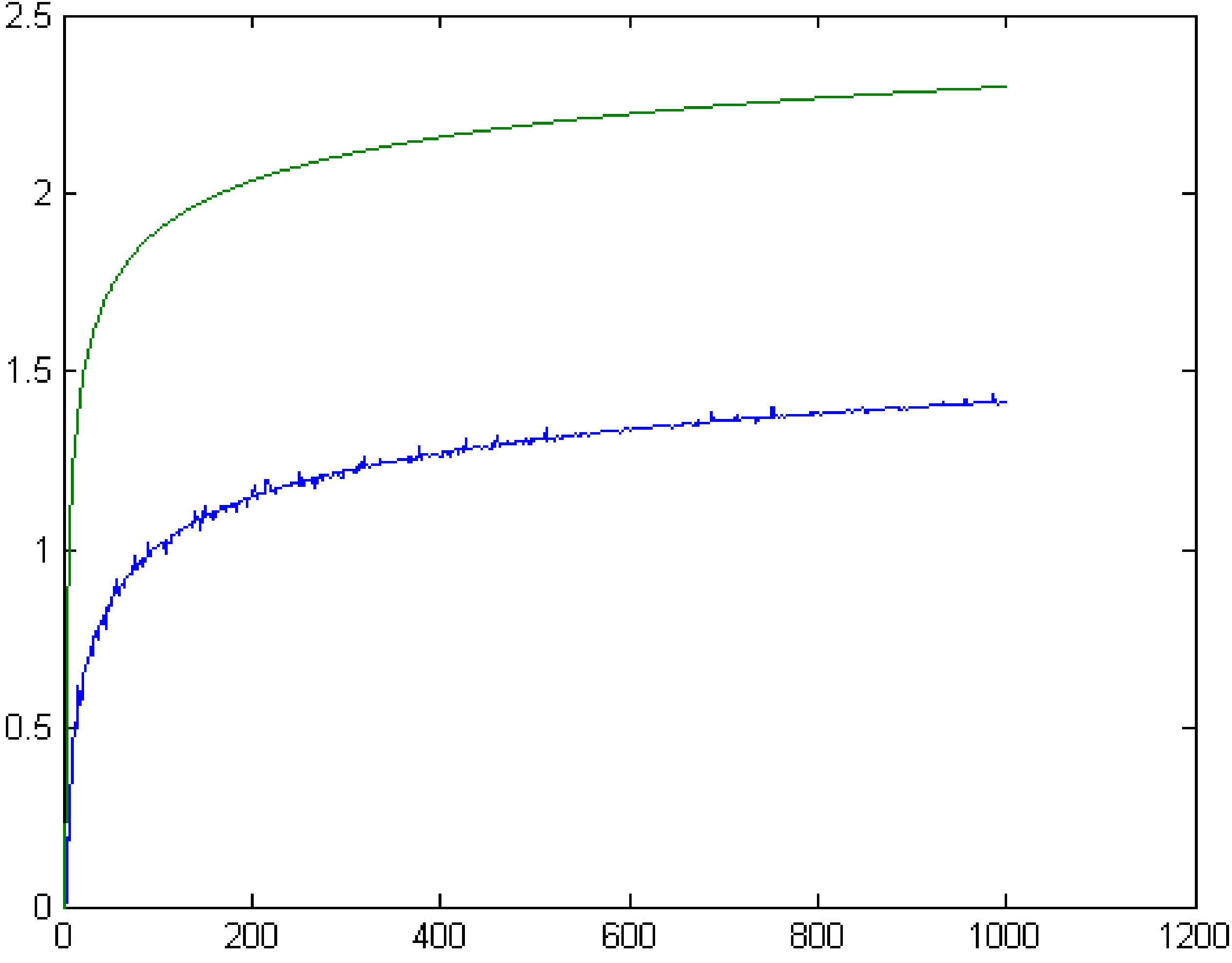


Figure 32



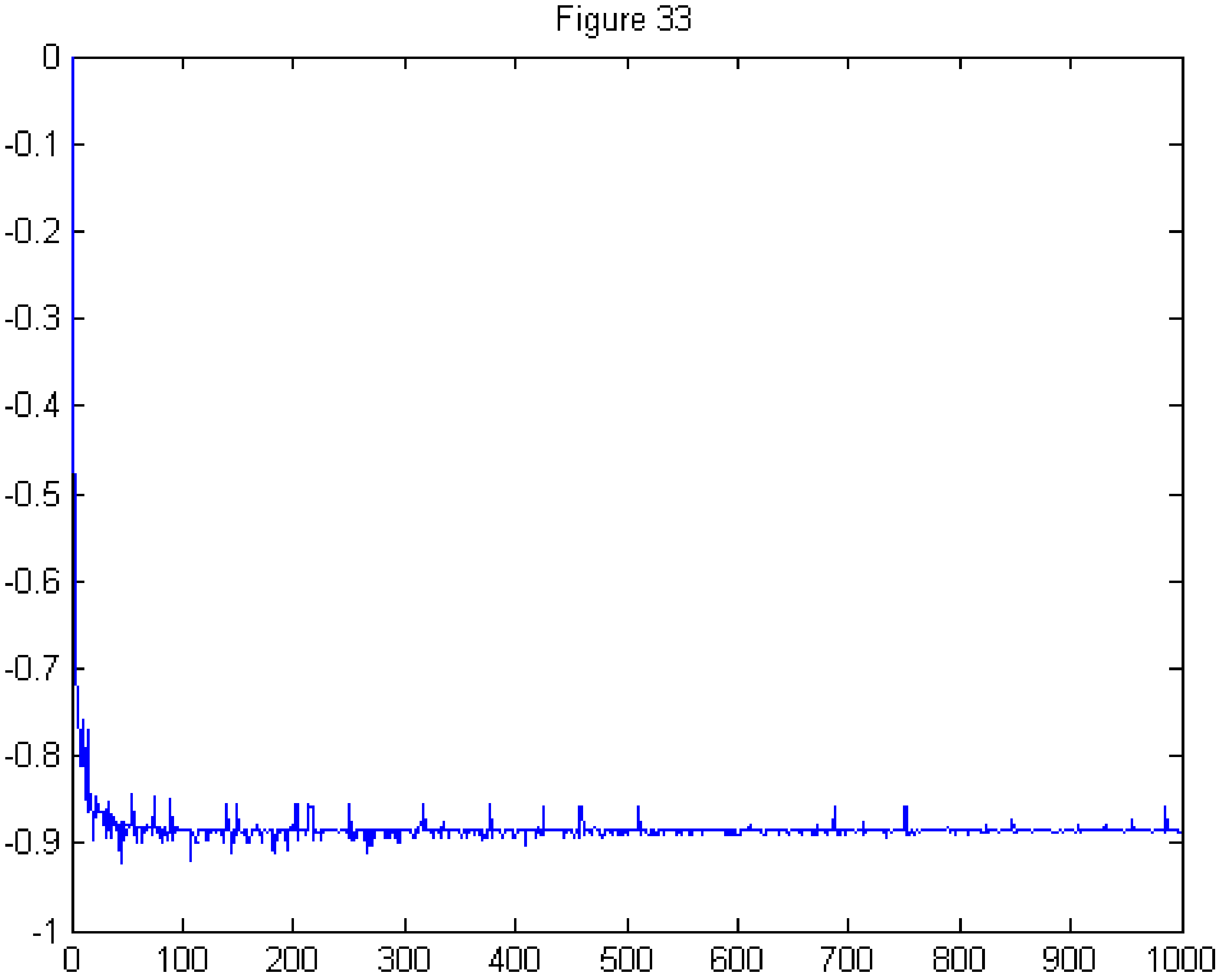
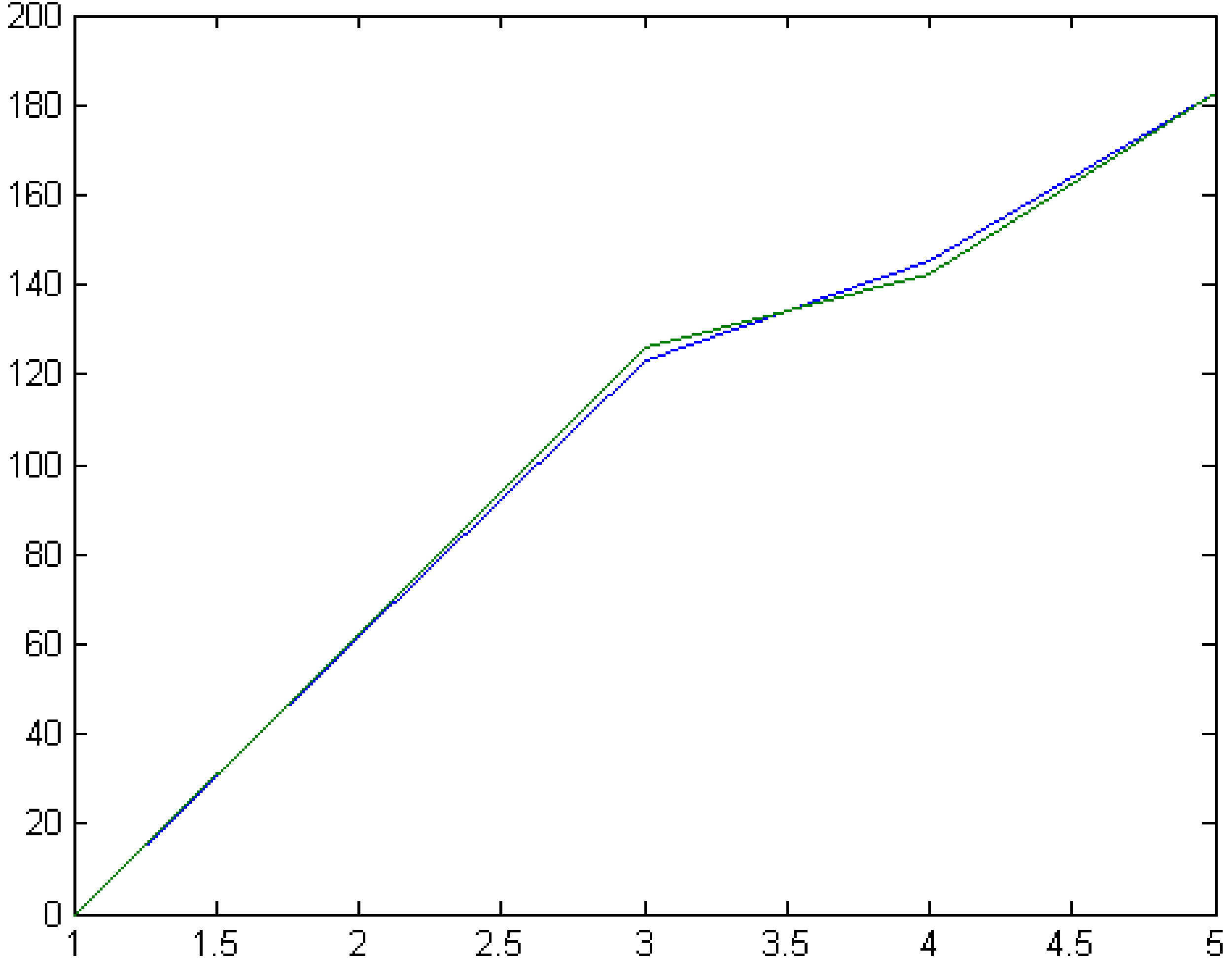


Figure 33
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
0
100
200
300
400
500
600
700
800
900
1000

Figure 34



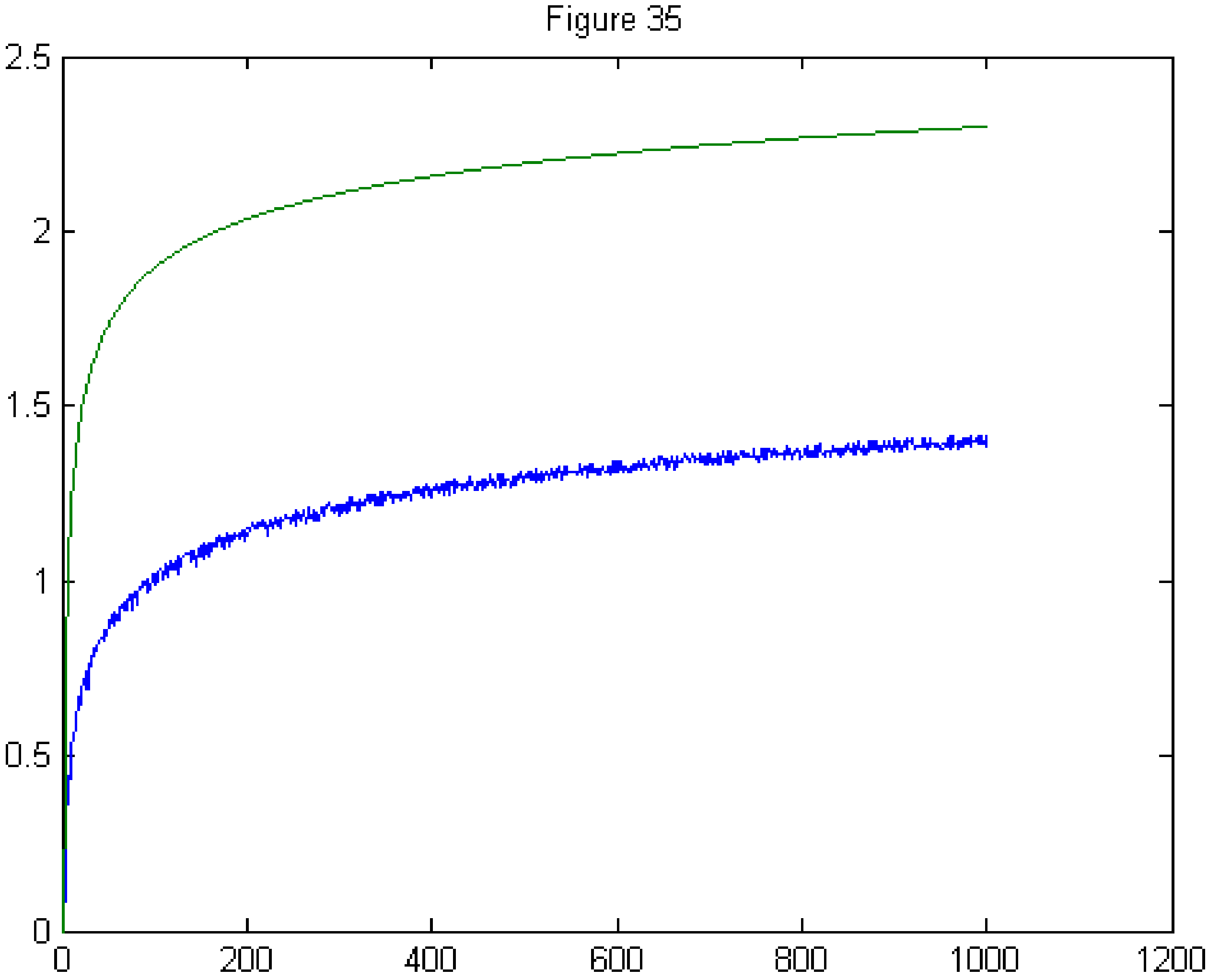


Figure 35
2.5
2
1.5
1
0.5
0
0
200
400
600
800
1000
1200

Figure 36

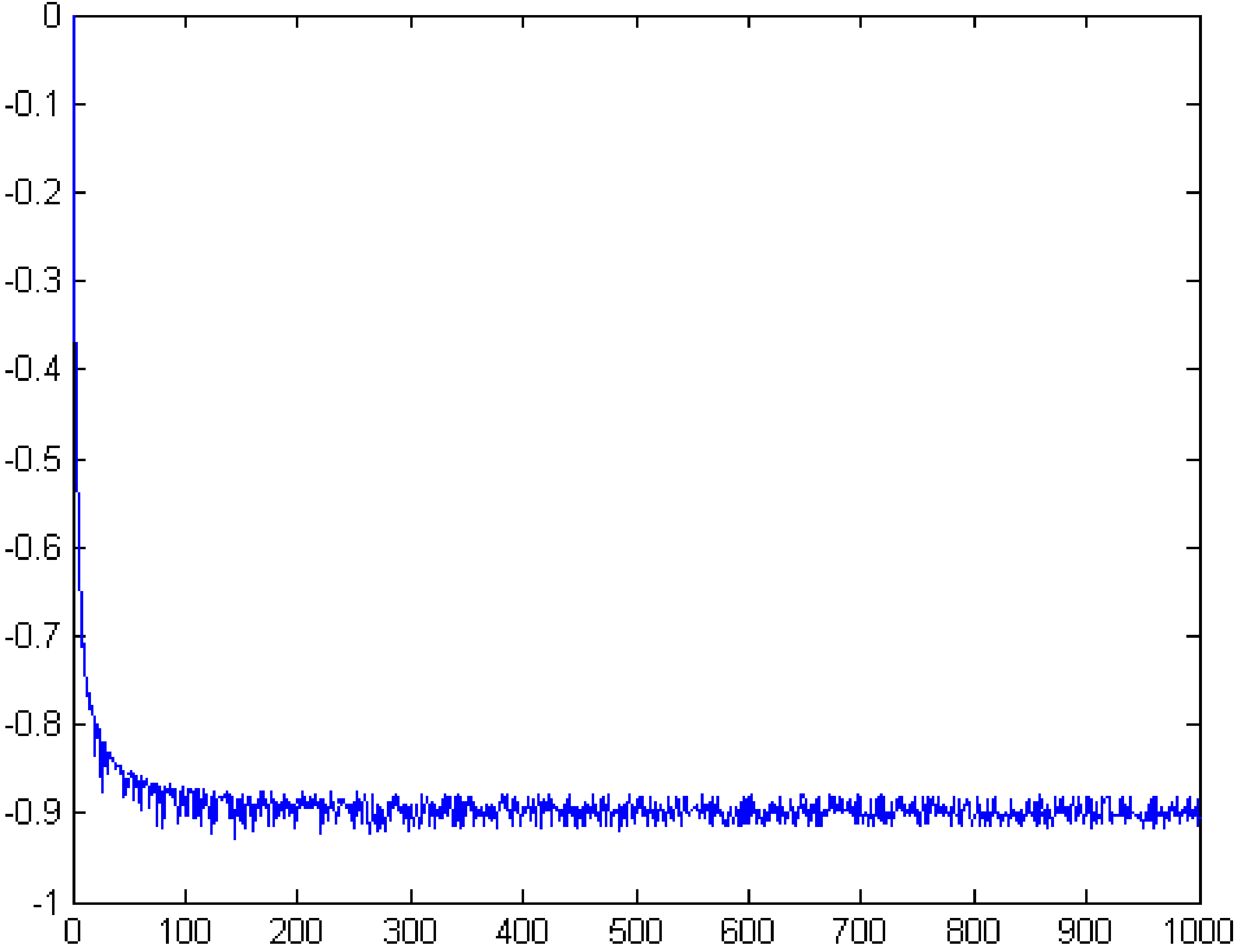
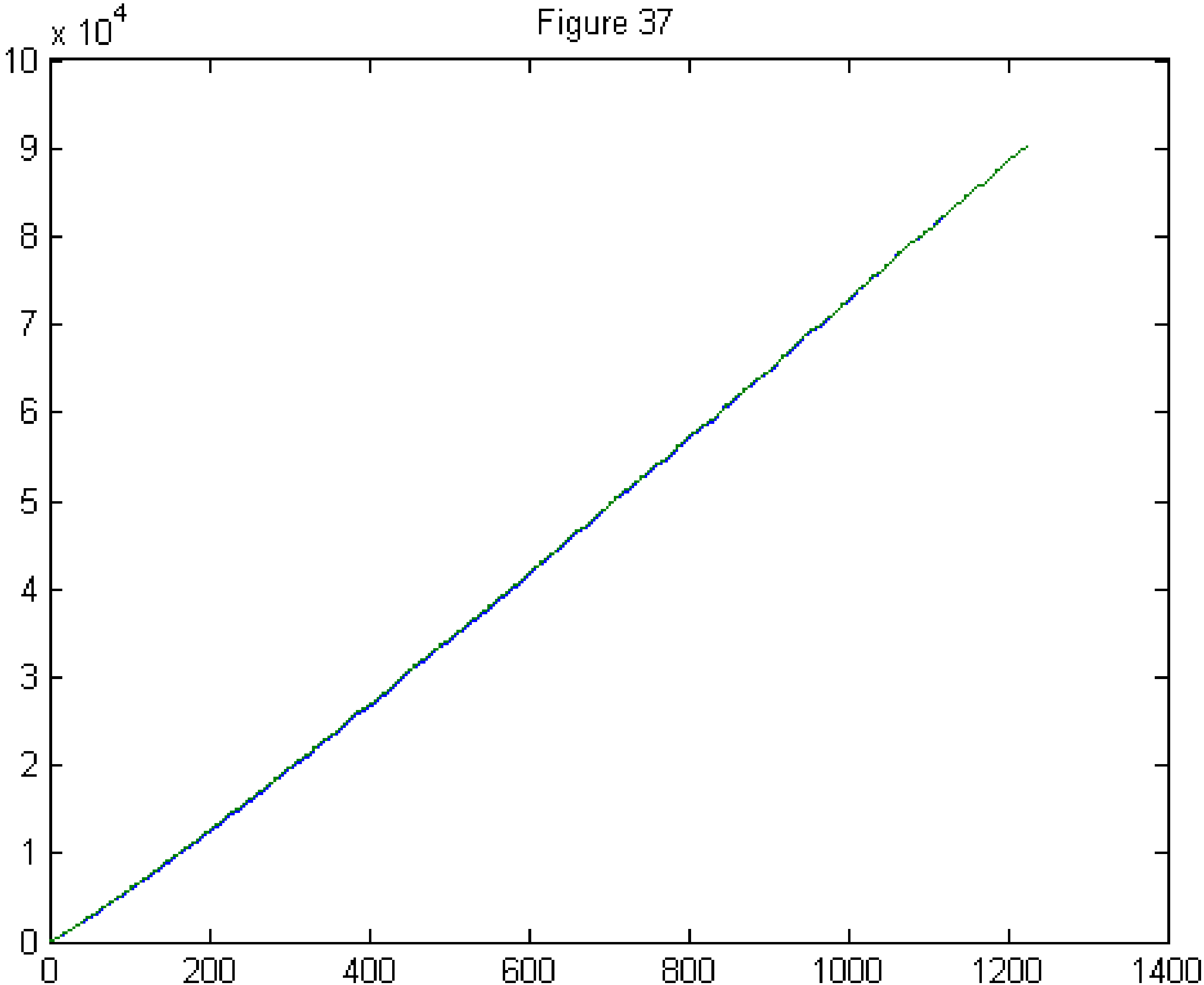

0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
0
100
200
300
400
500
600
700
800
900
1000

Figure 37



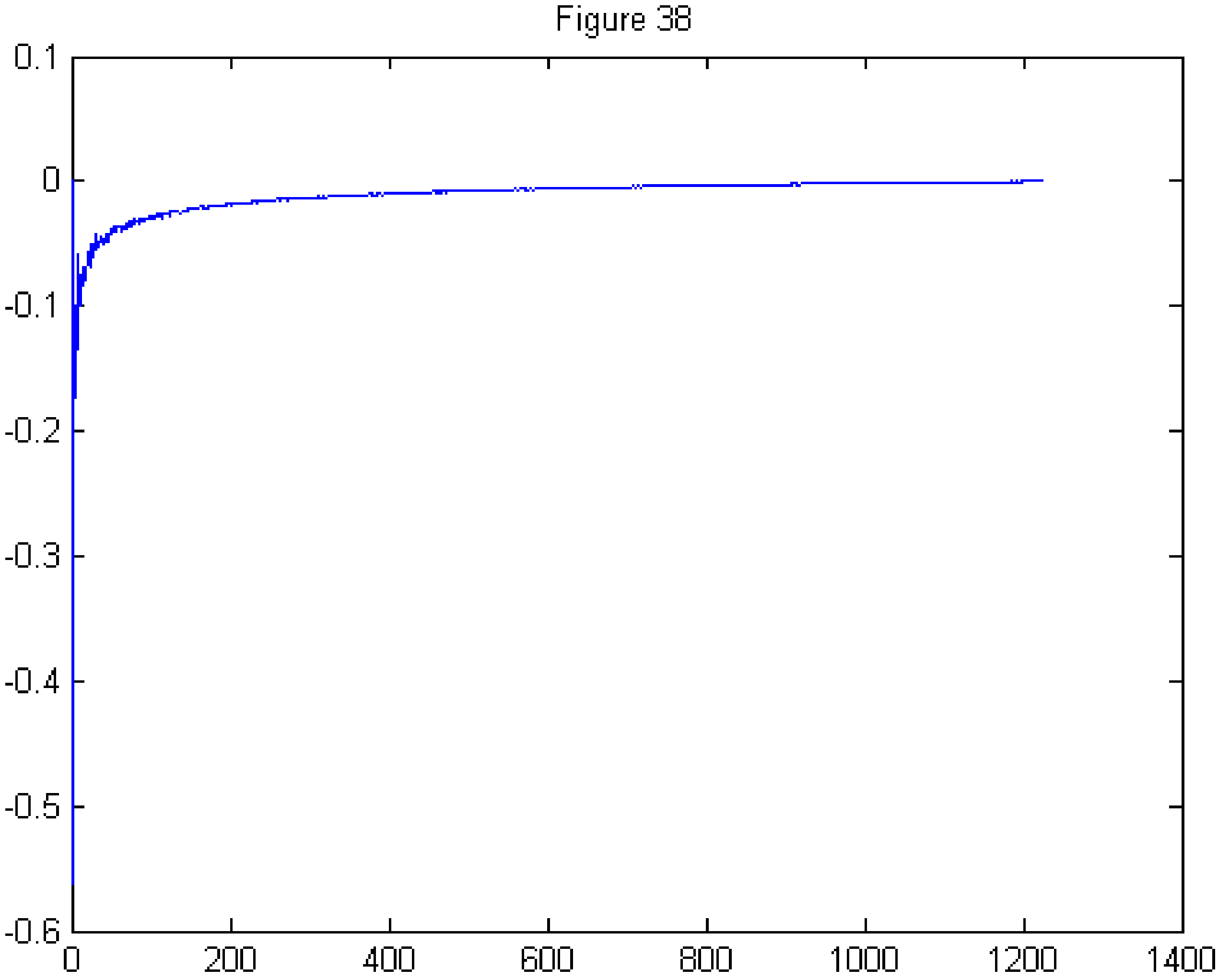


Figure 38
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
200
400
600
800
1000
1200
1400

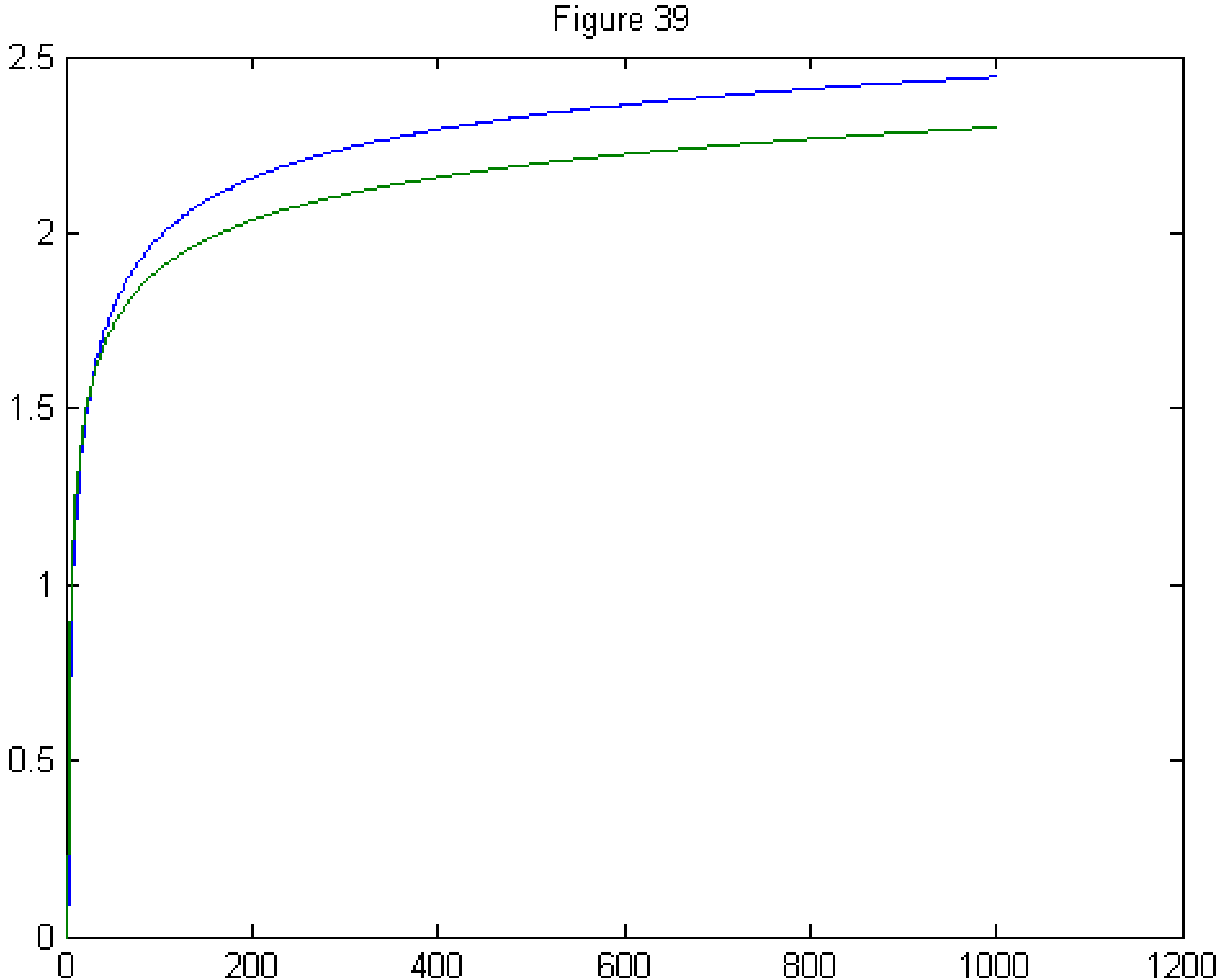


Figure 39
2.5
2
1.5
1
0.5
0
0
200
400
600
800
1000
1200

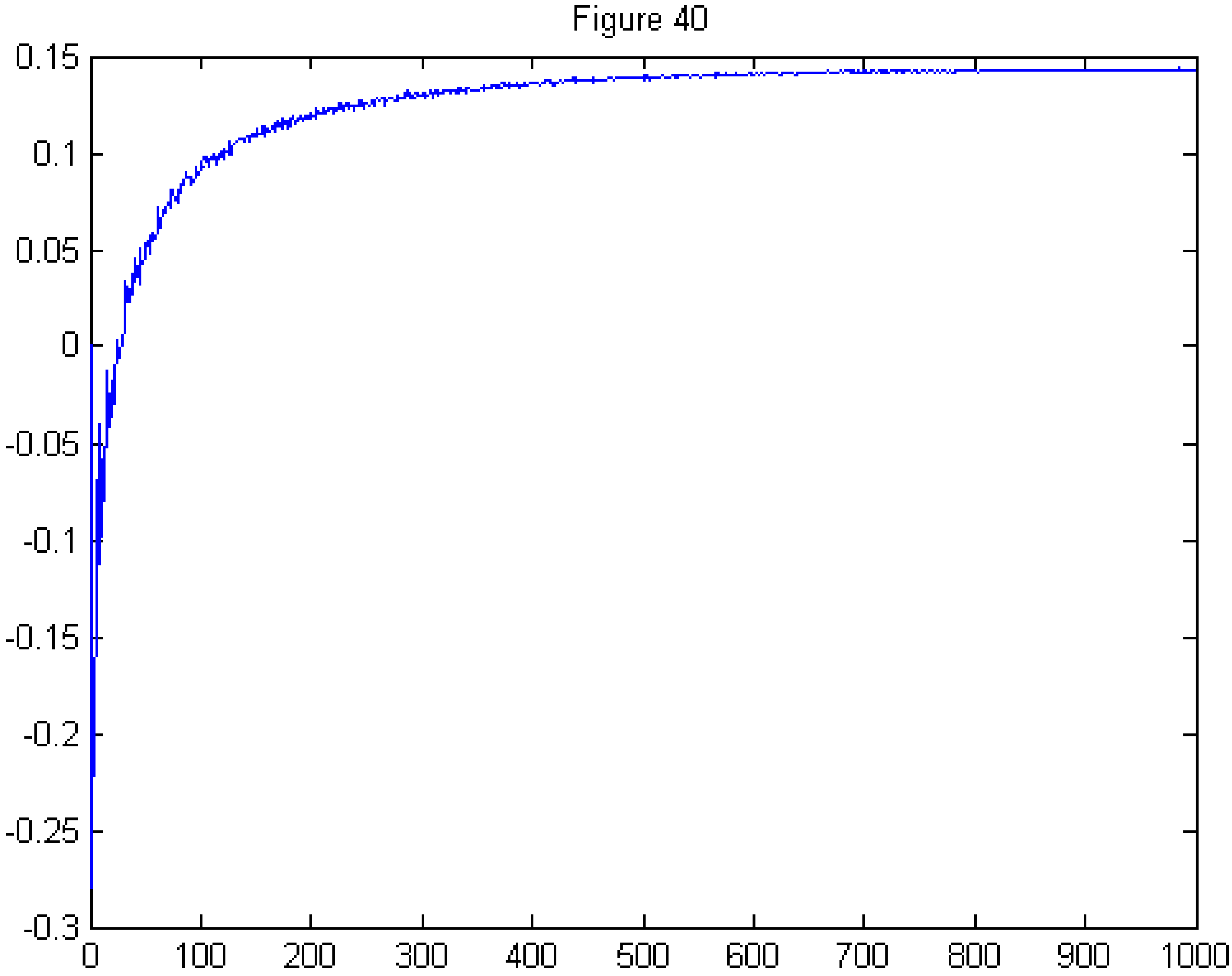


Figure 40
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
0
100
200
300
400
500
600
700
800
900
1000

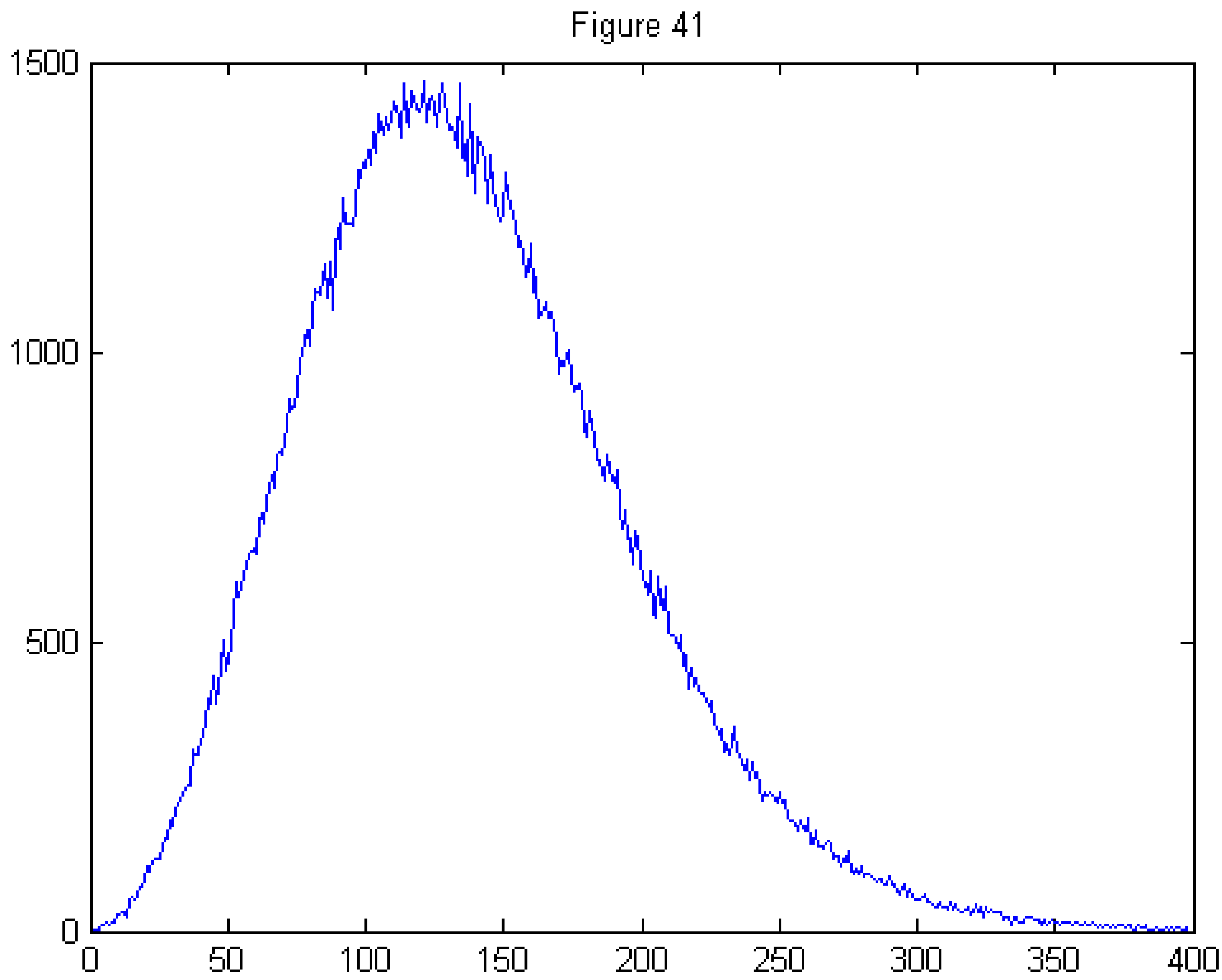
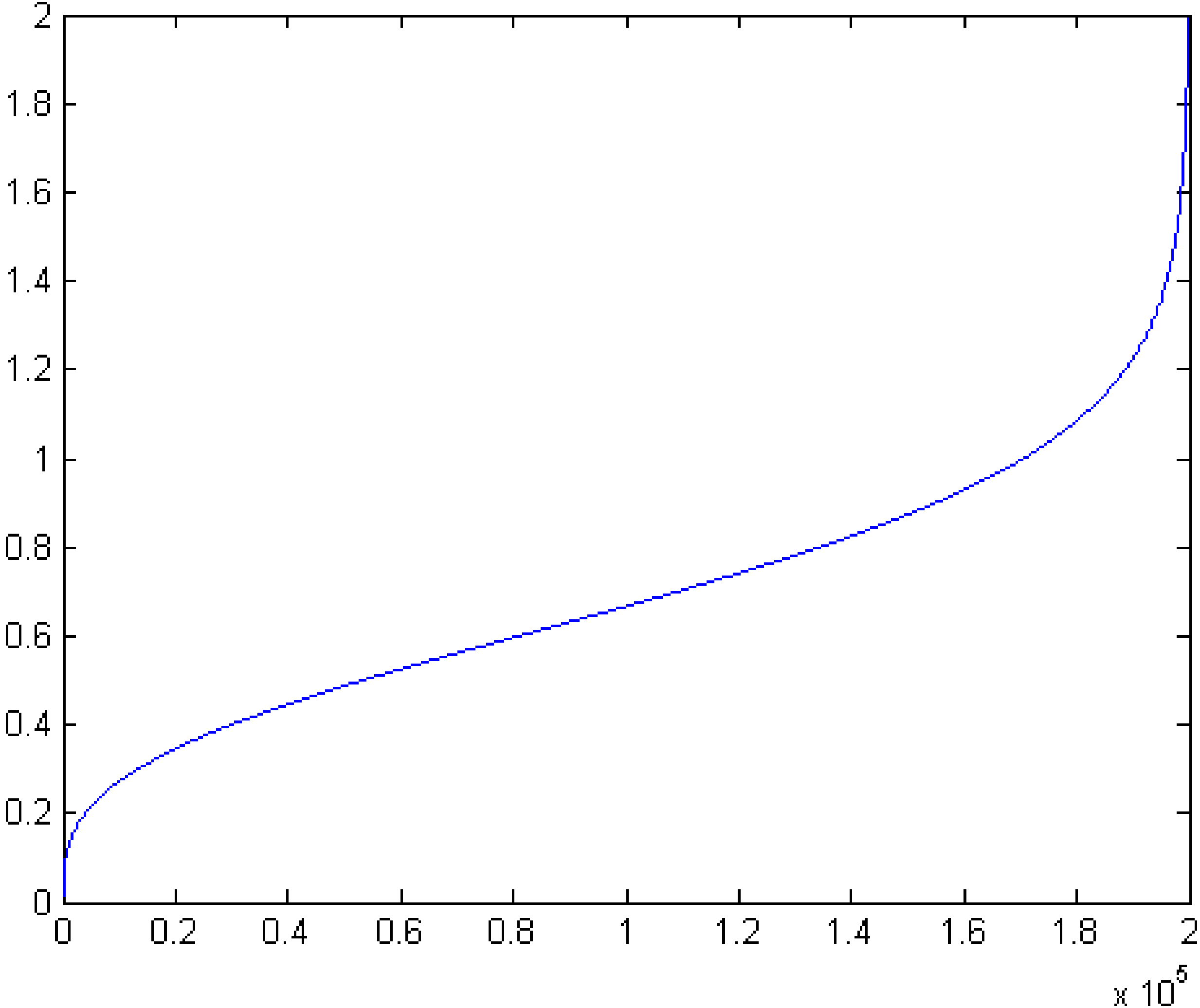
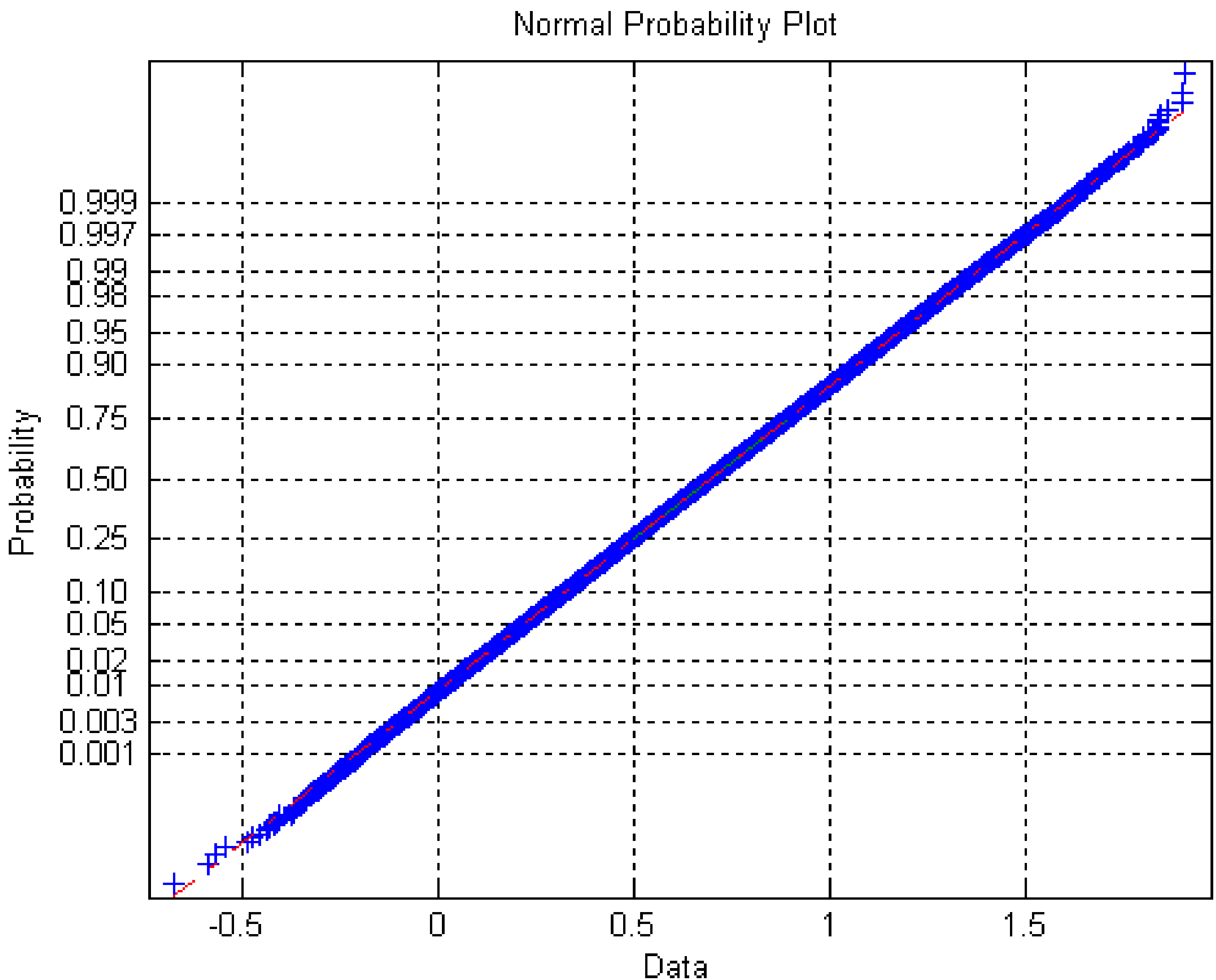


Figure 41
1500
1000
500
0
0
50
100
150
200
250
300
350
400

Figure 42




Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-0.5
0
0.5
1
1.5
Data

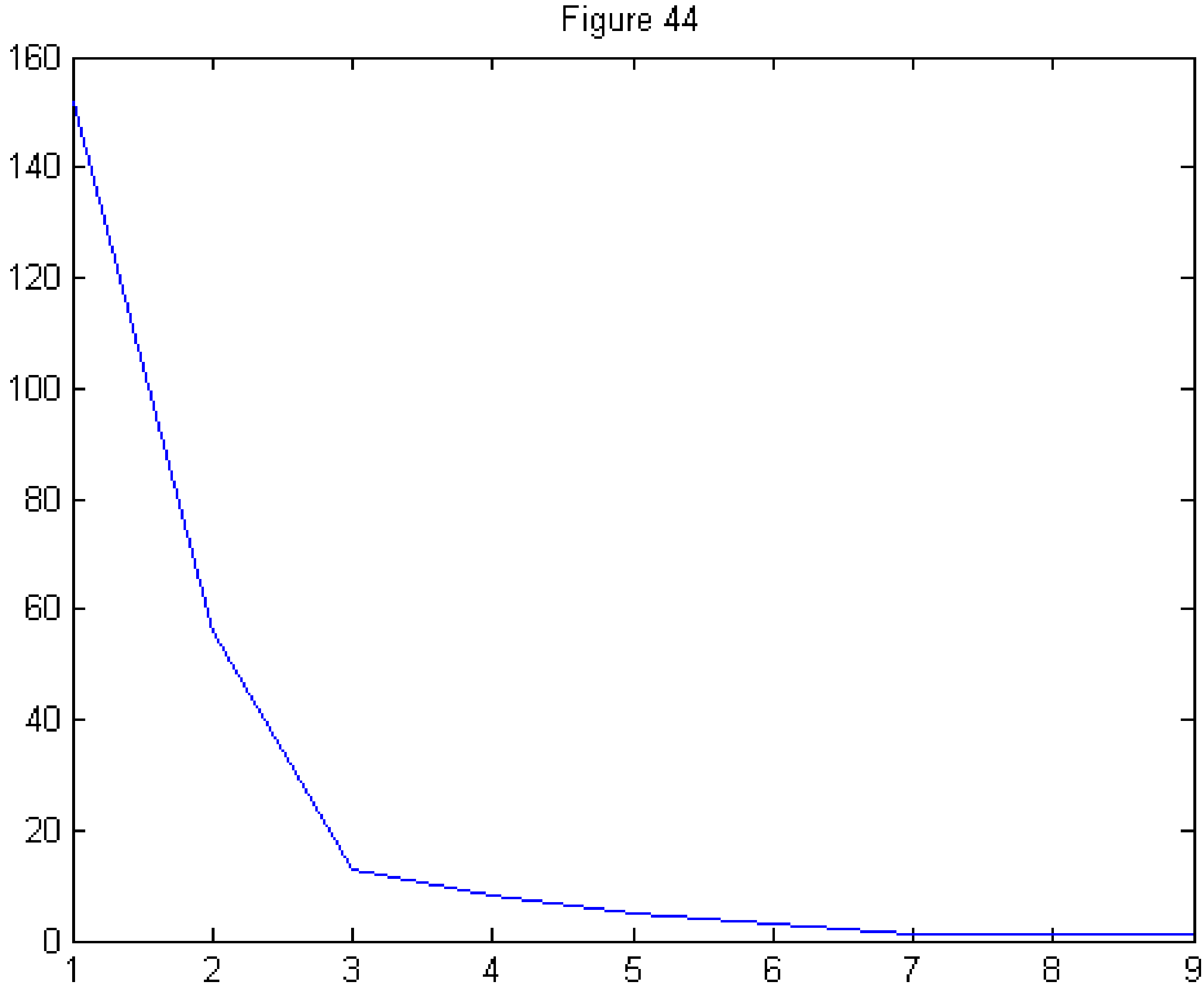


Figure 44
160
140
120
100
80
60
40
20
0
1
2
3
4
5
6
7
8
9

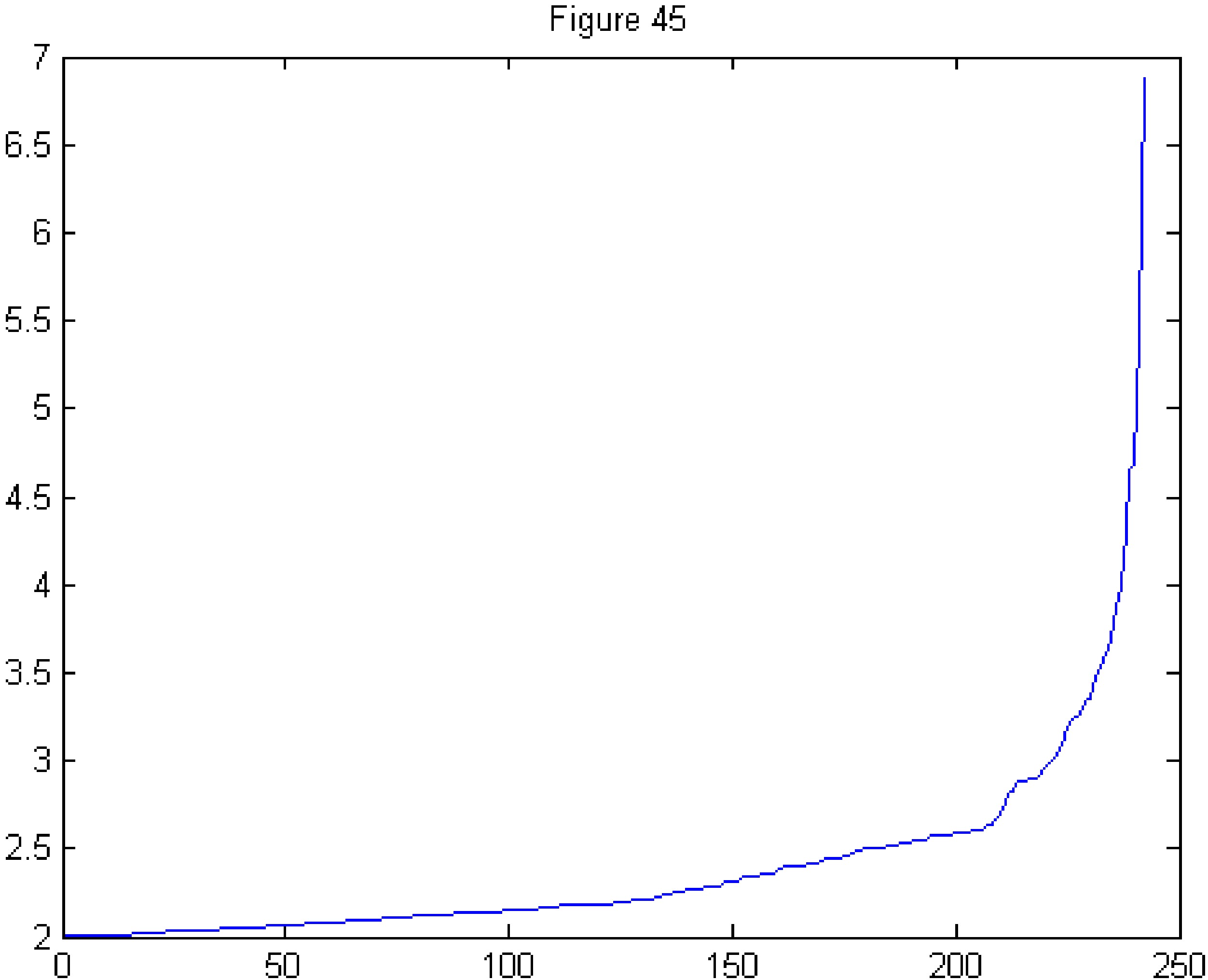
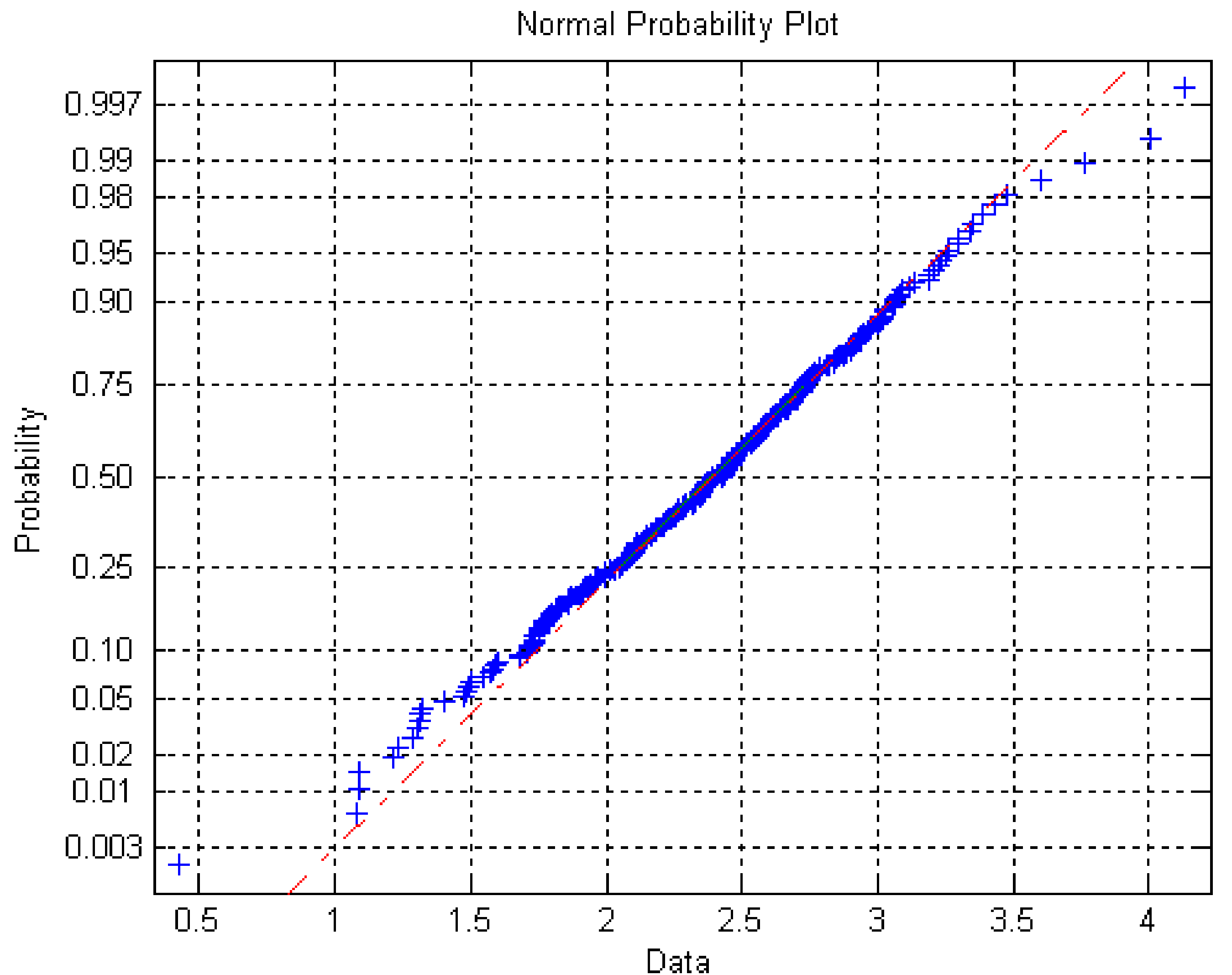
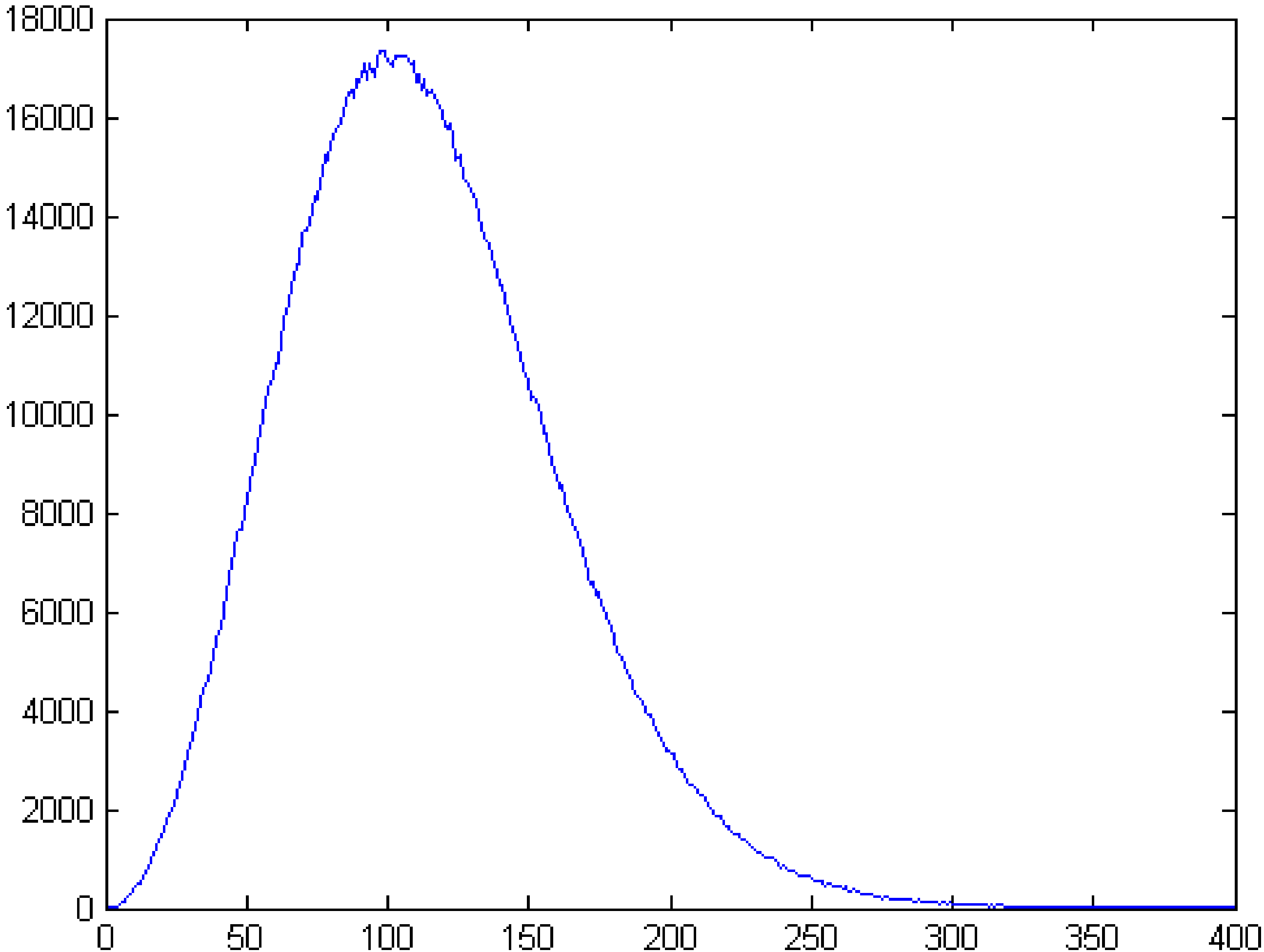


Figure 45
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
0
50
100
150
200
250



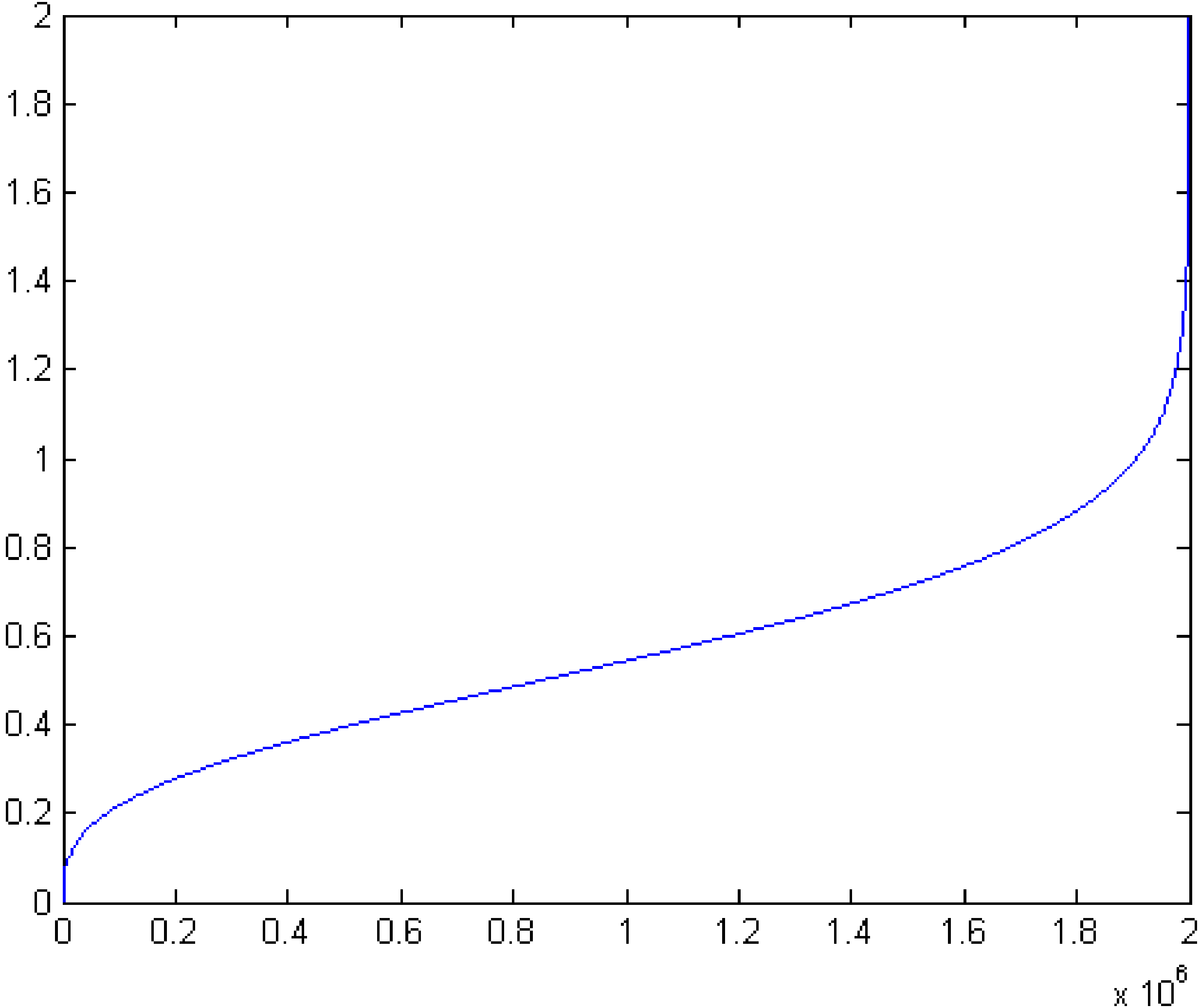
Normal Probability Plot
Probability
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.5
1
1.5
2
2.5
3
3.5
4
Data

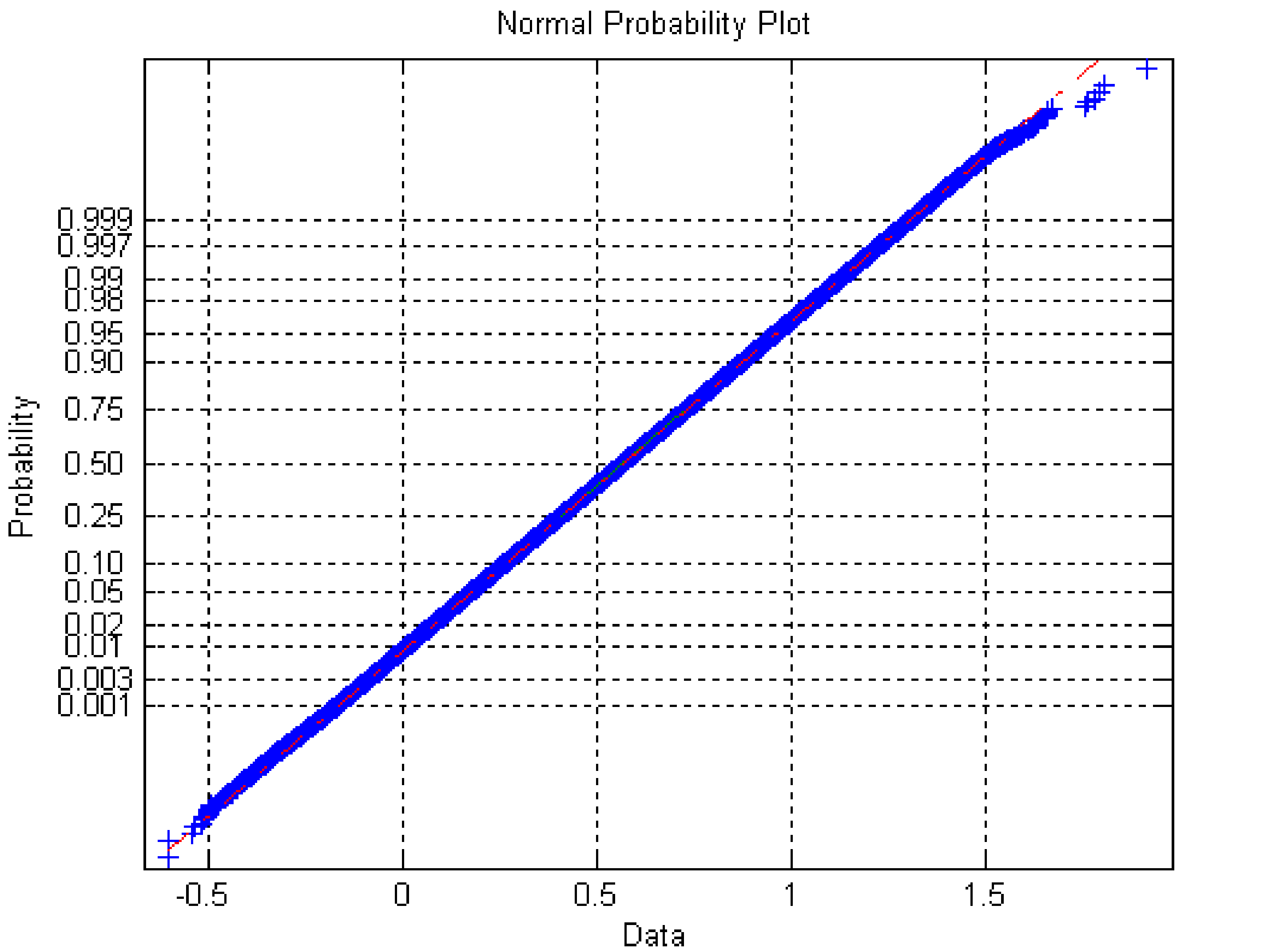
Figure 47



18000
16000
14000
12000
10000
8000
6000
4000
2000
0
0
50
100
150
200
250
300
350
400

Figure 48




Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-0.5
0
0.5
1
1.5
Data

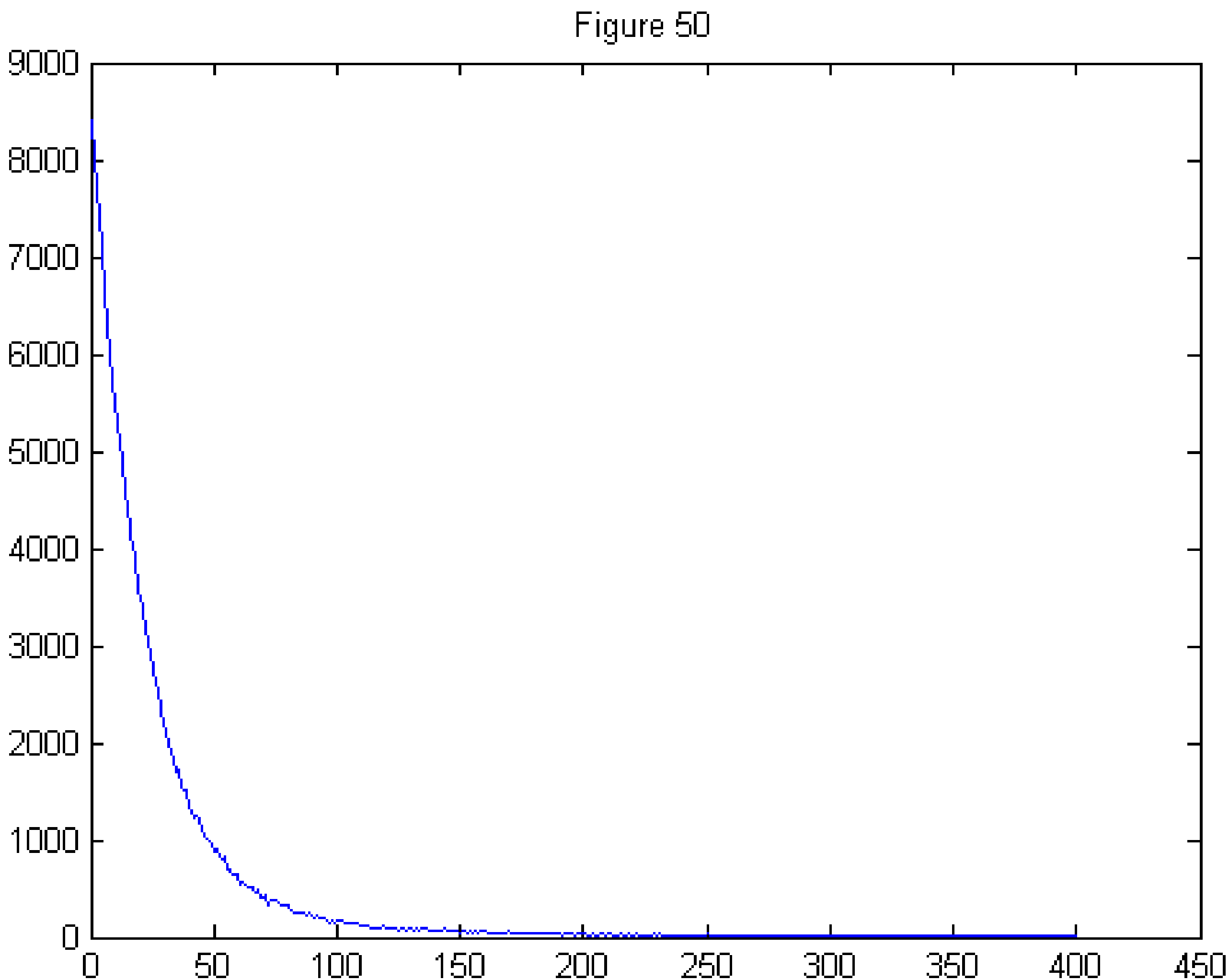


Figure 50
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
50
100
150
200
250
300
350
400
450

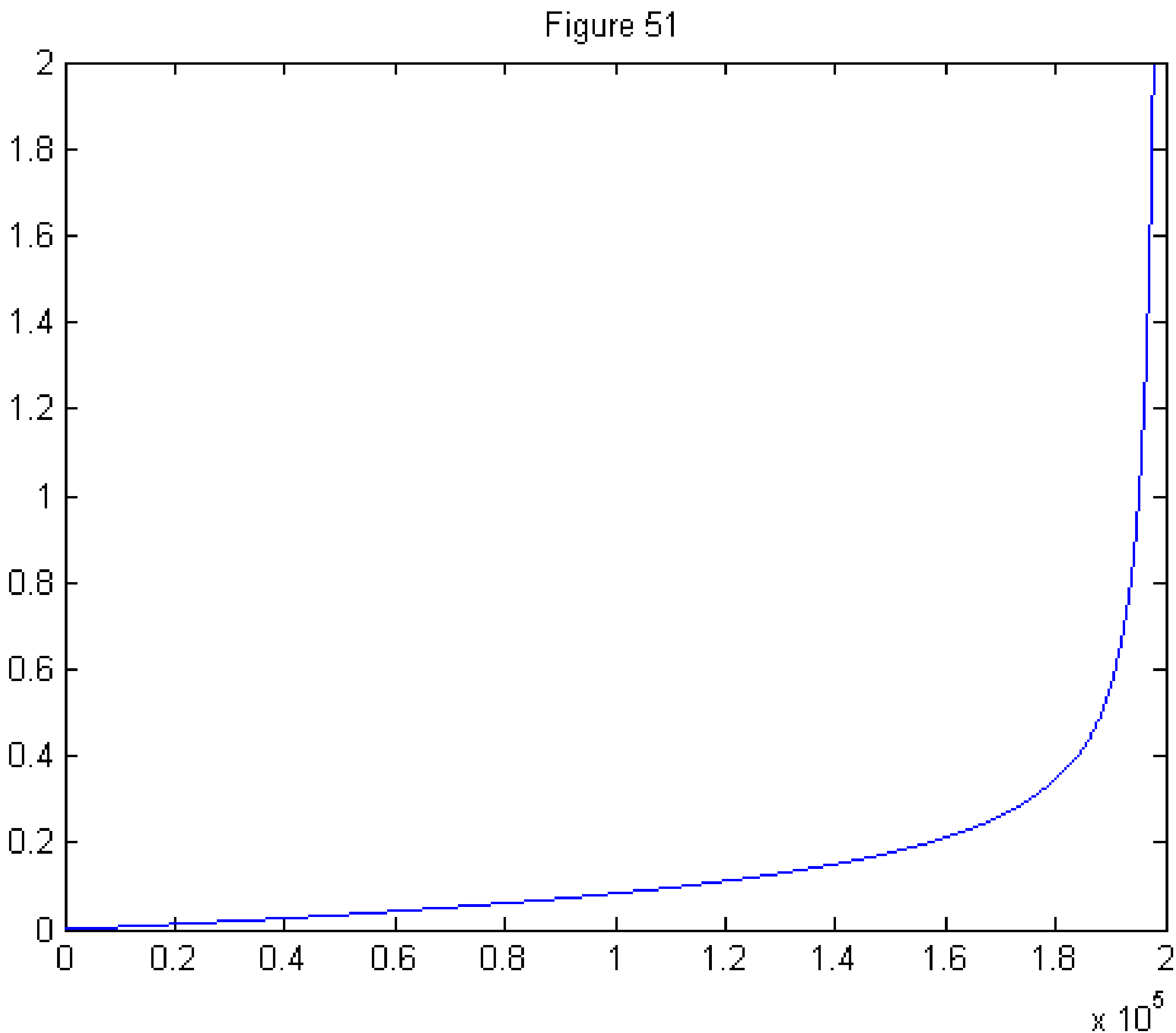
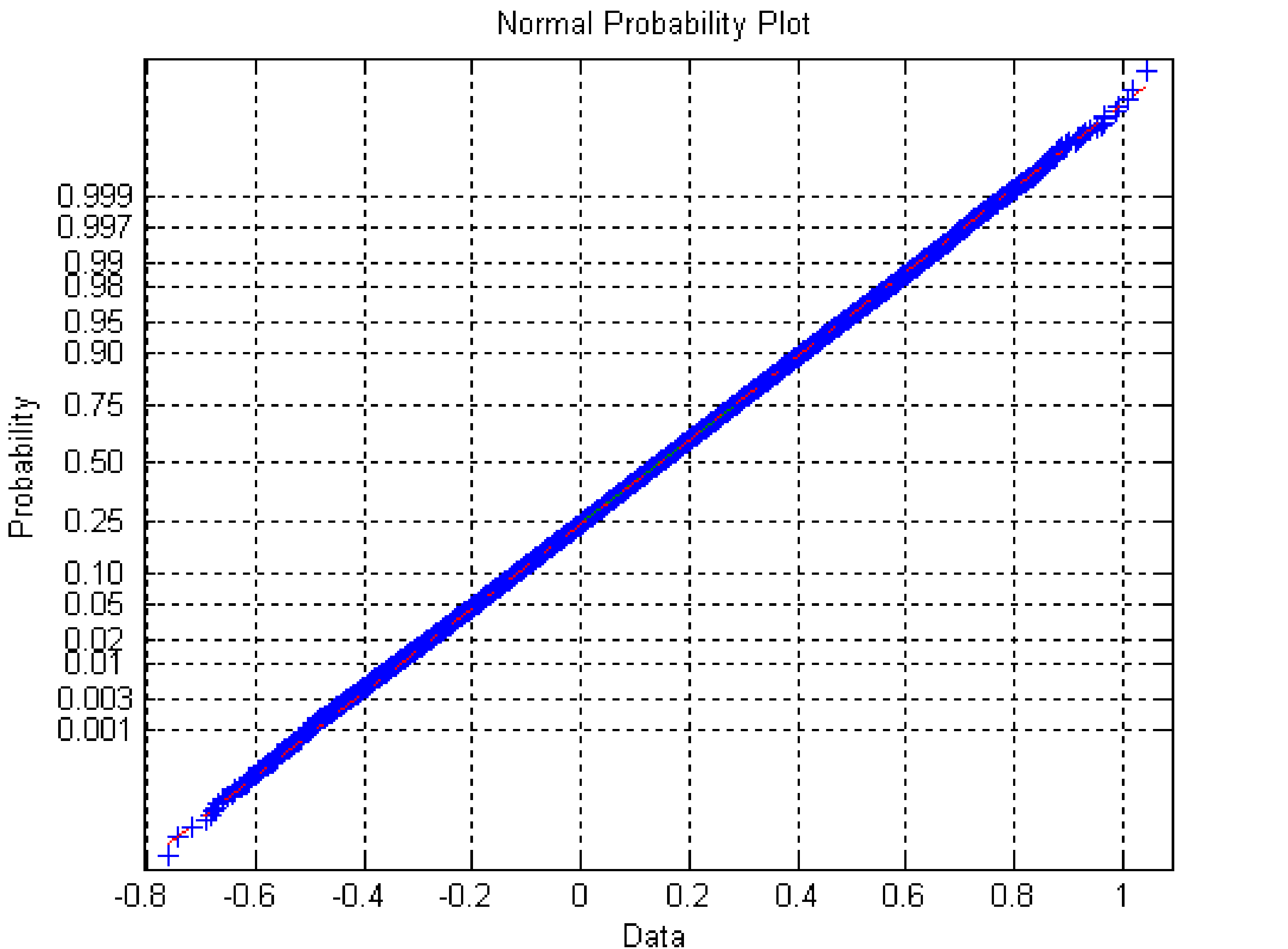


Figure 51
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^5



Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Data

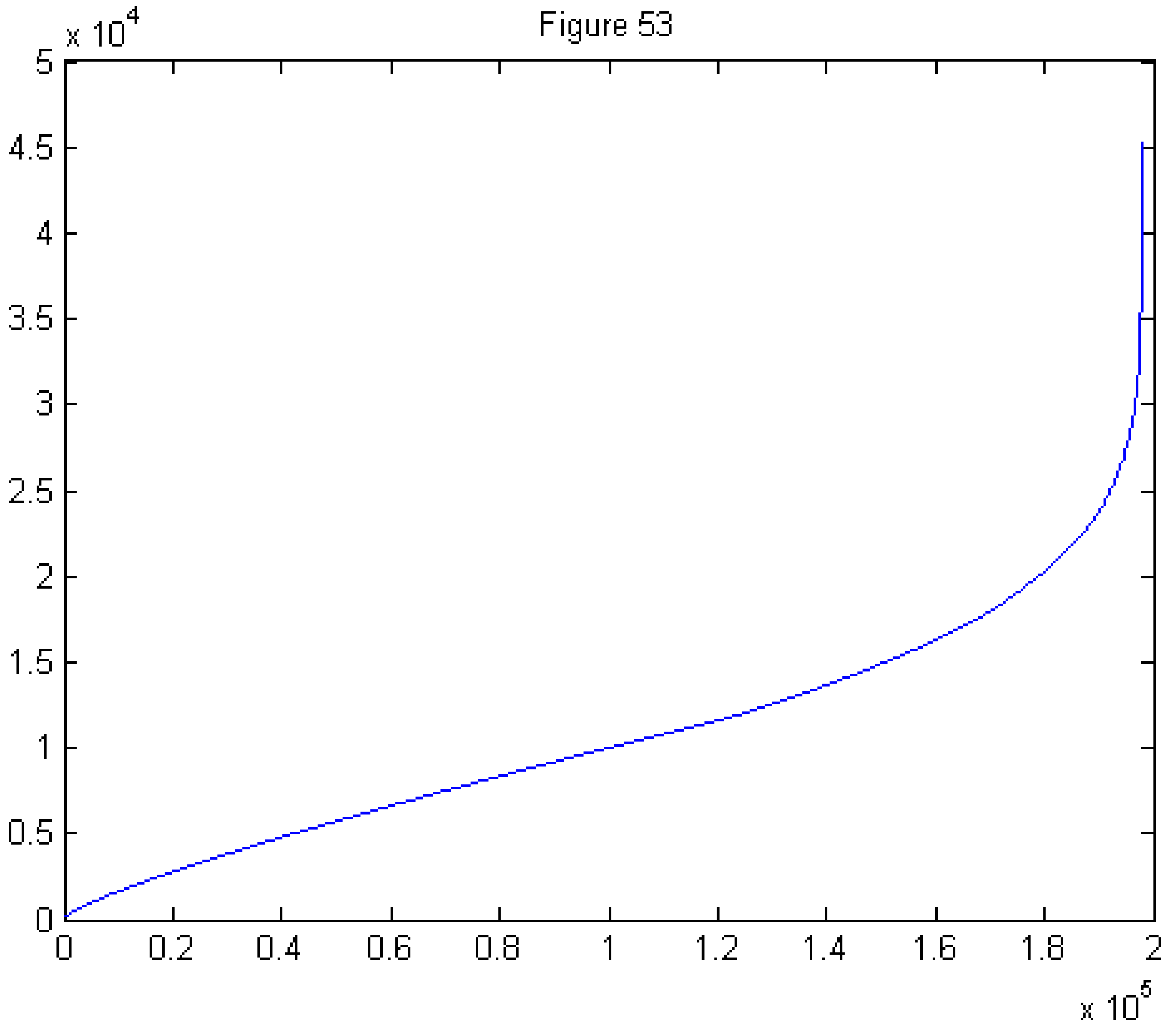
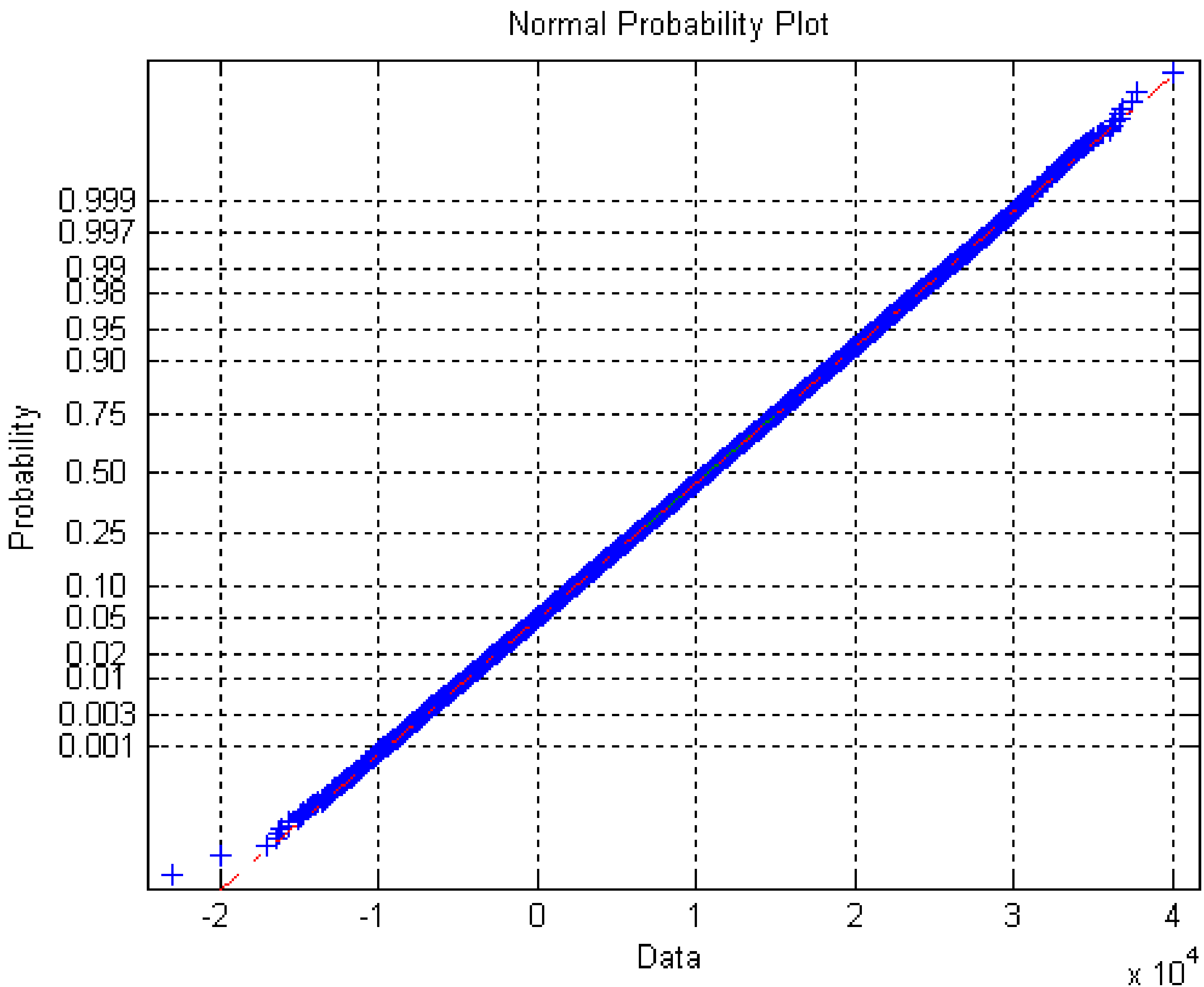


Figure 53
x 10^4
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^5



Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-2
-1
0
1
2
3
4
Data
x 10^4

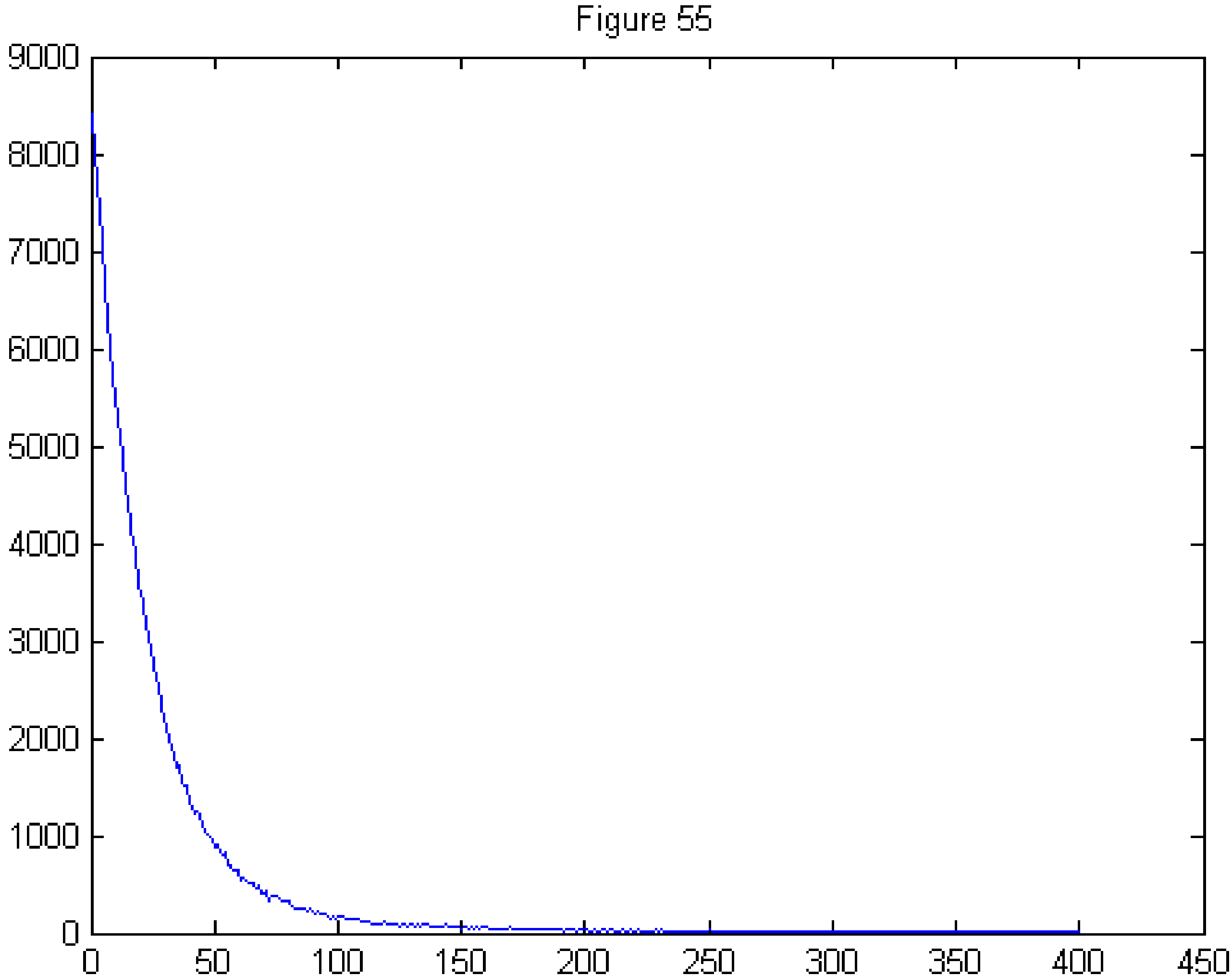
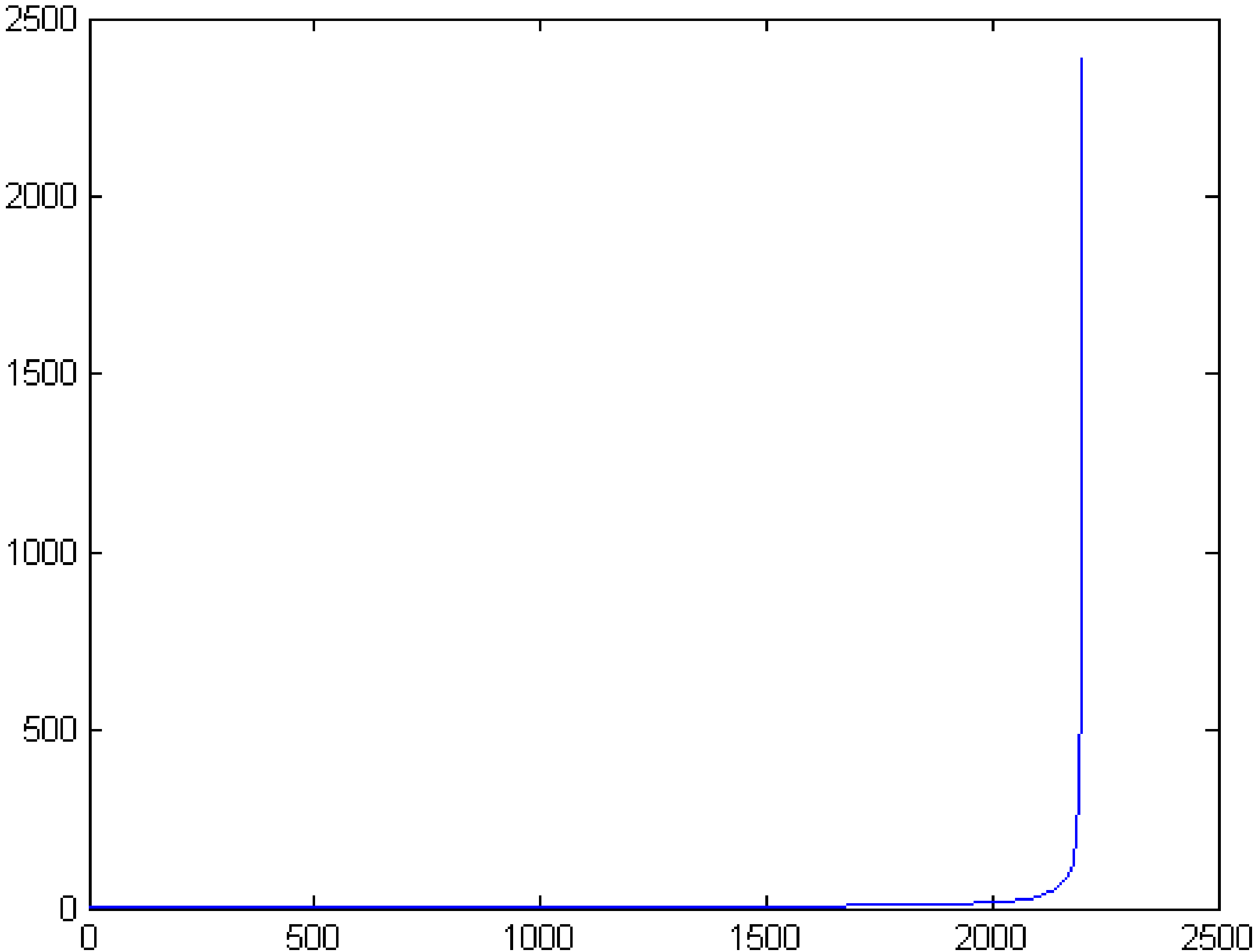
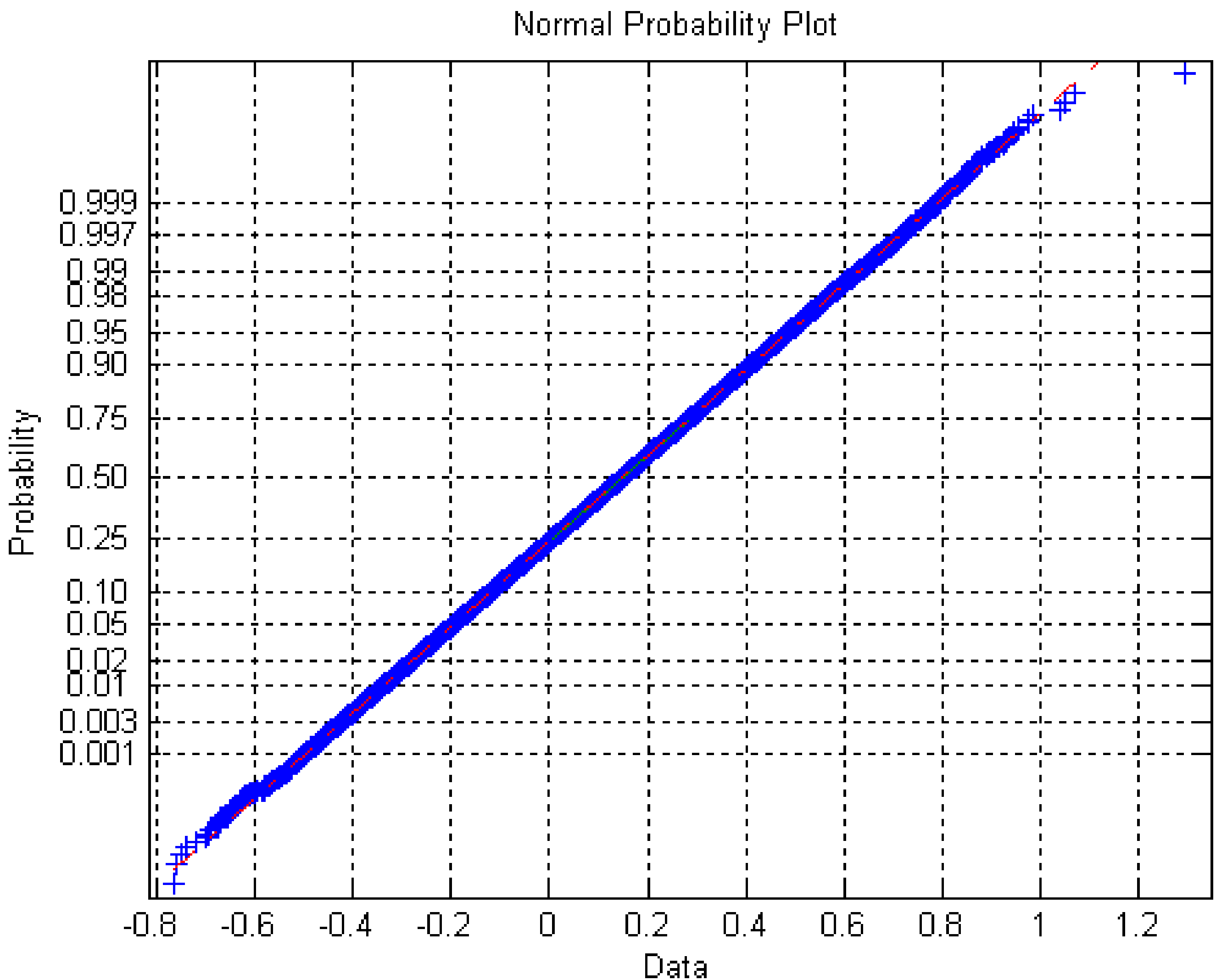


Figure 55
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
50
100
150
200
250
300
350
400
450

Figure 56





Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
Data

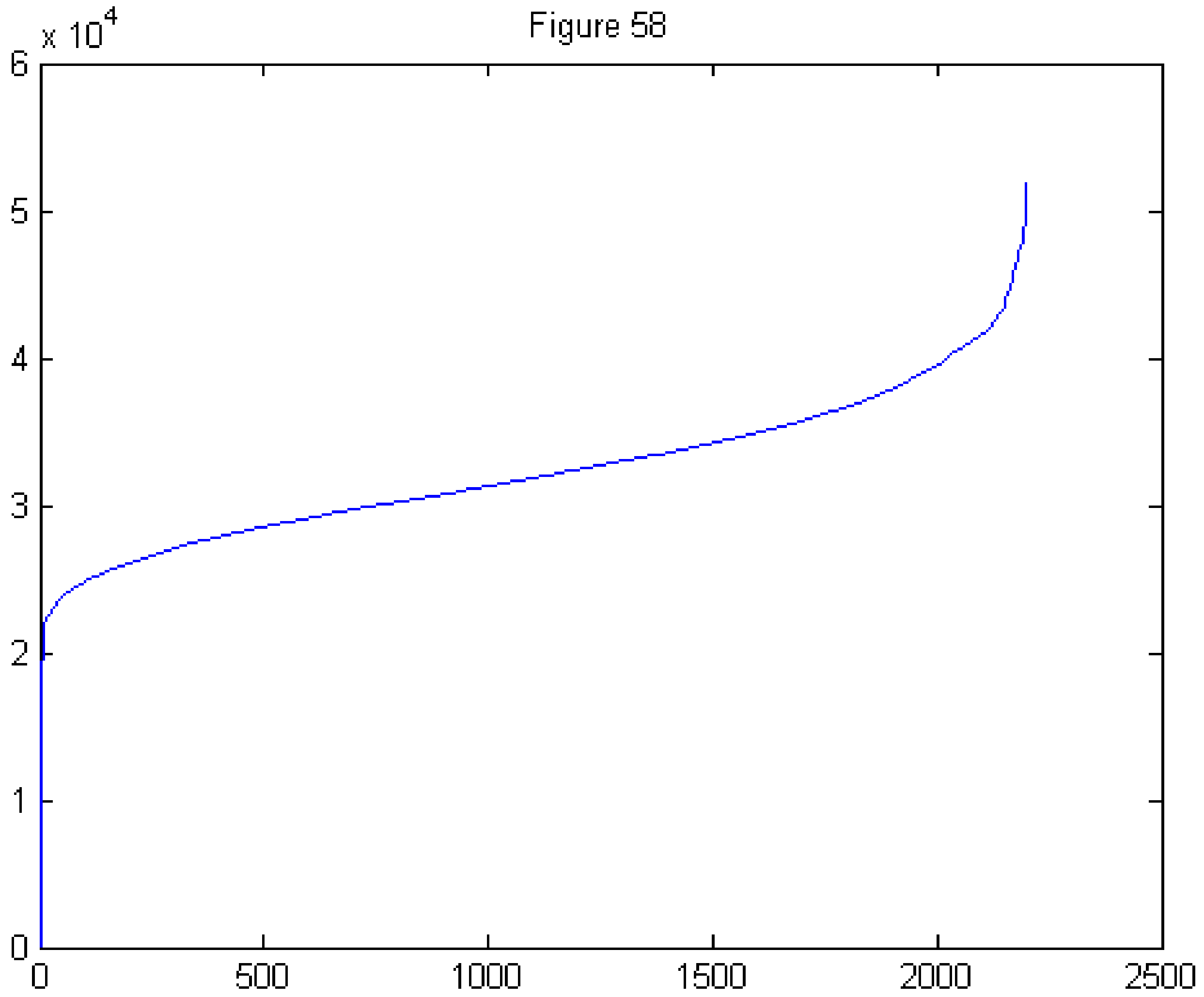


Figure 58
x 10^4
6
5
4
3
2
1
0
0
500
1000
1500
2000
2500

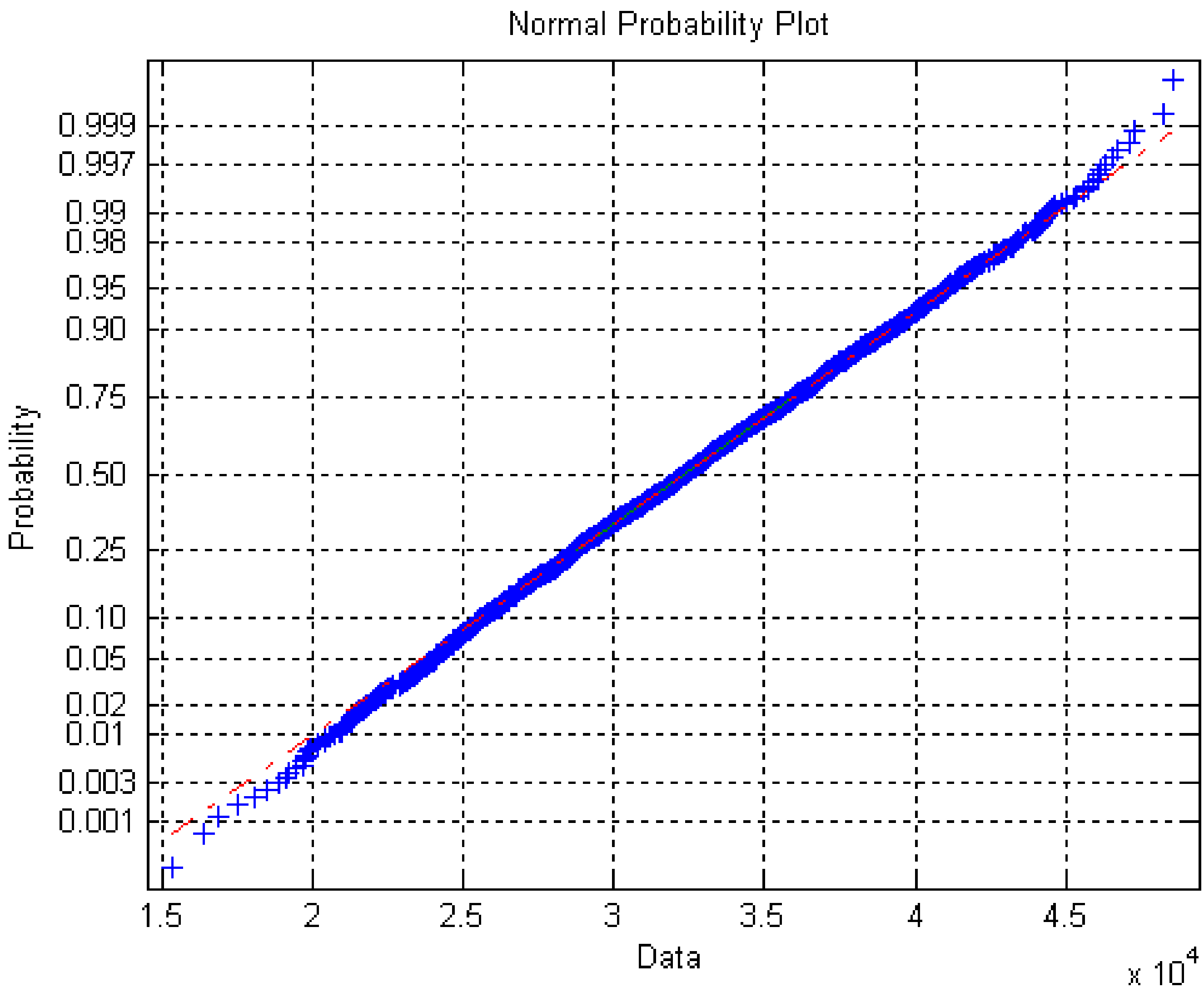
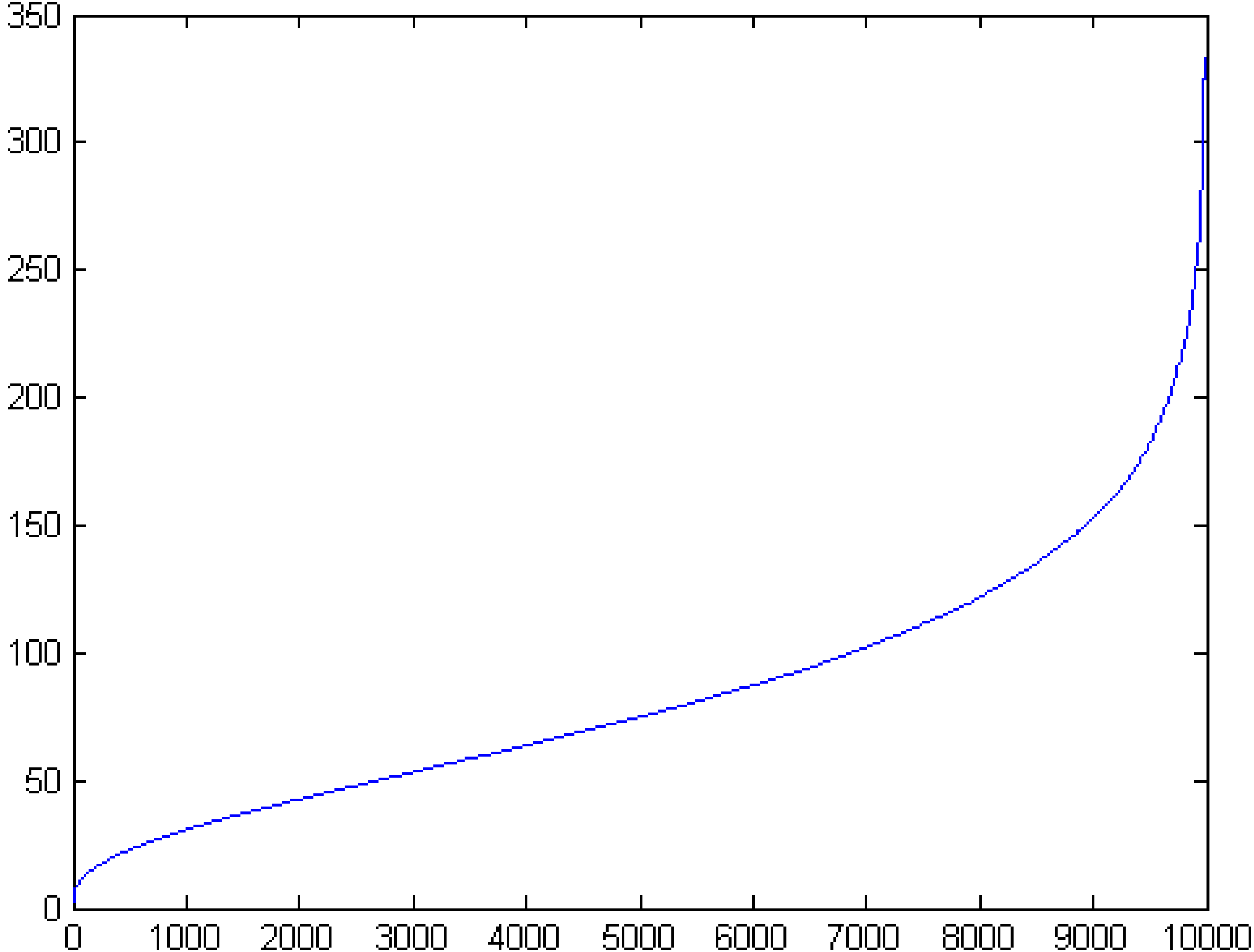
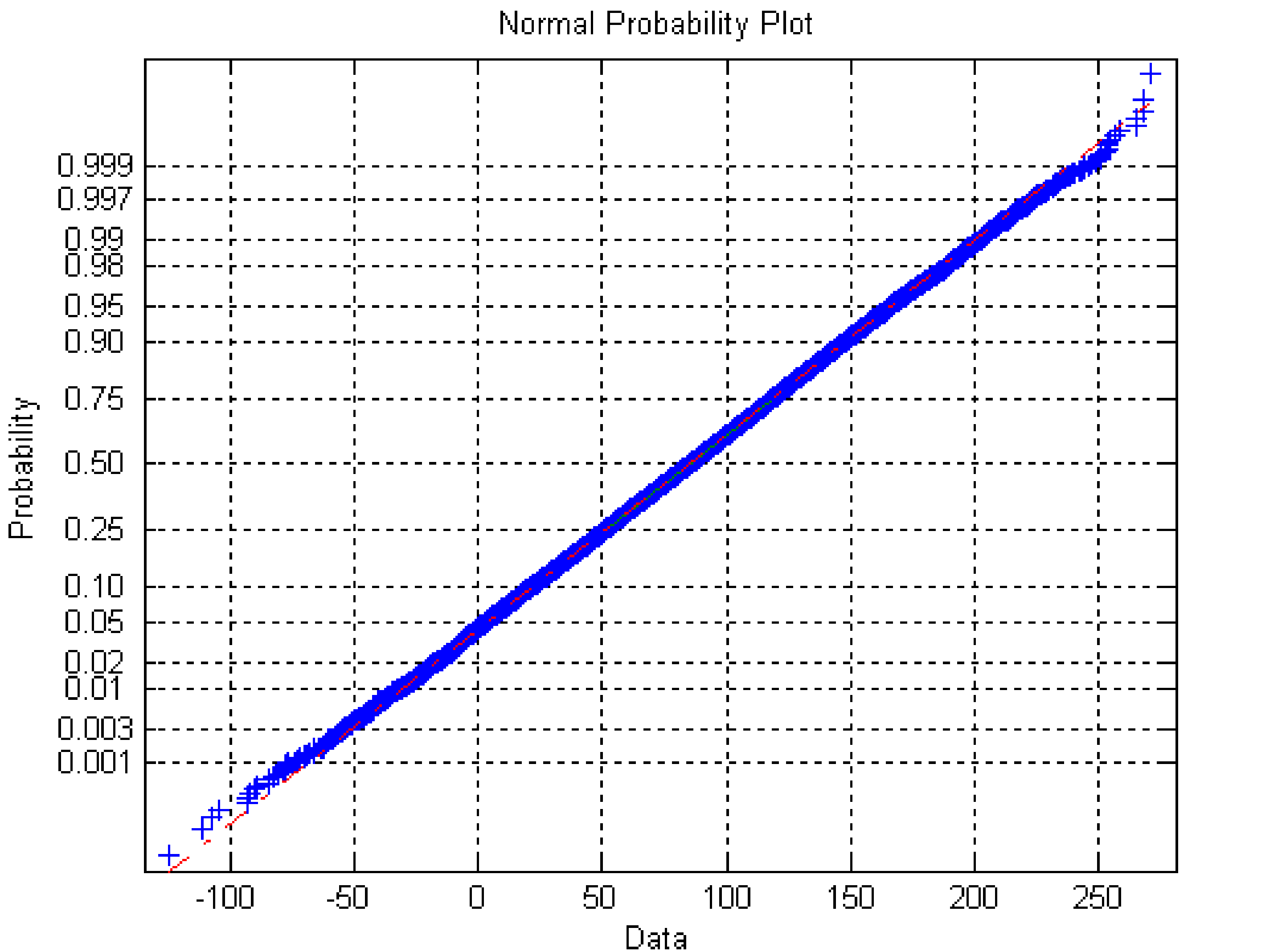

Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
1.5
2
2.5
3
3.5
4
4.5
Data
x 10^4

Figure 60



350
300
250
200
150
100
50
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000



Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-100
-50
0
50
100
150
200
250
Data

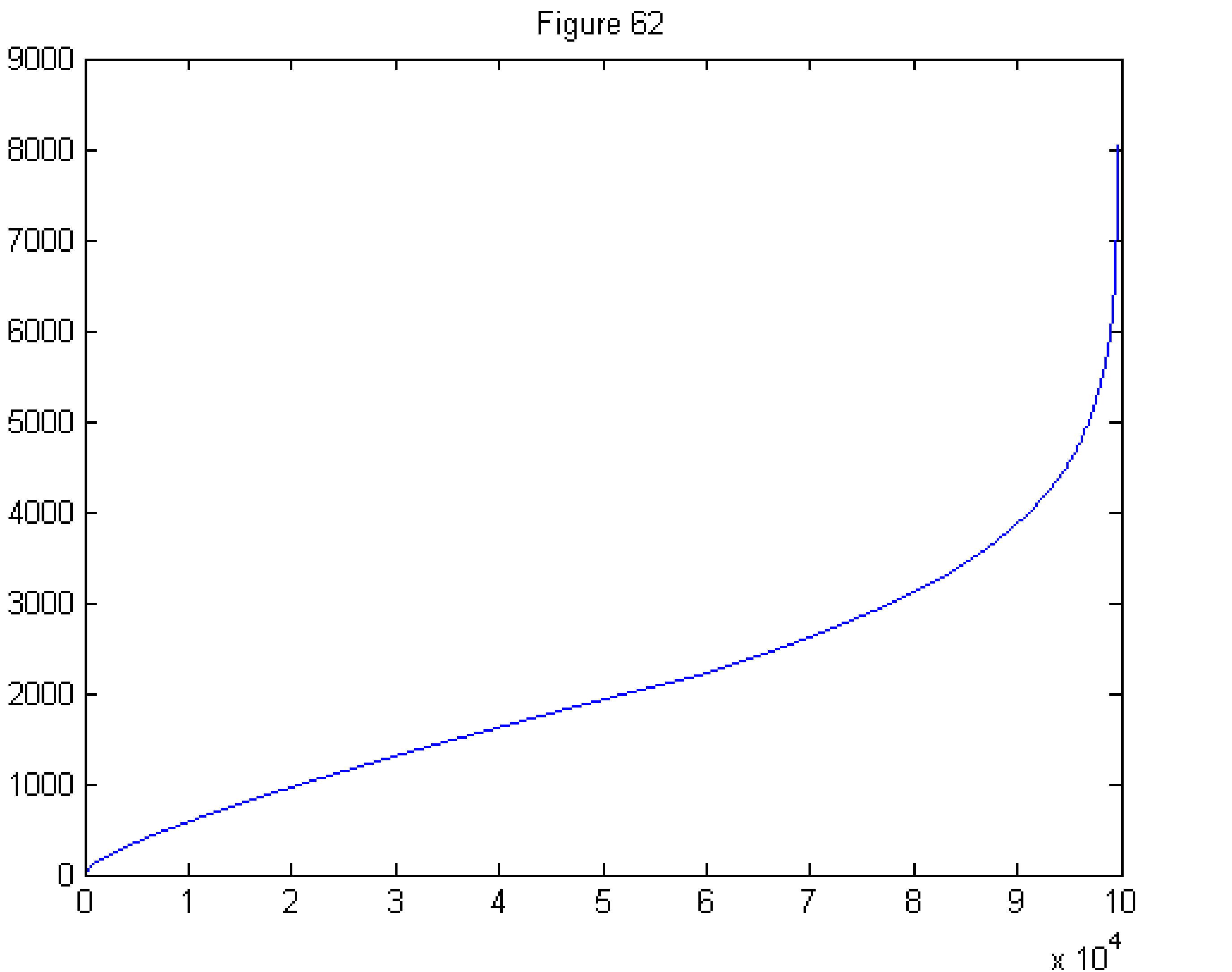


Figure 62
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
1
2
3
4
5
6
7
8
9
10
x 10^4

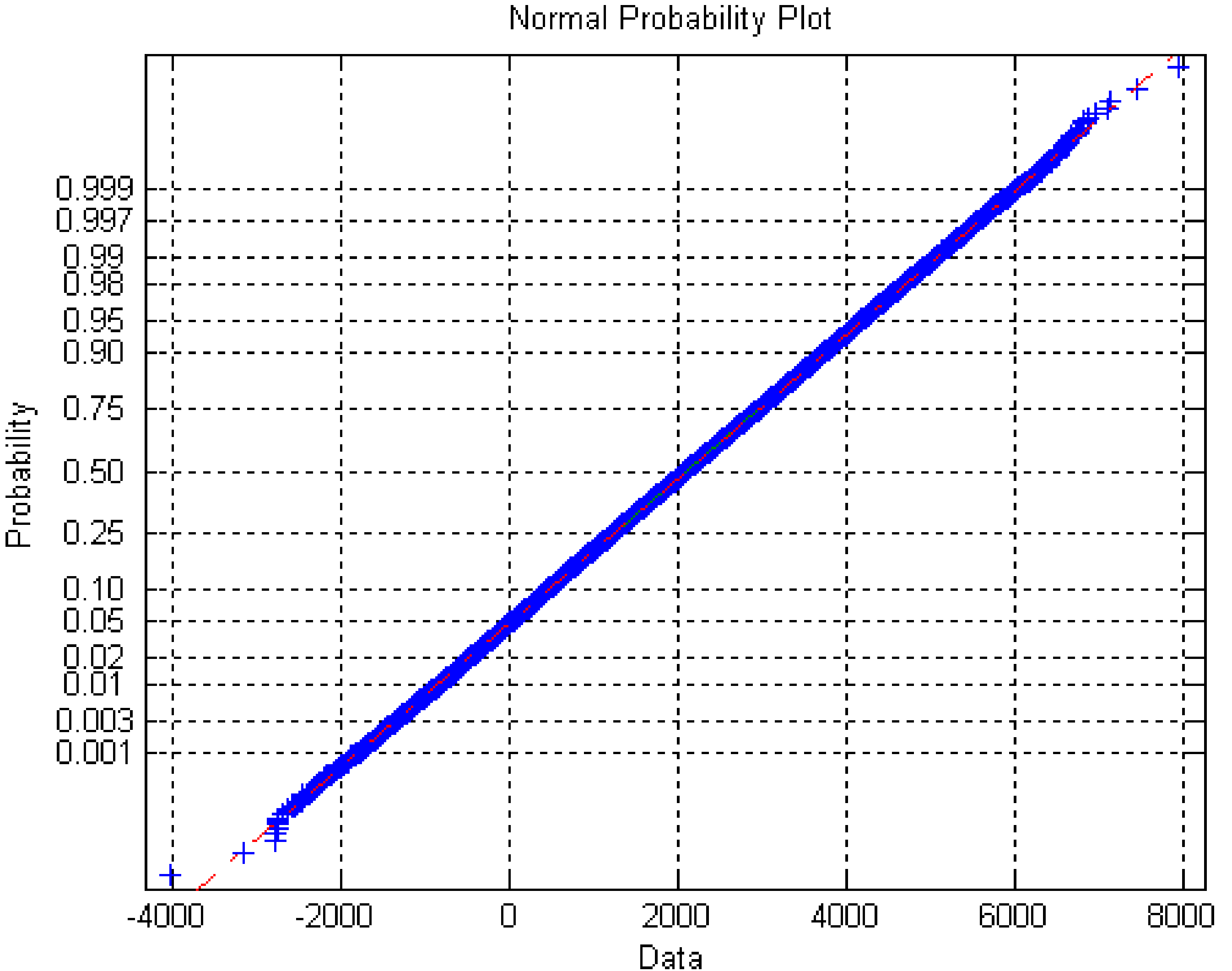
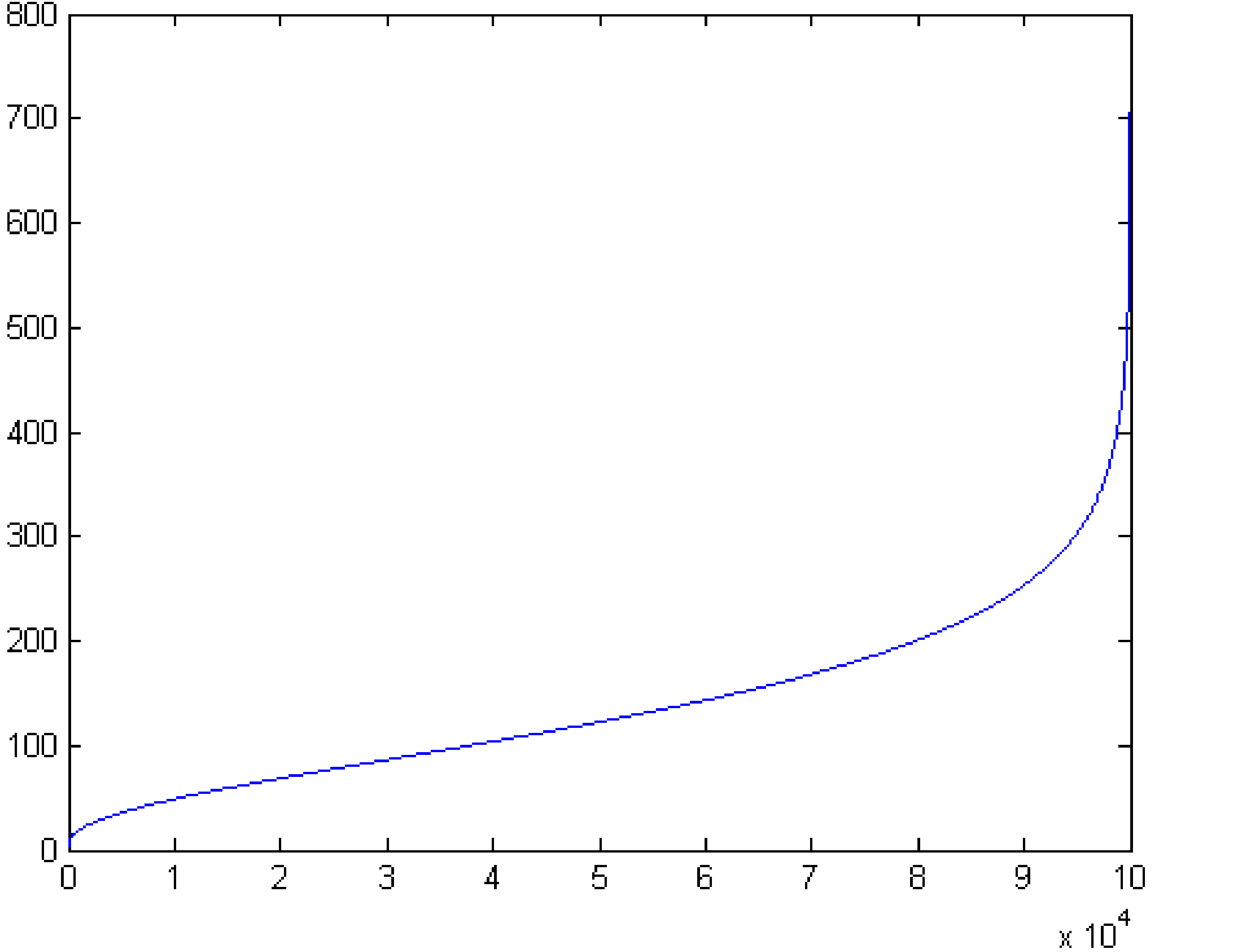
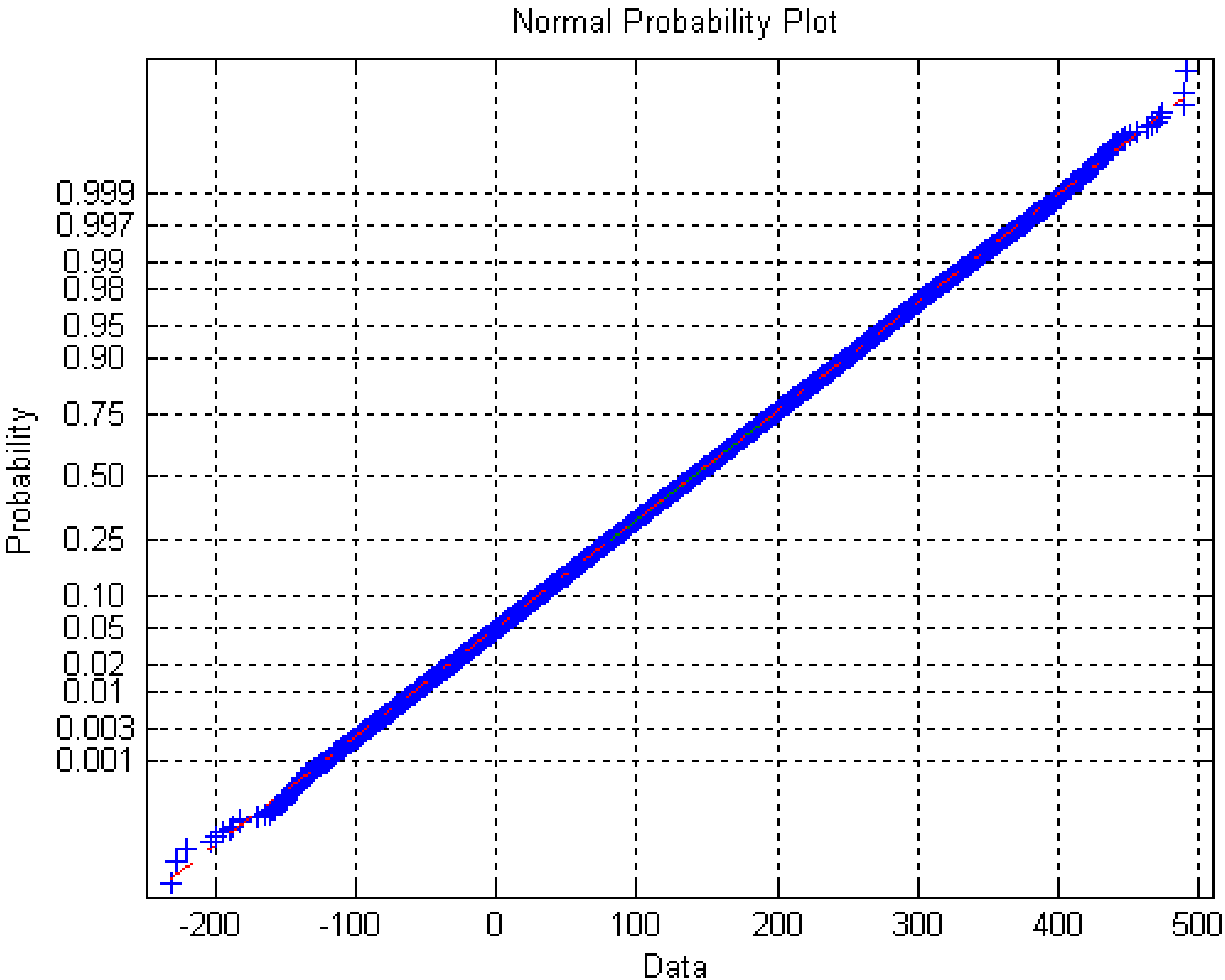

Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-4000
-2000
0
2000
4000
6000
8000
Data

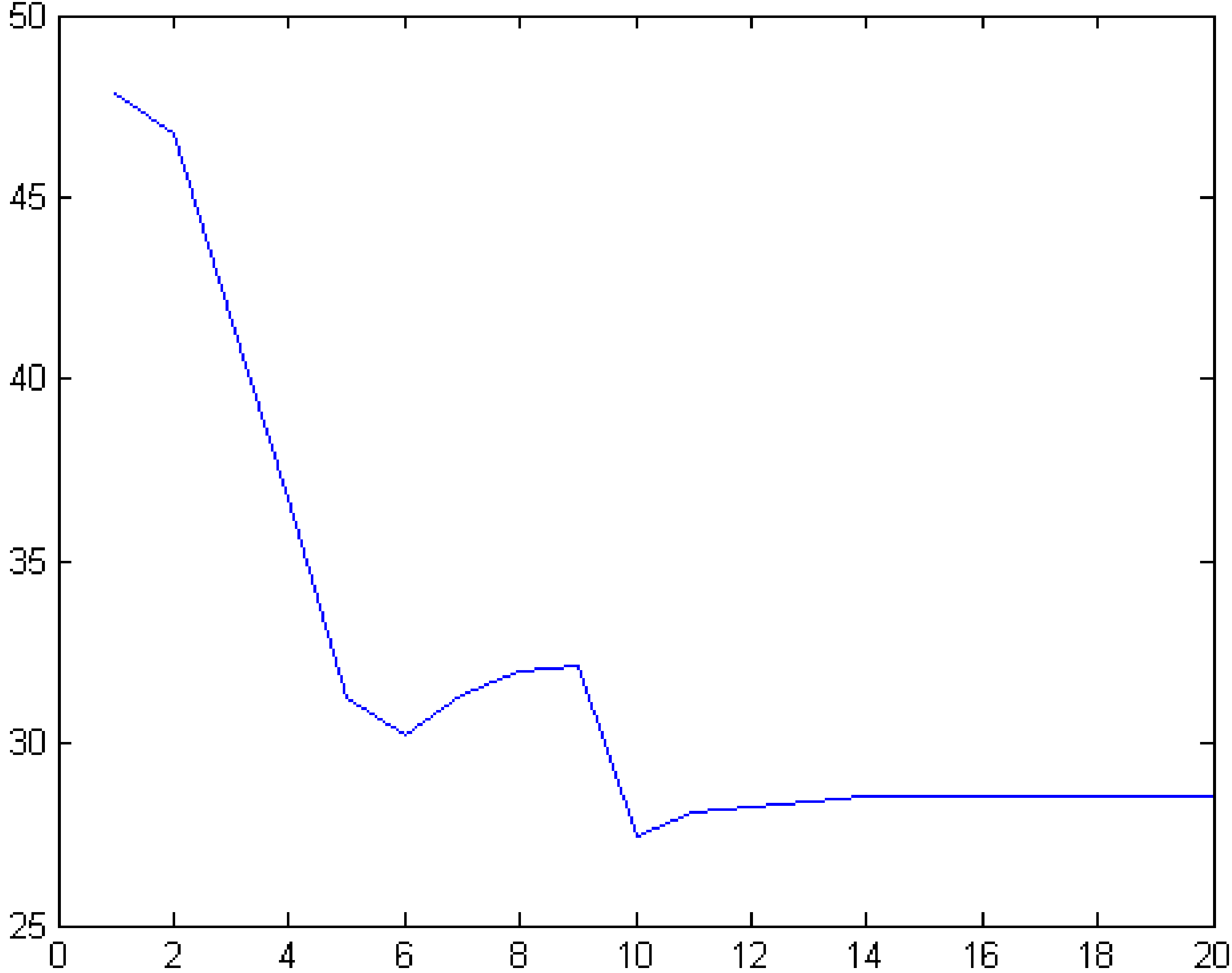
Figure 64





Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-200
-100
0
100
200
300
400
500
Data

Figure 66



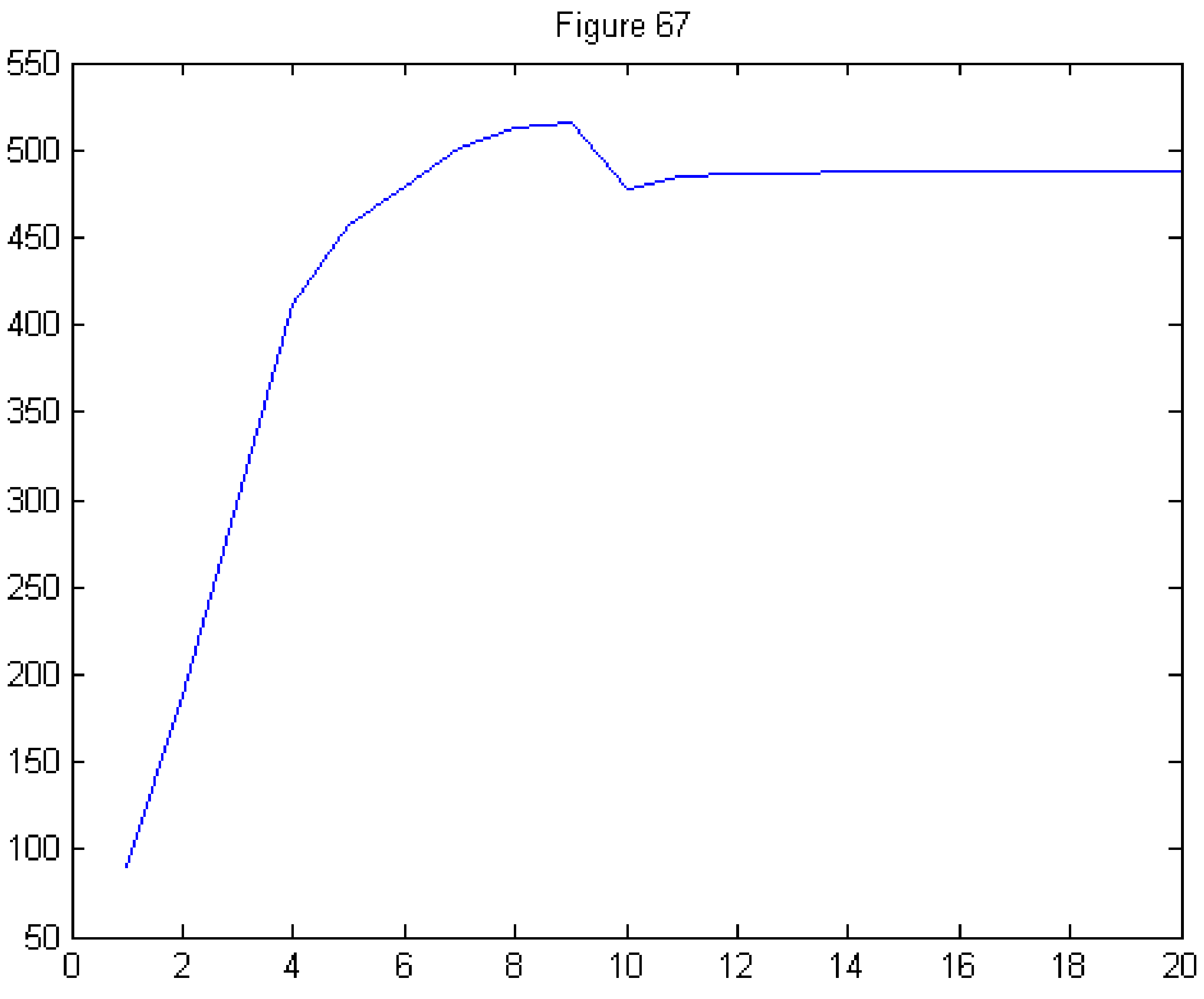


Figure 67
550
500
450
400
350
300
250
200
150
100
50
0
2
4
6
8
10
12
14
16
18
20

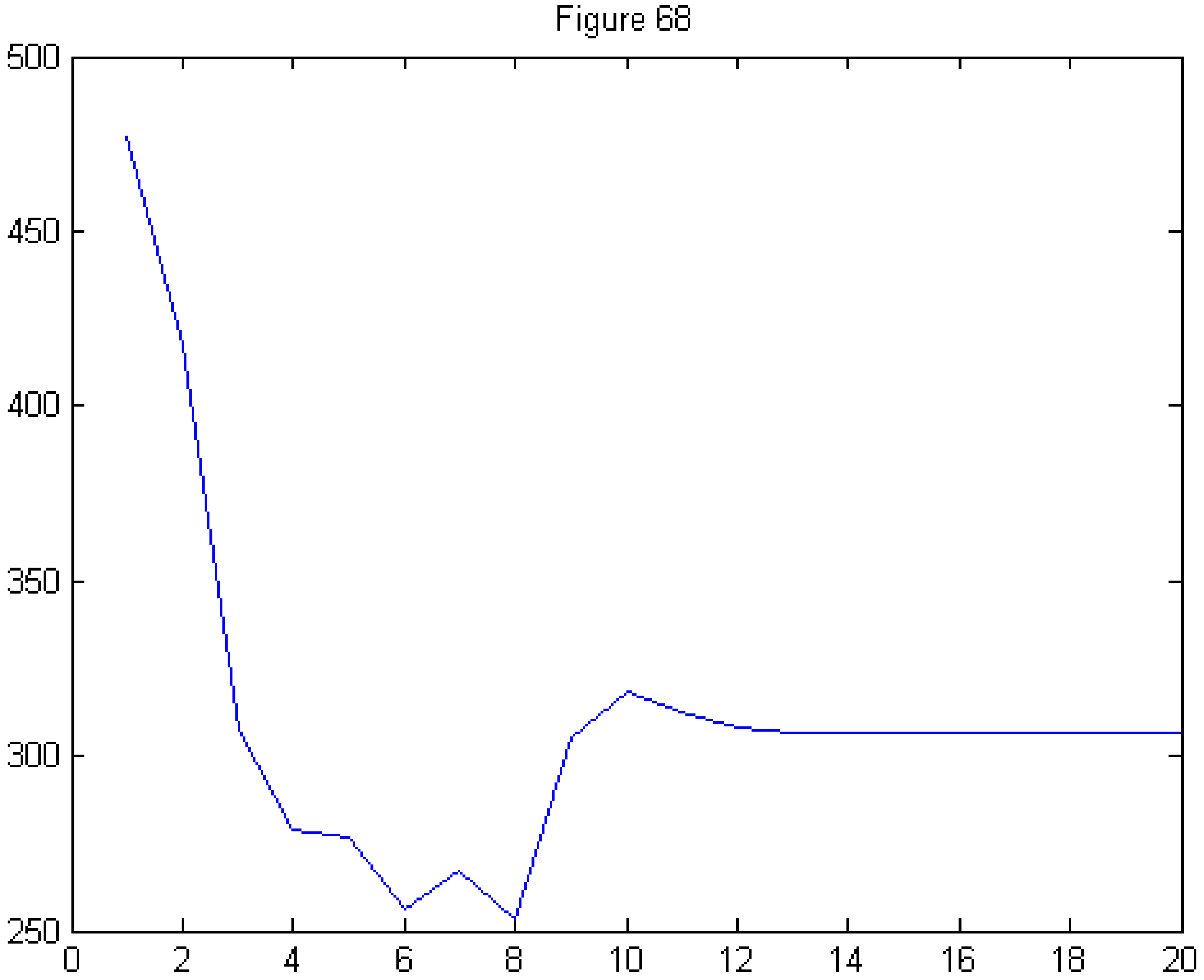
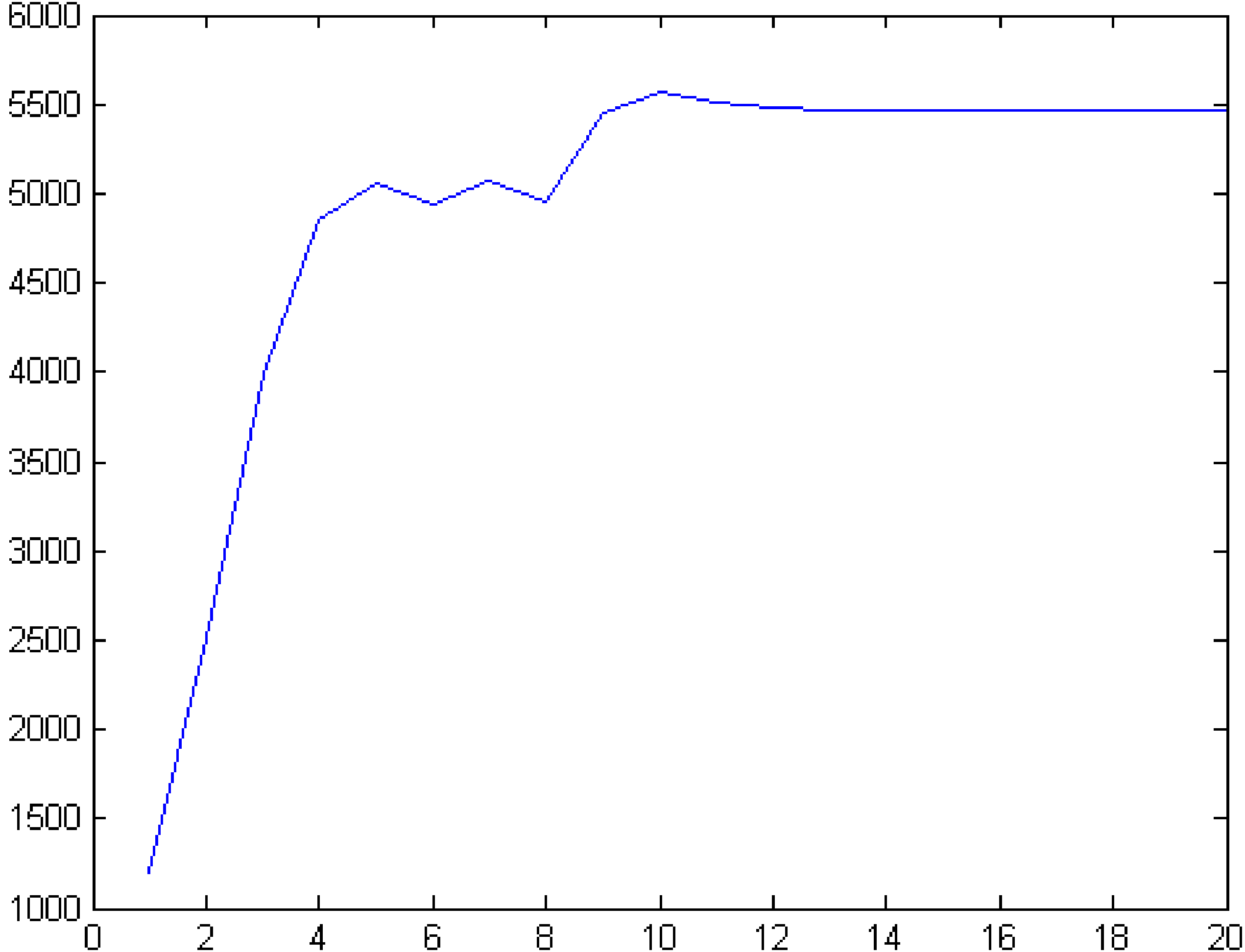


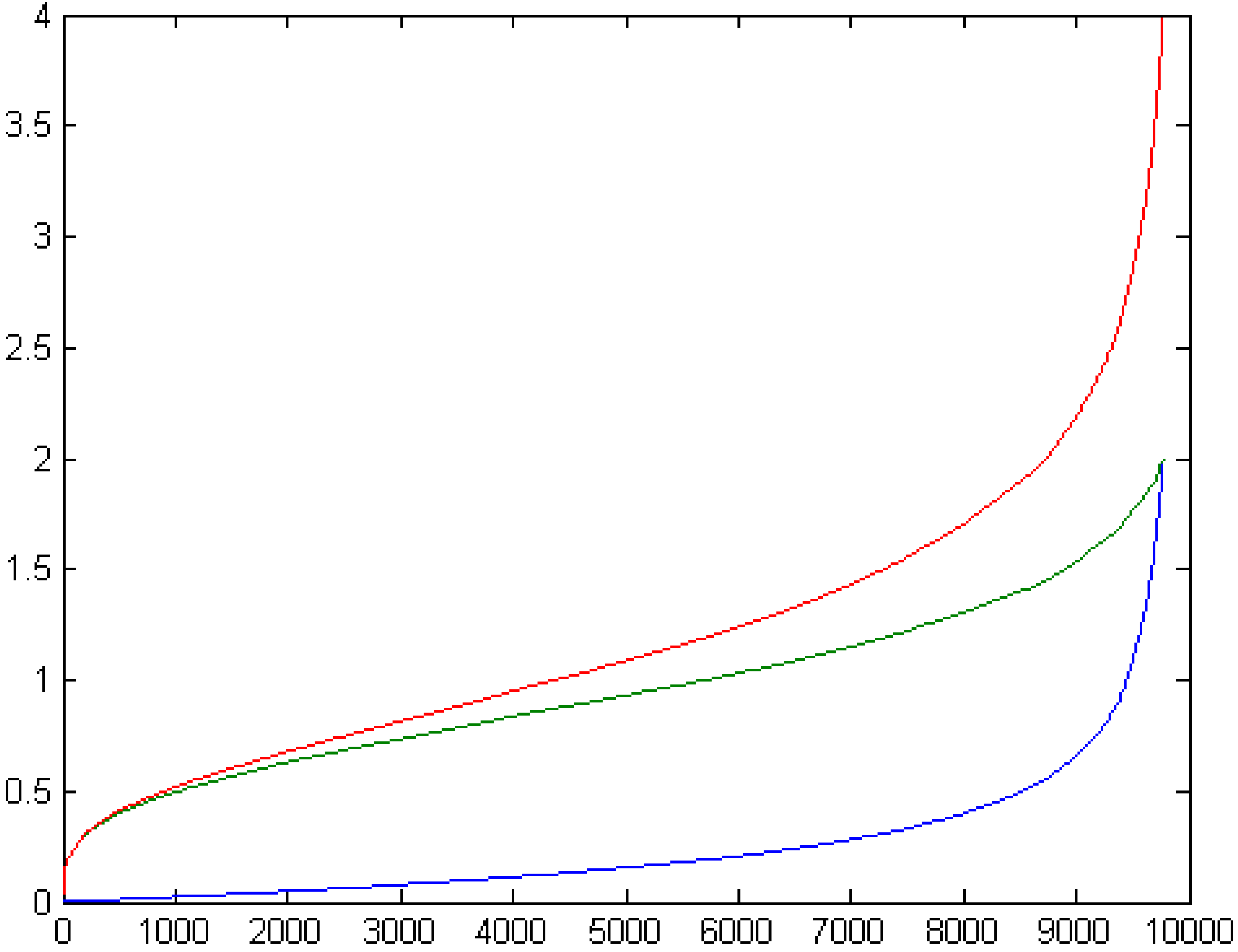
Figure 68
500
450
400
350
300
250
0
2
4
6
8
10
12
14
16
18
20

Figure 69

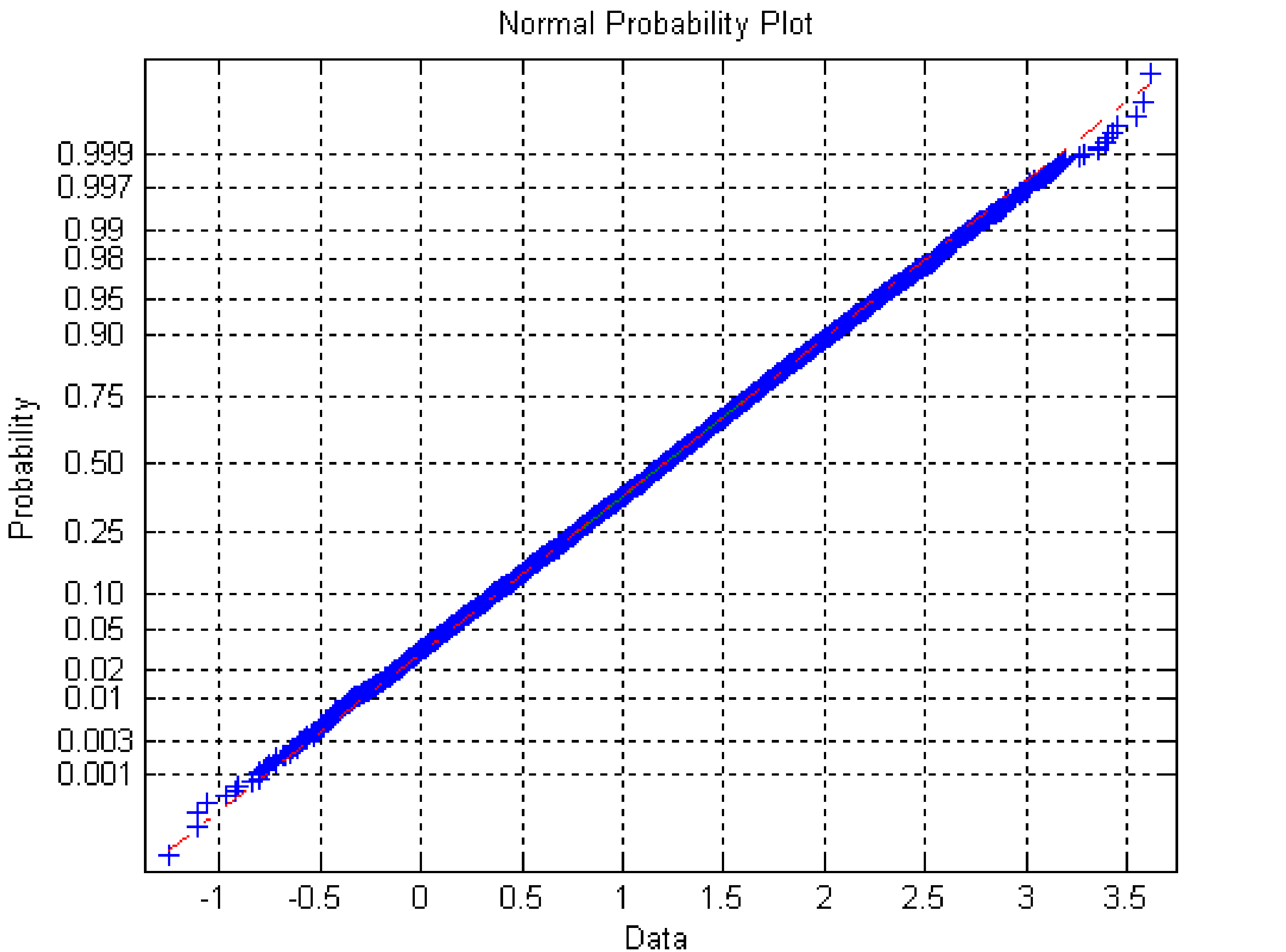


6000
5500
5000
4500
4000
3500
3000
2500
2000
1500
1000
0
2
4
6
8
10
12
14
16
18
20

Figure 70



4
3.5
3
2.5
2
1.5
1
0.5
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000



Normal Probability Plot
Probability
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
Data

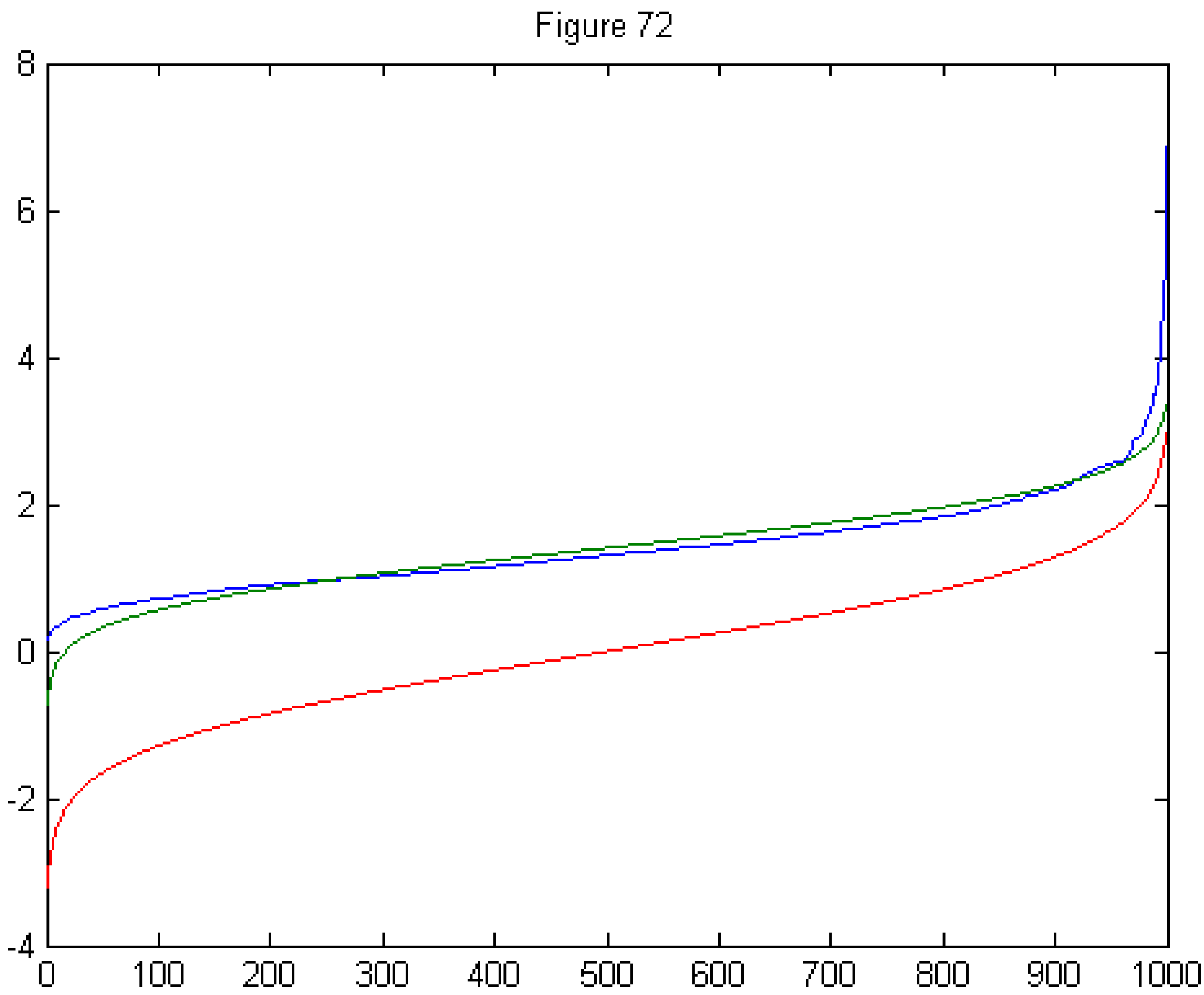


Figure 72
8
6
4
2
0
-2
-4
0
100
200
300
400
500
600
700
800
900
1000

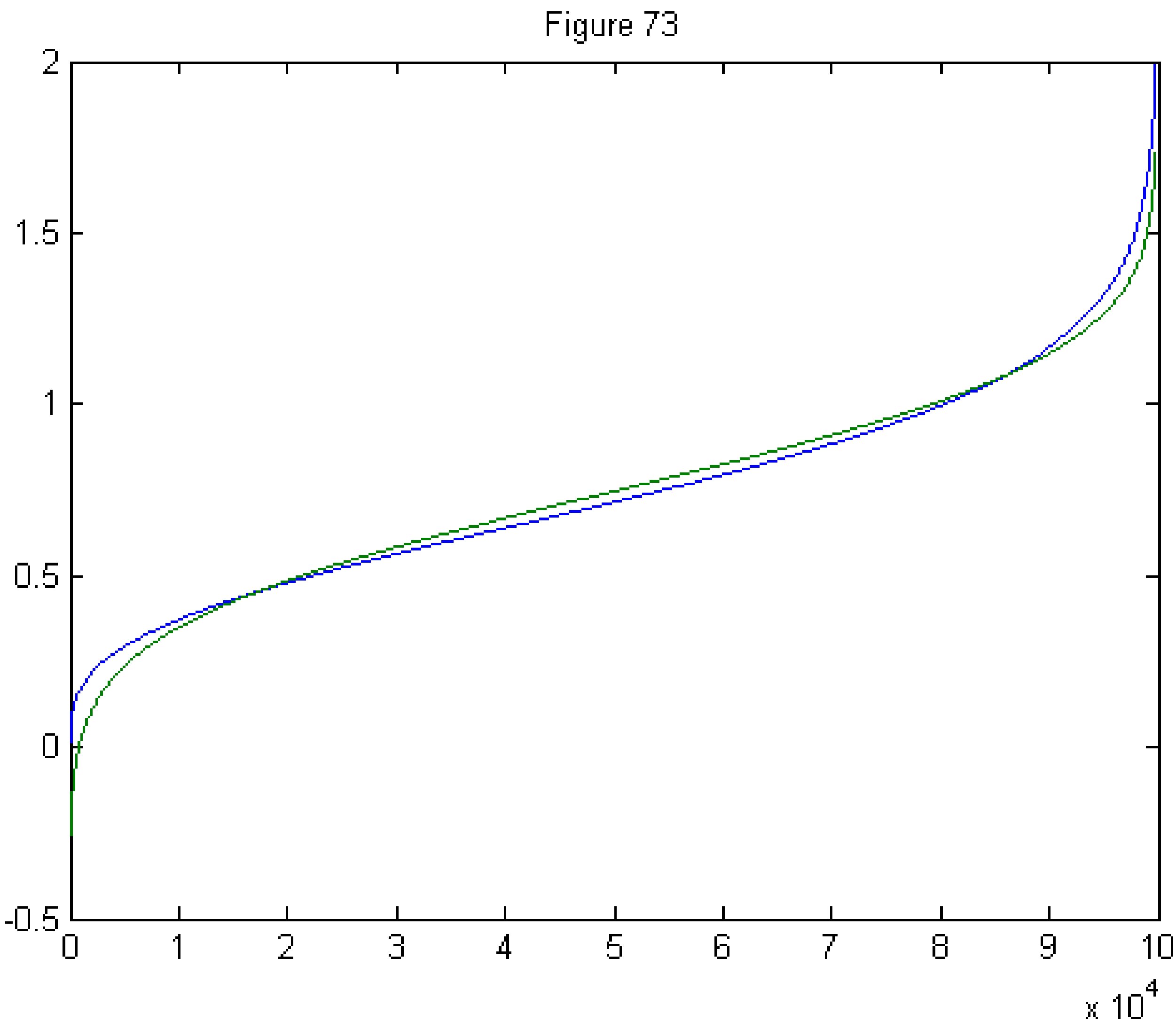


Figure 73
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
5
6
7
8
9
10
x 10^4

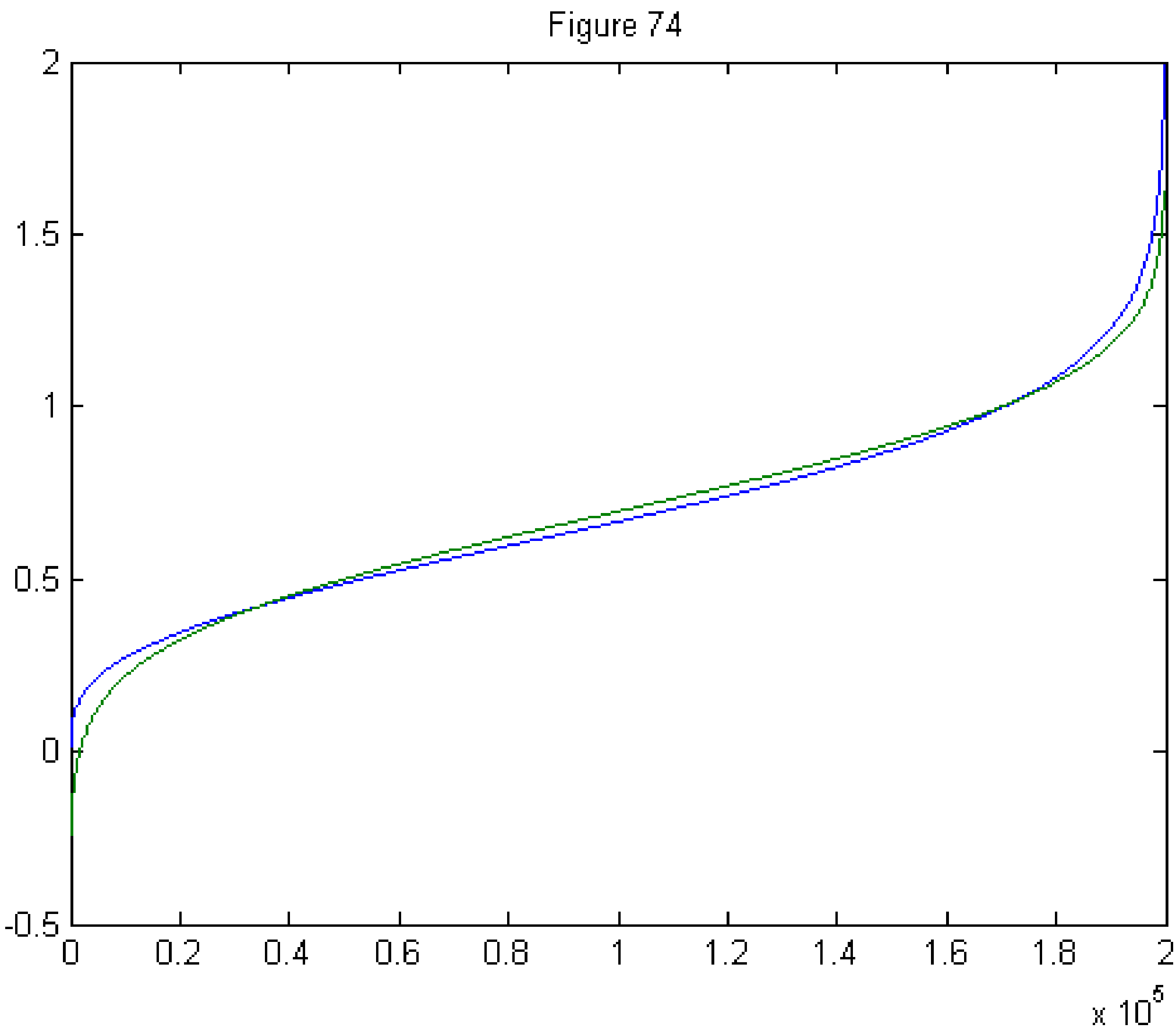


Figure 74
2
1.5
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^5

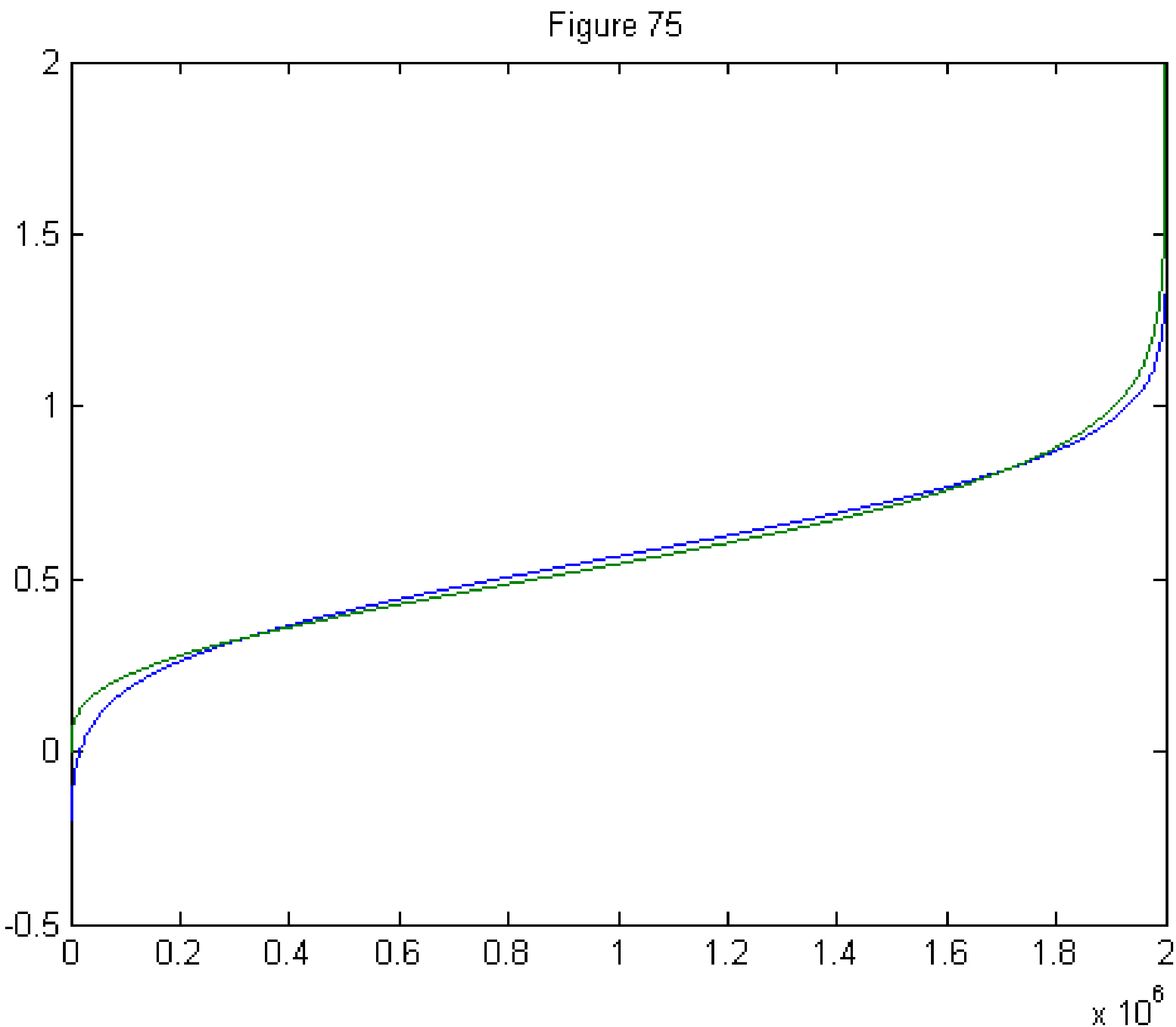


Figure 75
2
1.5
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^6

Figure 76

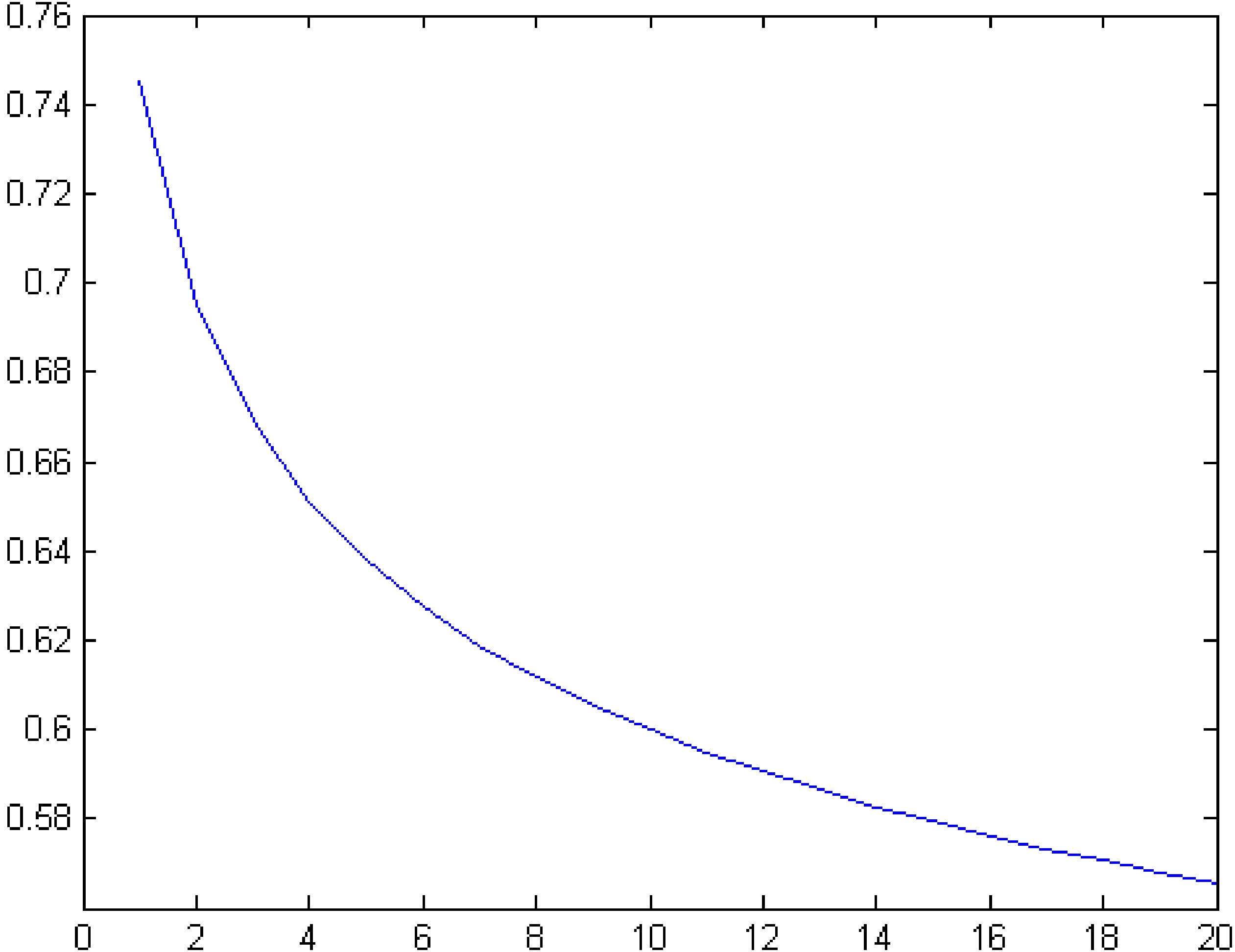


Figure 77

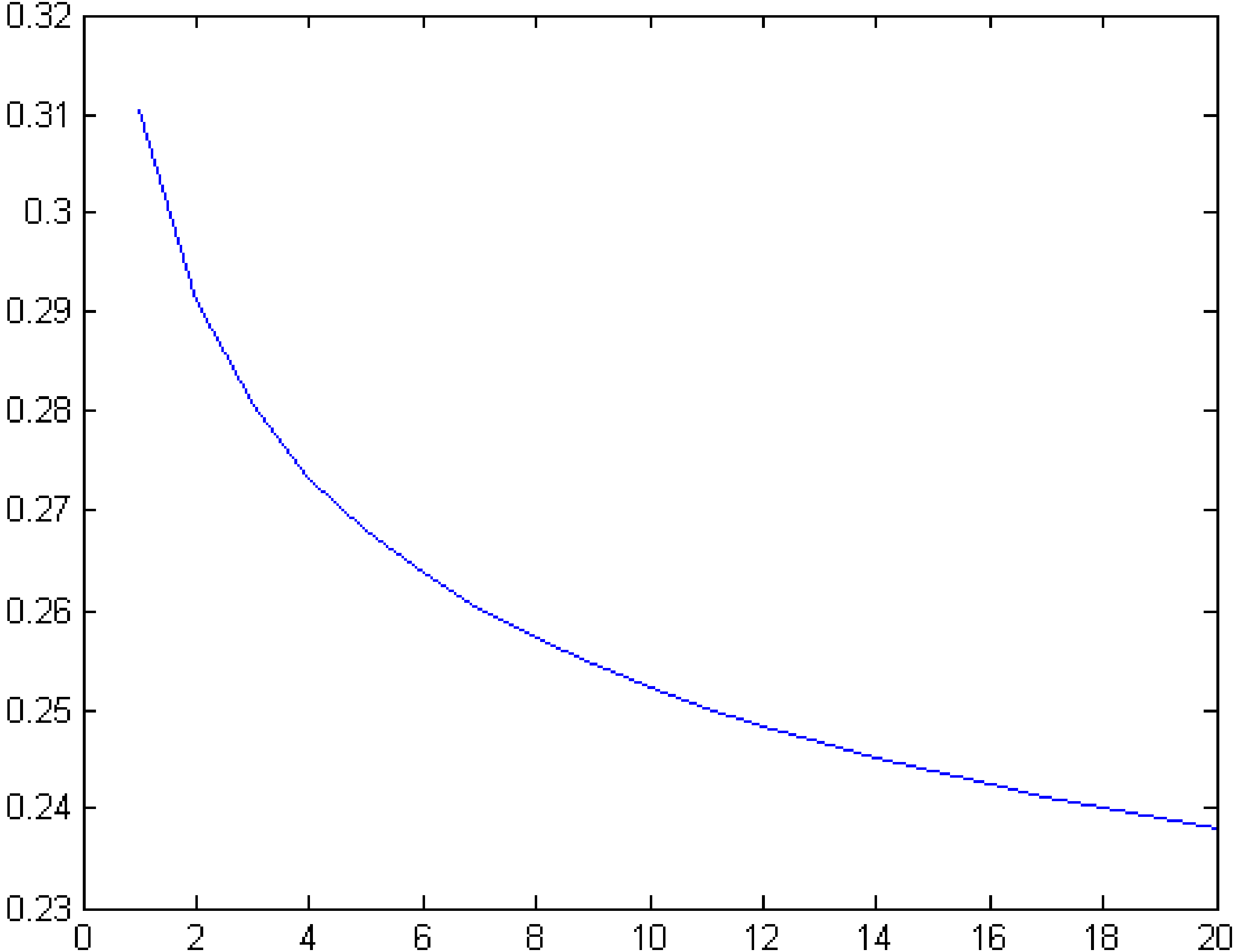


Figure 78

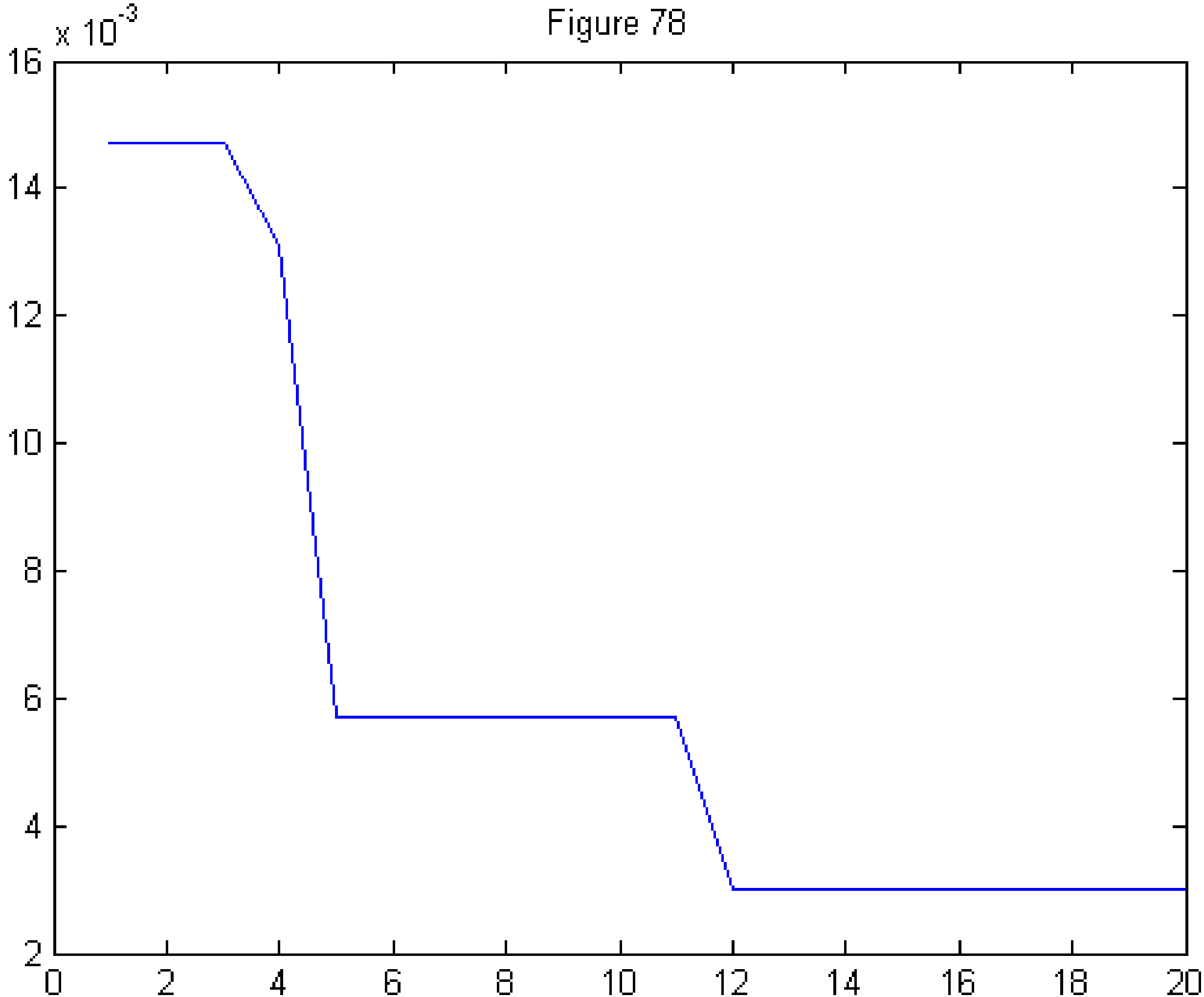
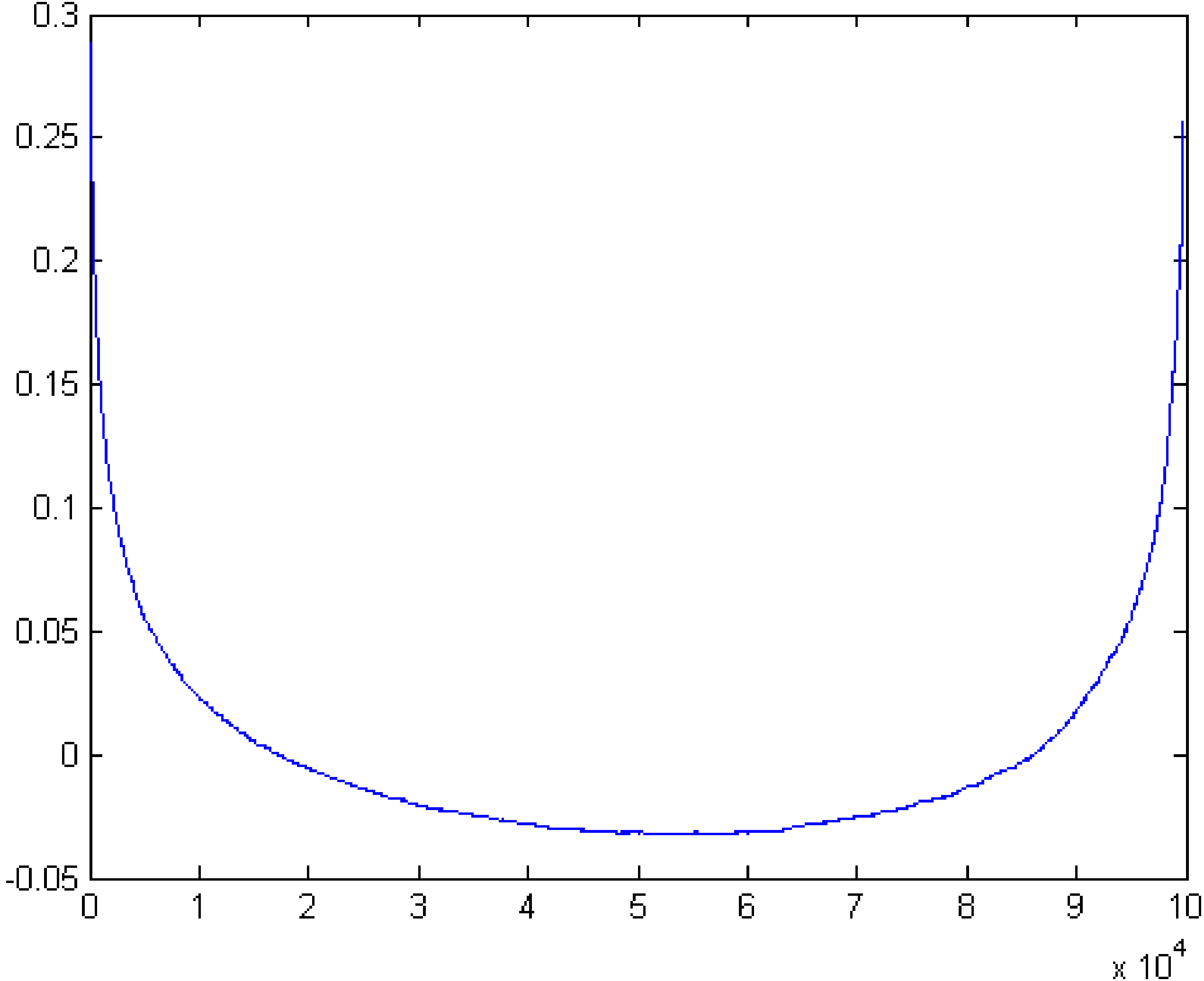


Figure 79



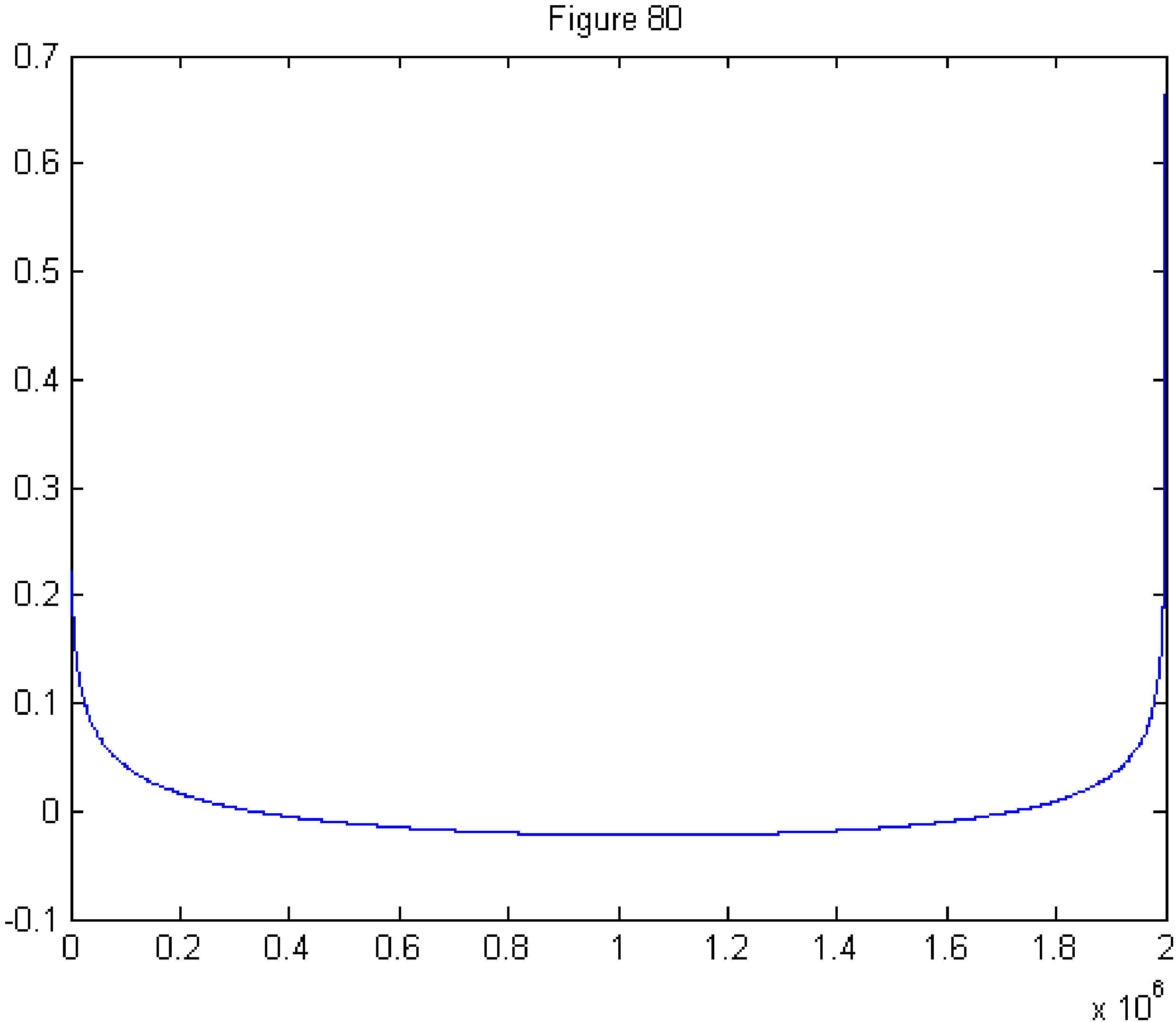


Figure 80
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x 10^6