Farey Sequences and the Riemann Hypothesis

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Abstract

Relationships between the Farey sequence and the Riemann hypothesis other than the Franel-Landau theorem are discussed.

1 Introduction

The Farey sequence F_x of order x is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed x. In this article, the fraction 0/1 is not considered to be in the Farey sequence. The number of fractions in F_x is $A(x) := \sum_{i=1}^{x} \phi(i)$ where ϕ is Euler's totient function. For $v = 1, 2, 3, ..., A(x)$ let δ_v denote the amount by which the vth term of the Farey sequence differs from $v/A(x)$. Franel (in collaboration with Landau) [1] proved that the Riemann hypothesis is equivalent to the statement that $|\delta_1| + |\delta_2| + ... + |\delta_{A(x)}| = o(x^{\frac{1}{2}+\epsilon})$ for all $\epsilon > 0$ as $x \to \infty$. The Stieltjes hypothesis states that $M(x) = O(x^{\frac{1}{2}})$ where $M(x)$ is the Mertens function ($M(x) := \sum_{k=1}^{x} \mu(k)$ where $\mu(k)$ is the Möbius function).

2 An Upper Bound of $|M(x)|$

Lehman [2] proved that $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) = 1$. In general, $\sum_{i=1}^{x} M(\lfloor x/(in) \rfloor) = 1$, $n = 1, 2, 3, ..., x$ (since $\lfloor \lfloor x/n \rfloor / i \rfloor = \lfloor x/(in) \rfloor$. Let T denote the x by x matrix where element (i, j) equals $\phi(j)$ if j divides i or 0 otherwise. Let U denote the matrix obtained from T by element-by-element multiplication of the columns by $M(\lfloor x/1 \rfloor)$, $M(\lfloor x/2 \rfloor)$, $M(\lfloor x/3 \rfloor)$, ..., $M(\lfloor x/x \rfloor)$. The sum of the columns of U then equals $A(x)$. $i = \sum_{d|i} \phi(d)$, so $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i$ (the sum of the rows of U) equals $A(x)$.

Theorem (1) $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i = A(x)$

By the Schwarz inequality, $A(x)/\sqrt{x(x+1)(2x+1)/6}$ is a lower bound of $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2}$. For $x \leq 1000000$, the "curve" of $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2$ values has been confirmed to be mostly linear. Also, for $x \leq 500$, the curve of $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2$ values resembles the curve of $8\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))$ values (the latter quantity equals $O(x)$).

Conjecture (1) $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 = O(x)$

Mertens [3] proved that $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\log(i) = \psi(x)$ where $\psi(x)$ denotes the second Chebyshev function. Let $d(i)$ denote half the number of positive divisors of i. Replacing $\phi(j)$ with $\log(j)$ in the T matrix gives a similar result.

Theorem (2) $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)\log(i)d(i) = \log(x!)$

The following conjecture is based on data collected for $x \leq 10000$.

Conjecture (2) $\log(x!) \geq \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \geq \psi(x)$

By Stirling's formula, $\log(x!) = x\log(x) - x + O(\log(x))$. Since $\log(x)$ increases more slowly than any positive power of x, this is a better upper bound of $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2$ than $x^{1+\epsilon}$ for any $\epsilon > 0$.

3 An $O(x^2)$ Function Similar to $A(x)$

Mertens [4] proved that $\sum_{m=1}^{G} \phi(m) = \frac{3}{\pi^2}G^2 + \Delta$ where $|\Delta| < G(\frac{1}{2}\log_e G + \frac{1}{2}C + \frac{5}{8}) + 1$ and C is Euler's constant 0.57721.... For a linear least squares fit of $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for x = 2, 3, 4, ..., 100000, $p_1 = 0.5513$ with a 95% confidence interval of (0.5513, 0.5513), $p_2 = 0.2757$ with a 95% confidence interval of (0.2741, 0.2772), SSE=1634, R-square=1, and RMSE=0.1278. Let $k = \lfloor x/6 \rfloor$ and $r = x - 6k$. Let $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$ equal 1, 2, 4, 6, and 11 respectively. If $k > 0$ and $r = 0$, let $g(x) = 12k + 23(k(k-1))/2$. If $k > 0$ and $r = 1$, let $g(x) = 12k + 23(k(k-1))/2 + 7 + 6(k-1)$. If $k > 0$ and $r = 2$, let $g(x) = 12k + 23(k(k-1))/2 + 11 + 9(k-1)$. If $k > 0$ and $r = 3$, let $g(x) = 12k + 23(k(k-1))/2 + 17 + 13(k-1)$. If $k > 0$ and $r = 4$, let $g(x) = 12k + 23(k(k-1))/2 + 22 + 16(k-1)$. If $k > 0$ and $r = 5$, let $g(x) = 12k + 23(k(k-1))/2 + 33 + 22(k-1)$.

Conjecture (3) $g(x) \leq \sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))i$.

$\sqrt{g(x)}$ increases almost linearly. For a linear least squares fit of $\sqrt{g(x)}$ versus x for x = 2, 3, 4, ..., 100000, $p_1 = 0.5652$ with a 95% confidence interval of (0.5652, 0.5652), $p_2 = 0.2826$ with a 95% confidence interval of (0.2809, 0.2843), SSE=1868, R-square=1, and RMSE=0.1367. The step height from the $\sqrt{g(x)}$ value where $r = 0$ to the value where $r = 1$ is approximately equal to the step height from the $\sqrt{g(x)}$ value where $r = 4$ to the value where $r = 5$ and the step height from the $\sqrt{g(x)}$ value where $r = 1$ to the value where $r = 2$ is approximately equal to the step height from the $\sqrt{g(x)}$ value where $r = 3$ to the value where $r = 4$. This accounts for there being essentially four different step sizes. Similarly, $\sqrt{\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))i}$ increases almost linearly and there are essentially four different step sizes. For a linear least squares fit of

$\sqrt{\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))i}$ versus x for $x = 2, 3, 4, ..., 100000$, $p_1 = 0.5653$ with a 95% confidence interval of (0.5653, 0.5653), $p_2 = 0.2826$ with a 95% confidence interval of (0.2809, 0.2840), SSE=1884, R-square=1, and RMSE=0.1373.

4 An $O(x)$ Function Similar to $\sqrt{g(x)}$

Let m_x denote the number of fractions in the Farey sequence of order x before $\frac{1}{4}$ and n_x the number of fractions between $\frac{1}{4}$ and $\frac{1}{2}$. The curve of $m_x - n_x$ values resembles that of the Mertens function in that the peaks and valleys occur roughly at the same places and have about the same heights and depths. Let $h(x)$ denote $\sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})$. $h(2)$, $h(3)$, $h(4)$, ..., and $h(13)$ equal 0, 0, 1, 1, 0, 1, 2, 1, 1, 2, 2, and 2 respectively.

Conjecture (4) $h(x + 12) = h(x) + 2$

$12h(x)^2$ is approximately equal to $g(x)$. For a linear least squares fit of $\sqrt{12}h(x)$ versus x for $x = 2, 3, 4, ..., 100000$, $p_1 = 0.5774$ with a 95% confidence interval of (0.5773, 0.5774), $p_2 = -0.5776$ with a 95% confidence interval of (-0.6383, -0.517), SSE=23880, R-square=1, and RMSE=1.546. The following conjecture is based on data collected for $x \leq 10000$.

Conjecture (5) $\psi(x) \geq \sum_{i=1}^{x} |sgn(M(\lfloor x/i \rfloor))| \geq 1 + \sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})^2 \geq \sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor)) \geq h(x) \geq \sum_{i=1}^{x} sgn(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})$

Also, $\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))$ is approximately equal to $\sum_{i=1}^{x} |sgn(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})|$.

Conjecture (6) $1 \geq |\sqrt{\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))} - \sqrt{\sum_{i=1}^{x} |sgn(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})|}|$

5 Corresponding $O(x^2)$ Functions and Miscellaneous

Other conjectures can be formulated where $n_x - m_x$ plays the role of $M(x)$.

Conjecture (7) If $x > 56$, $g(x) > 12\sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})i$.

Conjecture (8) If $x > 78$, $h(x)^2 > \sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})i$.

Conjecture (9) $\sum_{i=1}^{x}(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})i \geq \sum_{i=1}^{x} sgn(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor})i$

Also, upper bounds of partial sums of $\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))i$ can be found.

Conjecture (10) $(\lfloor x/n \rfloor(\lfloor x/n \rfloor + 1)/2 \geq -\sum_{i=1}^{\lfloor x/(j+1) \rfloor} sgn(M(\lfloor x/i \rfloor))i \geq 0$, $j = 1, 2, 3, ...$, where $n = 2 + \sum_{i=1}^{j} |sgn(M(i))|$

The loss in value of $\sum_{i=1}^{\lfloor x/2 \rfloor} sgn(M(\lfloor x/i \rfloor))i$ (compared to $\sum_{i=1}^{x} sgn(M(\lfloor x/i \rfloor))i$) is $\lfloor (x+1)/2 \rfloor$ (due to the $M(1)$ value being effectively set to 0). Let $d(2) = 0$ and $d(3)$, $d(4)$, $d(5)$, ..., $d(14)$ equal 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, and 1 respectively. Let $d(x + 12) = d(x) + 1$.

Conjecture (11) The gain in value of $\sum_{i=1}^{\lfloor x/4 \rfloor} sgn(M(\lfloor x/i \rfloor))i$ due to effectively setting $M(3)$ to 0 is $d(x)$.

Let $e(2) = e(3) = 0$ and $e(4)$, $e(5)$, $e(6)$, ..., $e(23)$ equal 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, and 1 respectively. Let $e(x + 20) = e(x) + 1$.

Conjecture (12) The gain in value of $\sum_{i=1}^{\lfloor x/5 \rfloor} sgn(M(\lfloor x/i \rfloor))i$ due to effectively setting $M(4)$ to 0 is $e(x)$.

Let $f(2) = f(3) = f(4) = 0$ and $f(5)$, $f(6)$, $f(7)$, ..., $f(34)$ equal 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, and 1 respectively. Let $f(x + 30) = f(x) + 1$.

Conjecture (13) The gain in value of $\sum_{i=1}^{\lfloor x/6 \rfloor} sgn(M(\lfloor x/i \rfloor))i$ due to effectively setting $M(5)$ to 0 is $f(x)$.

Other gains or losses can be computed similarly. $h(x)$ appears to be related to $d(x)$.

Conjecture (14) $h(x) + d(x)$, $x = 2, 3, 4, ...$, equals 0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, ...

Let $q(2) = q(3) = 0$ and $q(x)$, $x = 4, 5, 6, ...$, equal $1 - e(4)$, $1 - e(5)$, $1 - e(6)$, $1 - e(7)$, $1 - e(8)$, $2 - e(9)$, $2 - e(10)$, $2 - e(11)$, $2 - e(12)$, $2 - e(13)$, $3 - e(14)$, $3 - e(15)$, $3 - e(16)$, $3 - e(17)$, $3 - e(18)$, ..., respectively.

Conjecture (15) $2 + \sum_{i=1}^{x} sgn(n_{\lfloor x/i \rfloor} - m_{\lfloor x/i \rfloor}) \geq q(x)$.

References

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6 Miscellaneous

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