Farey Sequences and the Riemann Hypothesis

Darrell Cox

Abstract

Measures similar to those used in the Franel-Landau theorem are introduced. These measures are relevant to the Stieltjes hypothesis.

1 Introduction

The Farey sequence F_x of order x is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed x. In this article, the fraction 0/1 is not considered to be in the Farey sequence. The number of fractions in F_x is $A(x) := \sum_{i=1}^{x} \phi(i)$ where ϕ is Euler's totient function. For $v = 1, 2, 3, ..., A(x)$ let δ_v denote the amount by which the vth term of the Farey sequence differs from $v/A(x)$. Franel (in collaboration with Landau) [1] proved that the Riemann hypothesis is equivalent to the statement that $|\delta_1| + |\delta_2| + ... + |\delta_{A(x)}| = o(x^{\frac{1}{2}+\epsilon})$ for all $\epsilon > 0$ as $x \to \infty$. The Stieltjes hypothesis states that $M(x) = O(x^{\frac{1}{2}})$ where $M(x)$ is the Mertens function.

2 Variants of Franel's Measure

Franel proved that $2\pi \sum_{v=1}^{A(x)} |\delta_v| \geq |M(x)|$ (see section 12.2 of Edwards' [2] book). The quantity $\sqrt{A(x) \sum_{v=1}^{A(x)} \delta_v^2}$ is used in the proof that the Riemann hypothesis implies $\sum_{v=1}^{A(x)} |\delta_v| = o(x^{\frac{1}{2}+\epsilon})$. $\sqrt{A(x) \sum_{v=1}^{A(x)} \delta_v^2}$ is greater than or equal to $\sum_{v=1}^{A(x)} |\delta_v|$ by the Schwarz inequality. For a linear least-squares fit of $\sqrt{A(x) \sum_{v=1}^{A(x)} \delta_v^2}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 2560$, p_1 (the slope) equals 0.4542 with a 95% confidence interval of (0.4539, 0.4545), p_2 (the y-intercept) equals -0.4419 with a 95% confidence interval of $(-0.4528, -0.4309)$, SSE=22.61, R-square=0.9997, and RMSE=0.09403. (In this and similar computations, double-precision floating-point arithmetic is used. Cumulative errors due to the large number of adds, subtracts, and divides are assumed to have not occurred.) The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 0.5 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 0.5 less than that of the least-squares fit. This indicates that the

Stieltjes hypothesis is true. For a linear least-squares fit of $\sqrt{A(x)\sum_{v=1}^{A(x)} \delta_v^2}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.4533$ with a 95% confidence interval of (0.4531, 0.4535), $p_2 = -0.4045$ with a 95% confidence interval of (-0.4138, -0.3952), SSE=65.71, R-square=0.9998, and RMSE=0.1132. The values are bounded below by a line having the same slope and a y-intercept 0.5 less than that of the least-squares fit. Other than five x values (2803, 2804, 2806, 2810, and 2837), the values are bounded above by a line having the same slope and a y-intercept 0.5 more than that of the least-squares fit. A y-intercept of 0.55244 is required to bound these values. As will be shown, the quadratic nature of the "curve" of $\sqrt{A(x)\sum_{v=1}^{A(x)} \delta_v^2}$ values is due to the last δ_v values (where $v = A(x)$). Let r_v denote the vth fraction in the Farey sequence and let β_v denote $[r_v] - v/A(x)$ where the brackets denote the fractional portion of r_v. $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2}$ increases almost linearly with x (the deviations from a straight line are very small). For a linear least-squares fit of $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2}$ versus x for $x = 2, 3, 4, ..., 900$, $p_1 = 0.5514$ with a 95% confidence interval of (0.5513, 0.5514), $p_2 = 0.4187$ with a 95% confidence interval of (0.4017, 0.4357), SSE=15.07, R-square=1, and RMSE=0.1296. For a linear least-squares fit of $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2}$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.5513$ with a 95% confidence interval of (0.5513, 0.5513), $p_2 = 0.444$ with a 95% confidence interval of (0.4369, 0.451), SSE=84.73, R-square=1, and RMSE=0.1286. Let γ_v denote $r_v - [2v/A(x)]$. The $|\gamma_v|$ values corresponding to the fractions symmetrical about the fraction 1/2 are then equal. For a linear least-squares fit of $\sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|}$ versus x for $x = 2, 3, 4, ..., 900$, $p_1 = 0.5513$ with a 95% confidence interval of (0.5513, 0.5513), $p_2 = 0.283$ with a 95% confidence of (0.2661, 0.2999), SSE=14.86, R-square=1, and RMSE=0.1287. For a linear least-squares fit of $\sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|}$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.5513$ with a 95% confidence interval of (0.5513, 0.5513), $p_2 = 0.2728$ with a 95% confidence of (0.2657, 0.2799), SSE=85.37, R-square=1, and RMSE=0.1291. Based on this data, $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2}$ is approximately equal to $\sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|}$. For $x > 346$, $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2} - \sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|}$ is between 0.16 and 0.20. The smallest difference (for $x = 2$) is -0.5858 and the largest difference (for $x = 199$) is 0.2057. The differences appear to be approaching $\frac{1}{\pi\sqrt{3}}$ as x increases. Except for small x values, the differences are less than $\frac{1}{\pi\sqrt{3}}$ about as often as they are greater than $\frac{1}{\pi\sqrt{3}}$. $8\sum_{v=1}^{A(x)/2} |\gamma_v|$ is also approximately equal to $A(x)$. For x less than or equal to 5128, $8\sum_{v=1}^{A(x)/2} |\gamma_v| - A(x)$ ranges from about 16 to about -24. For $x = 5128$, $8\sum_{v=1}^{A(x)/2} |\gamma_v| = 7994259.2$ and $A(x) = 7994266$. For $x > 24$, $\sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|} - \sqrt{A(x)}$ is between 0.05 and -0.05. The differences appear to be approaching 0 as x increases. For a linear least-squares fit of $\sqrt{\sum_{v=1}^{A(x)/2} |\gamma_v|}$

versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.1949$ with a 95% confidence interval of (0.1949, 0.1949), $p_2 = 0.097$ with a 95% confidence interval of (0.09452, 0.09947), SSE=10.48, R-square=1, and RMSE=0.04523. For a linear least-squares fit of $\sqrt{\sqrt{(A(x)/2)\sum_{v=1}^{A(x)/2}\gamma_v^2}}$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.2095$ with a 95% confidence interval of (0.2095, 0.2095), $p_2 = 0.1083$ with a 95% confidence interval of (0.1057, 0.111), SSE=12.14, R-square=1, and RMSE=0.04867. As expected, $\sqrt{(A(x)/2)\sum_{v=1}^{A(x)/2}\gamma_v^2}$ is greater than or equal to $\sum_{v=1}^{A(x)/2}|\gamma_v|$. For $x > 136$ and $x \leq 5128$, $\sqrt{A(x)}\sum_{v=1}^{A(x)}\delta^2$ is between 0.3 and 0.4. $\sqrt{A(x)}\sum_{v=1}^{A(x)}\delta^2$ is usually slightly less than $\frac{2}{\pi\sqrt{3}}$. $\sqrt{A(x)}\sum_{v=1}^{A(x)}\delta^2$ is greater than or equal to $\frac{2}{\pi\sqrt{3}}$ for 158 x values and the maximum difference is 0.00996847. These measures are useful as additional evidence that $\sqrt{A(x)}$ increases linearly.

3 The Number of Fractions in the Farey Sequence Before 1/4 and Between 1/4 and 1/2

Let $L(x)$ denote $\sum_{i=1}^{x}\lfloor\phi(i)/4\rfloor$ and $U(x)$ denote $\sum_{i=1}^{x}(\lfloor\phi(i)/4\rfloor + \lceil(\phi(i) - \lfloor\phi(i)/4\rfloor 4)/\phi(i)\rceil)$. (In the latter sum, $\lfloor\phi(i)/4\rfloor$ is incremented by 1 if 4 does not divide $\phi(i)$.) $L(x)$ is a lower bound of the number of fractions less than $\frac{1}{4}$ in the Farey sequence F_x and $U(x)$ is an upper bound. Similarly, $L(x)$ is a lower bound of the number of fractions greater than $\frac{1}{4}$ and less than $\frac{1}{2}$ in the Farey sequence F_x and $U(x)$ is an upper bound. $\sqrt{U(x) - L(x)}$ is roughly equal to $\frac{3}{4}\pi\sum_{v=1}^{A(x)}|\delta_v|$ for x values up to about 1500 and then appears to gradually become larger than $\frac{3}{4}\pi\sum_{v=1}^{A(x)}|\delta_v|$. For x values up to 5128, $\sqrt{U(x) - L(x)}$ has been confirmed to be greater than $\frac{1}{2}|M(x)|$. For a quadratic least-squares fit of $\sqrt{U(x) - L(x)}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 900$, $p_1 = -0.002733$ with a 95% confidence interval of $(-0.002841, -0.002625)$, $p_2 = 0.4359$ with a 95% confidence interval of (0.432, 0.4399), $p_3 = 1.292$ with a 95% confidence interval of (1.258, 1.325), SSE=6.321, R-square=0.9988, and RMSE=0.08399. As x becomes larger, the "curve" of $U(x) - L(x)$ values becomes more linear. For a quadratic least-squares fit of $\sqrt{U(x) - L(x)}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 5128$, $p_1 = -0.0005552$ with a 95% confidence interval of $(-0.0005635, -0.000547)$, $p_2 = 0.3433$ with a 95% confidence interval of (0.3426, 0.344), $p_3 = 2.136$ with a 95% confidence interval of (2.121, 2.15), SSE=40.45, R-square=0.9997, and RMSE=0.08884. 4 doesn't divide $\phi(i)$ if i is a power of a prime of the form $4k + 3$ or twice a power of a prime of the form $4k + 3$. $U(x) - L(x)$ then equals $3 + \sum\lfloor\log_q x\rfloor + \sum\lfloor\log_q \frac{x}{2}\rfloor$ where the summation is over the primes q of the form $4k + 3$ that are less than or equal to x. The first Chebyshev function $\vartheta(x)$ is defined to equal $\sum_{p\leq x}\log p$. The second Chebyshev function $\psi(x)$ equals $\sum_{p\leq x}\lfloor\log_p x\rfloor\log p$. Rosser and Schoenfeld [3] proved that $\psi(x) - \vartheta(x) < 1.42620x^{\frac{1}{2}}$. For a quadratic least-squares fit of $\psi(x) - \vartheta(x)$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 25000$, $p_1 = -0.0008752$ with a 95% confidence interval of $(-0.000986, -0.0008543)$, $p_2 = 1.327$ with a 95% confidence interval of $(1.322, 1.331)$, $p_3 = -1.865$ with a 95% confidence interval of $(-2.046, -1.685)$, SSE=1.469+5, R-square=0.9969, and RMSE=2.424. The plot of $\psi(x) - \vartheta(x)$ versus $\sqrt{x}$ becomes more linear as x increases. For a quadratic least-squares fit of $\psi(x) - \vartheta(x)$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 75000$, $p_1 = -0.000355$ with a 95% confidence interval of $(-0.0003606, -0.0003493)$, $p_2 = 1.24$ with a 95% confidence interval of $(1.238, 1.242)$, $p_3 = 1.127$ with a 95% confidence interval of $(0.9807, 1.274)$, SSE=8.731e+5, R-square=0.9978, and RMSE=3.412. For a quadratic least-squares fit of $\sqrt{U(x) - L(x)}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 75000$, the p_2 parameter equals 0.2869, giving a normalization factor of 4.322 (1.24/0.2869). For $x \leq 75000$, $|(\psi(x) - \vartheta(x)) - (4.322\sqrt{U(x) - L(x)} - 14.983)| < 15$. For a quadratic least-squares fit of $4.322\sqrt{U(x) - L(x)} - 14.983$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 75000$, $p_1 = -0.0003655$ with a 95% confidence interval of $(-0.0003663, -0.0003647)$, $p_2 = 1.24$ with a 95% confidence interval of $(1.24, 1.24)$, $p_3 = 1.123$ with a 95% confidence interval of $(1.102, 1.143)$, SSE=1.711e+4, R-square=1, and RMSE=0.4776. These parameter values are almost equal to those of the least-squares fit for $\psi(x) - \vartheta(x)$.

Mertens [4] proved that $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) \log i = \psi(x)$. For a linear least-squares fit of $\vartheta(x) - \sum_{i=1}^{x} (-1)^{i+1} M(\lfloor x/i \rfloor) \log i$ versus x for $x = 2, 3, 4, ..., 2000$, $p_1 = 0.9707$ with a 95% confidence interval of $(0.9692, 0.9722)$, $p_2 = -12.86$ with a 95% confidence interval of $(-13.72, -12)$, SSE=4.75e+4, R-square=0.9994, and RMSE=6.903. For a linear least-squares fit of $2(M(\lfloor x/2 \rfloor) \log 2 + M(\lfloor x/4 \rfloor) \log 4 + M(\lfloor x/6 \rfloor) \log 6 + ...)$ versus x for $x = 2, 3, 4, ..., 2000$, $p_1 = 0.9982$ with a 95% confidence interval of $(0.9976, 0.9989)$, $p_2 = -0.6153$ with a 95% confidence interval of $(-1.363, 0.1322)$, SSE=1.447e+5, R-square=0.9998, and RMSE=8.511. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 40 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 40 less than that of the least-squares fit. For a linear least-squares fit of $2(M(\lfloor x/2 \rfloor) \log 2 + M(\lfloor x/4 \rfloor) \log 4 + M(\lfloor x/6 \rfloor) \log 6 + ...)$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 1.0$ with a 95% confidence interval of $(1.0, 1.001)$, $p_2 = -2.427$ with a 95% confidence interval of $(-3.267, -1.588)$, SSE=1.204e+6, R-square=0.9999, and RMSE=15.33. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 80 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 80 less than that of the least-squares fit. In general, $n(M(\lfloor x/n \rfloor) \log n + M(\lfloor x/(2n) \rfloor) \log (2n) + M(\lfloor x/(3n) \rfloor) \log (3n) + ...)$ appears to be a step function where the width of a step is a multiple of n.

$\sum_{i=1}^{x} M(\lfloor x/i \rfloor) d(i) \log i = \sum_{i=1}^{x} \log i$ where $d(i)$ denotes half the number of positive divisors of i (this can be proved rigorously). Based on empirical evidence, $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) \log i$ is less than $\sum_{i=1}^{x} \log i / d(i)$. For a quadratic least-squares fit of $\sum_{i=1}^{x} \log i / d(i)$ versus x for $x = 2, 3, 4, ..., 1000$, $p_1 = 0.0002933$

with a 95% confidence interval of (0.0002878, 0.0002988), $p_2 = 2.307$ with a 95% confidence interval of (2.301, 2.313), $p_3 = -37.02$ with a 95% confidence interval of $(-38.26, -35.79)$, SSE=4.326e+4, R-square=0.9999, and RMSE=6.591. $\sqrt{\sum_{i=1}^{x} \log i/d(i)}$ has been confirmed to be greater than $|M(x)|$ for $x = 2, 3, 4, ..., 5128$. Let $e(x)$ denote $\sum_{i=1}^{x} \log i/d(i) - \vartheta(x)$. For a quadratic least-squares fit of $e(x)$ versus x for $x = 2, 3, 4, ..., 10000$, p_1=2.376e-5 with a 95% confidence interval of (2.362e-5, 2.389e-5), $p_2 = 1.86$ with a 95% confidence interval of (1.858, 1.861), $p_3 = -270$ with a 95% confidence interval of $(-273.1, -266.9)$, SSE=2.758e+7, R-square=0.9999, and RMSE=52.53. $\sqrt{e(x)}$ becomes consistently greater than $\psi(x) - \vartheta(x)$ for about $x > 2500$.

In the proof that $\sum_{v=1}^{A(x)} |\delta_v| = o(x^{\frac{1}{2}+\epsilon})$ implies the Riemann hypothesis, the function $f(u) = e^{2\pi i u}$ is substituted into the equation $\sum_{v=1}^{A(x)} f(r_v) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f(j/k)M(x/k)$. The function $\lceil(\phi(d) - \lfloor\phi(d)/4\rfloor 4)/\phi(d)\rceil/\phi(d)$ where d denotes the denominator of a fraction can be used to find a direct relationship between $U(x) - L(x)$ and the Mertens function since the sum of this function over the fractions in the Farey sequence equals $U(x) - L(x)$. Substituting the function into the right-hand side of the above equation and using the floor of x/k gives a useable result. Let $N(x)$ denote $-\sum_{i=1}^{x} M(\lfloor x/i \rfloor) i \lceil(\phi(i) - \lfloor\phi(i)/4\rfloor 4)/\phi(i)\rceil/\phi(i)$. ($N(x)$ can be viewed as being an approximation of $U(x) - L(x)$ or just an *ad hoc* function. Since $\lceil(\phi(i) - \lfloor\phi(i)/4\rfloor 4)/\phi(i)\rceil/\phi(i)$ is a rational number and $M(\lfloor x/i \rfloor)$ is an integer, it's not likely that $-N(x)$ will be a natural number. As will be shown, using the floor of x/k and substituting an integer-valued function into the above equation does give the expected result on occasion.) For $x > 80$, $N(x)$ appears to be consistently smaller than $U(x) - L(x)$. For x values up to 5128, $\sqrt{N(x)}$ has been confirmed to be greater than $\frac{1}{2}|M(x)|$ (although $\sqrt{N(x)} - \frac{1}{2}|M(x)| = 0.1662$ for $x = 2837$). For a linear least-squares fit of $N(x)$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 5128$, $p_1 = 2.893$ with a 95% confidence interval of (2.876, 2.91), $p_2 = -5.654$ with a 95% confidence interval of $(-6.52, -4.789)$, SSE=5.675e+5, R-square=0.9555, and RMSE=10.52. $N(x)$ is analogous to the quantity $2\pi\sqrt{A(x)\sum_{v=1}^{A(x)} \delta_v^2}$. For a linear least-squares fit of $2\pi\sqrt{A(x)\sum_{v=1}^{A(x)} \delta_v^2}$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 5128$, $p_1 = 2.848$ with a 95% confidence interval of (2.847, 2.849), $p_2 = -2.542$ with a 95% confidence interval of $(-2.6, -2.483)$, SSE=2594, R-square=0.9998, and RMSE=0.715.

Let m_x denote the number of fractions in the Farey sequence before $\frac{1}{4}$ and n_x the number of fractions between $\frac{1}{4}$ and $\frac{1}{2}$. The "curve" of $m_x - n_x$ values resembles that of the Mertens function in that the peaks and valleys occur roughly at the same places and have about the same heights and depths. The following is a partial explanation for the relationship between these two "curves". Franel proved that $M(x) = \sum_{v=1}^{A(x)} e^{2\pi i r_v}$. The sines cancel out and the cosines are symmetrical about the x axis, so it is only necessary to compute the cosines for the fractions up to $\frac{1}{2}$. (By Euler's formula, $e^{ix} = \cos(x) + i\sin(x)$.) Suppose $\frac{h}{k}$ and $\frac{h'}{k'}$ are successive fractions in a Farey sequence of order n. Let l

denote $\lfloor (n-k')/(k'-k) \rfloor$ if $k' > k$, or $\lfloor \lceil (2k'-n-1)/(k-k') \rceil /2 \rfloor$ otherwise. If $l \neq 0$, the next l fractions in the Farey sequence correspond to the remaining lattice points on the line through (h, k) and (h', k') and are $\frac{h'+(h'-h)i}{k'+(k'-k)i}$, $i = 1, 2, 3, ..., l$. This property of the Farey sequence (where the numerators and denominators increase or decrease linearly) can be used to compute a full-order Farey sequence by interpolating between the fractions in a half-order sequence (if n is odd, a half-order of $(n+1)/2$ is used). The corresponding interpolation of sums of cosines (of 2π times the fractions) is almost linear. For example, for $x = 29$ (the half-order), the interpolated fractions between $\frac{7}{25}$ and $\frac{2}{7}$ are $\frac{16}{57}$, $\frac{9}{32}$, $\frac{11}{39}$, $\frac{13}{46}$, and $\frac{15}{53}$ and the respective sums of cosines (starting with the sum for $\frac{7}{25}$) are 155.3513, 155.1596, 154.9645, 154.7645, 154.5610, 154.3551, and 154.1325. The change in the value of the Mertens function from the half-order Farey sequence to the full-order sequence is then mostly dependent on the changes in the number of fractions before $\frac{1}{4}$ and between $\frac{1}{4}$ and $\frac{1}{2}$. Let a_x denote the sum of $\cos(2\pi r_v)$ for r_v up to $\frac{1}{4}$ and let b_x denote the sum of $\cos(2\pi r_v)$ for r_v between $\frac{1}{4}$ and $\frac{1}{2}$. For example, $a_{255} = 2455.446$, $m_{255} = 3858$, $b_{255} = -2453.946$, $n_{225} = 3854$, $a_{450} = 9807.467$, $m_{450} = 15405$, $b_{450} = -9810.967$, and $n_{450} = 15410$. Then $a_{450}(m_{255}/m_{450}) = 2456.164$, $b_{450}(n_{225}/n_{450}) = -2453.6966$ and the sum of these two quantities is 2.4674, close to the expected value of 1.5. The Mertens function can be similarly approximated using third-order, fourth-order, etc. Farey sequences. Let $P(x)$ denote $M(x) - \sum_{i=1}^{x}(a_x(m_{\lfloor x/i \rfloor}/m_x) + b_x(n_{\lfloor x/i \rfloor}/n_x))i\lceil(\phi(i) - \lfloor \phi(i)/4 \rfloor 4)/\phi(i)\rceil/\phi(i)$. $P(x)$ is analogous to the quantity $N(x) - M(x)$. The "curve" of $P(x)$ values has well-defined peaks and valleys that become larger and broader as x increases (for x values less than about 400, the peaks and valleys oscillate between 0 and $U(x) - L(x)$). $P(x)$ appears to be negative for only x equal to 287, 288, 289, 290, 291, 292, and 293 (the respective $P(x)$ values are -0.6266, -2.1199, -2.1067, -1.2984, -2.1241, -2.1239, and -1.2914). $P(x)$ appears to be greater than $U(x) - L(x)$ for only x equal to 41, 94, 95, 96, 97, and 98 (the respective differences in values are -0.0199, -0.1662, -3.1855, -3.2144, -3.9493, and -0.0486). For a quadratic least-squares fit of $P(x)$ versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.02648$ with a 95% confidence interval of (0.02344, 0.02952), $p_2 = 1.093$ with a 95% confidence interval of (0.8262, 1.36), and $p_3 = 6.057$ with a 95% confidence interval of (0.6468, 11.47). SSE=5.506e+6, R-square=0.754, and RMSE=32.78.

Let $L(x)$ denote $\sum_{i=1}^{x} a_x(m_{\lfloor x/i \rfloor}/m_x)$ and $R(x)$ denote $\sum_{i=1}^{x} b_x(n_{\lfloor x/i \rfloor}/n_x)$. For a quadratic least-squares fit of $L(x)$ versus x for $x = 2, 3, 4, ..., 1280$, $p_1 = 0.07958$ with a 95% confidence interval of (0.07958, 0.07958), $p_2 = -0.318$ with a 95% confidence interval of (-0.3189, -0.3171), and $p_3 = -0.6552$ with a 95% confidence interval of (-0.9089, -0.4014). SSE=3005, R-square=1, and RMSE=1.535. For a quadratic least-squares fit of $R(x)$ versus x for $x = 2, 3, 4, ..., 1280$, $p_1 = -0.07958$ with a 95% confidence interval of (-0.07958, -0.07958), $p_2 = 0.2144$ with a 95% confidence interval of (0.2132, 0.2155), and $p_3 = -0.7061$ with a 95% confidence interval of (-1.034, -0.378). SSE=5026, R-square=1, and RMSE=1.985. For a quadratic least-squares fit of $L(x)$ versus x for $x = 2, 3, 4, ..., 101$, $p_1 = 0.07973$ with a 95% confidence interval of (0.07962, 0.07985), $p_2 = -0.3372$ with a 95% confidence interval of (-0.3493, -0.325), and $p_3 = -0.1597$ with a 95% confidence interval of (-0.4305, 0.1112). SSE=17.81, R-square=1, and RMSE=0.4285. For a quadratic least-squares fit of $R(x)$ versus x for $x = 2, 3, 4, ..., 101$, $p_1 = -0.07958$ with a 95% confidence interval of (-0.07973, -0.07943), $p_2 = 0.2088$ with a 95% confidence interval of (0.1932, 0.2244), and $p_3 = -0.06974$ with a 95% confidence interval of (-0.418, 0.2785). SSE=29.45, R-square=1, and RMSE=0.551. $L(x)$ and $R(x)$ appear to have fixed probability distributions. For quadratic least-squares fits of $L(x)$ versus x where $x = 1000$, 2000, 3000, 4000, and 5000, p_1 equals 0.07958, 0.07958, 0.07958, 0.07958, and 0.07958 respectively, p_2 equals -0.319, -0.3186, -0.3172, -0.3182, and -0.3173 respectively, and p_3 equals -0.4803, -0.528, -1.012, -0.5513, and -1.186 respectively. For quadratic least-squares fits of $R(x)$ versus x where $x = 1000$, 2000, 3000, 4000, and 5000, p_1 equals -0.07957, -0.07958, -0.07958, -0.07958, and -0.07958 respectively, p_2 equals 0.2086, 0.214, 0.2136, 0.211, and 0.2135 respectively, and p_3 equals 0.1585, -0.8939, -1.022, 0.1643, and -1.342 respectively. Let $S(x)$ denote $\sum_{i=1}^{x} a_x(m_{\lfloor x/i \rfloor}/m_x)i$ and $T(x)$ denote $\sum_{i=1}^{x} b_x(n_{\lfloor x/i \rfloor}/n_x)i$. For a quadratic least-squares fit of $\sqrt{S(x)}$ versus x for $x = 2, 3, 4, ..., 1280$, p_1=4.404e-5 with a 95% confidence interval of (4.335e-5, 4.474e-5), $p_2 = 0.4826$ with a 95% confidence interval of (0.4817, 0.4835), and $p_3 = -9.014$ with a 95% confidence interval of (-9.269, -8.759). SSE=3031, R-square=0.9999, and RMSE=1.541. For a quadratic least-squares fit of $\sqrt{-T(x)}$ versus x for $x = 2, 3, 4, ..., 1280$, p_1=4.303e-5 with a 95% confidence interval of (4.236e-5, 4.371e-5), $p_2 = 0.4977$ with a 95% confidence interval of (0.4968, 0.4986), and $p_3 = -8.589$ with a 95% confidence interval of (-8.838, -8.34). SSE=2897, R-square=0.9999, and RMSE=1.507. The "curves" of $\sqrt{S(x)}$ and $\sqrt{-T(x)}$ values become more linear as x increases. For quadratic least-squares fits of $\sqrt{S(x)}$ versus x where $x = 1000$, 2000, 3000, 4000, and 5000, p_1 equals 5.738e-5, 2.729e-5, 1.768e-5, 1.3e-5, and 1.025e-5 respectively, p_2 equals 0.4705, 0.5039, 0.5226, 0.5355, and 0.5453 respectively, and p_3 equals -7.293, -13.32, -19.1, -24.72, and -30.24 respectively. For quadratic least-squares fits of $\sqrt{-T(x)}$ versus x where $x = 1000$, 2000, 3000, 4000, and 5000, p_1 equals 5.603e-5, 2.668e-5, 1.729e-5, 1.272e-5, and 1.003e-5 respectively, p_2 equals 0.486, 0.5185, 0.5368, 0.5494, and 0.559 respectively, and p_3 equals -6.913, -12.79, -18.43, -23.92, and -29.31 respectively.

In this section, a_1 (equal to 0.5) and b_3 (equal to -0.5) are set to 0. The rationale for doing this is that $\frac{1}{4}$ is not in a Farey sequence of order less than 4. For a quadratic least-squares fit of $\sum_{i=1}^{x} a_{\lfloor x/i \rfloor}$ versus x for $x = 2, 3, 4, ..., 1280$, $p_1 = 0.07958$ with a 95% confidence interval of (0.07958, 0.07958), $p_2 = -0.0426$ with a 95% confidence interval of (-0.4206, -0.4206), and $p_3 = 0.3793$ with a 95% confidence interval of (0.3768, 0.3819). SSE=0.3006, R-square=1, and RMSE=0.1535. For a quadratic least-square fit of $\sum_{i=1}^{x} b_{\lfloor x/i \rfloor}$ versus x for $x = 2, 3, 4, ..., 1280$, $p_1 = -0.07958$ with a 95% confidence interval of (-0.07958,

-0.07958), $p_2 = 0.2123$ with a 95% confidence interval of (0.2122, 0.2124), and $p_3 = 0.01705$ with a 95% confidence interval of (-0.01531, 0.0494). SSE=48.87, R-square=1, and RMSE=0.1957. For quadratic least-squares fits of $\sum_{i=1}^{x} a_{\lfloor x/i \rfloor}$ versus x where x = 1000, 2000, 3000, 4000, and 5000, p_1 equals 0.07958, 0.07958, 0.07958, 0.07958, and 0.07958 respectively, p_2 equals -0.4207, -0.4206, -0.4205, -0.4205, and -0.4205 respectively, and p_3 equals 0.3389, 0.3684, 0.3586, 0.3472, and 0.3211 respectively. For quadratic least-squares fits of $\sum_{i=1}^{x} b_{\lfloor x/i \rfloor}$ versus x where x = 1000, 2000, 3000, 4000, and 5000, p_1 equals -0.07958, -0.07958, -0.07958, -0.07958, and -0.07958 respectively, p_2 equals 0.2123, 0.2122, 0.2122, 0.2122, and 0.2121 respectively, and p_3 equals 0.008158, 0.02777, 0.03777, 0.05075, and 0.05669 respectively. $\sum_{i=1}^{x} a_{\lfloor x/i \rfloor}$ has almost the same probability distribution as $L(x)$ except that the p_2 parameter is somewhat smaller and the p_3 parameter is somewhat larger. $\sum_{i=1}^{x} b_{\lfloor x/i \rfloor}$ has about the same probability distribution as $R(x)$. For a quadratic least-squares fit of $\sqrt{\sum_{i=1}^{x} a_{\lfloor x/i \rfloor} i}$ versus x for $x = 2, 3, 4, ..., 1280$, p_1=4.482e-5 with a 95% confidence interval of (4.411e-5, 4.553-5), $p_2 = 0.4729$ with a 95% confidence interval of (0.472, 0.4739), and $p_3 = -9.22$ with a 95% confidence interval of (-9.481, -8.959). SSE=3184, R-square=0.9999, and RMSE=1.58. For a quadratic least-squares fit of $\sqrt{-\sum_{i=1}^{x} b_{\lfloor x/i \rfloor} i}$ versus x for $x = 2, 3, 4, ..., 1280$, p_1=4.31e-5 with a 95% confidence interval of (4.242e-5, 4.378e-5), $p_2 = 0.4965$ with a 95% confidence interval of (0.4956, 0.4974), and $p_3 = -8.625$ with a 95% confidence interval of (-8.875, -8.375). SSE=2914, R-square=0.9999, and RMSE=1.511. For quadratic least-squares fits of $\sqrt{\sum_{i=1}^{x} a_{\lfloor x/i \rfloor} i}$ versus x where x = 1000, 2000, 3000, 4000, and 5000, p_1 equals 5.847e-5, 2.772e-5, 1.794e-5, 1.318e-5, and 1.039e-5 respectively, p_2 equals 0.4606, 0.4946, 0.5137, 0.5268, and 0.5368 respectively, and p_3 equals -7.46, -13.61, -19.49, -25.21, and -30.82 respectively. For quadratic least-squares fits of $\sqrt{-\sum_{i=1}^{x} b_{\lfloor x/i \rfloor} i}$ versus x where x = 1000, 2000, 3000, 4000, and 5000, p_1 equals 5.615e-5, 2.672e-5, 1.732e-5, 1.275e-5, and 1.005e-5 respectively, p_2 equals 0.487, 0.5173, 0.5365, 0.5483, and 0.5579 respectively, and p_3 equals -6.943, -12.83, -18.48, -23.99, and -29.39 respectively. As expected, $\sum_{i=1}^{x} a_{\lfloor x/i \rfloor} i$ has about the same distribution of values as $S(x)$ and $\sum_{i=1}^{x} b_{\lfloor x/i \rfloor} i$ has about the same distribution of values as $T(x)$.

Based on empirical evidence $\sum_{i=1}^{x} M(\lfloor x/(in) \rfloor) = 1$ for $n = 1, 2, 3, ..., x$ and $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) i = A(x)$. (The second result follows from the first. Let T denote the x by x matrix where element (i, j) equals $\phi(j)$ if j divides i or 0 otherwise. Let U denote the matrix obtained from T by element-by-element multiplication of the columns by $M(\lfloor x/1 \rfloor)$, $M(\lfloor x/2 \rfloor)$, $M(\lfloor x/3 \rfloor)$, ..., $M(\lfloor x/x \rfloor)$. By the first result, the sum of the columns of U equals $A(x)$. $i = \sum_{d|i} \phi(d)$, so $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) i$ (the sum of the rows of U) equals $A(x)$. Also, the function $\lceil 1/d \rceil$ where d denotes the denominator of a fraction can be substituted into the equation $\sum_{v=1}^{A(x)} f(r_v) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f(j/k) M(x/k)$. The first result then follows from the second if it can be shown that each of $\sum_{i=1}^{x} M(\lfloor x/(in)\rfloor)$, $n = 1, 2, 3, ..., x$, is positive. Another approach is to use the definition of the Mertens function $[M(x) = \sum_{k=1}^{x} \mu(k)]$ to prove that $\sum_{i=1}^{x} M(\lfloor x/i\rfloor) - \sum_{i=1}^{x-1} M(\lfloor (x-1)/i\rfloor) = 0$. Let n denote the number of distinct prime factors of x. In essence, $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n \binom{n}{n} = 0$ implies $\sum_{i=1}^{x} M(\lfloor x/i\rfloor) = 1$.) $A(x)$ is approximately equal to $3x^2/\pi^2$. (Mertens [5] proved that $\sum_{m=1}^{G} \phi(m) = \frac{3}{\pi^2}G^2 + \Delta$ where $|\Delta| < G(\frac{1}{2}\log_e G + \frac{1}{2}C + \frac{5}{8}) + 1$ and C is Euler's constant 0.57721....) For $x > 13$ and $x \le 5128$, $\sqrt{\sqrt{A(x)}}$ has been confirmed to be greater than $|M(x)|$. For a linear least-squares fit of $\sqrt{\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i\rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.3898$ with a 95% confidence interval of (0.3898, 0.3898) and $p_2 = 0.1952$ with a 95% confidence interval of (0.1902, 0.2001). SSE=41.87, R-square=1, and RMSE=0.09038. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 0.5 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 0.5 less than that of the least-squares fit. (If $a > b > 0$, $\sqrt{a} + b/(2\sqrt{a}) > \sqrt{a+b}$. Mertens' result is not strong enough to guarantee that these upper and lower bounds won't fail for very large x due to the growth of $\log_e x$. For $x = 5128$, $\sqrt{A(x)} = 2827.4133$, $\sqrt{\frac{3}{\pi^2}x^2 + |\Delta|} = 2831.913$, $\sqrt{\frac{3}{\pi^2}x^2} + |\Delta|/(2\sqrt{\frac{3}{\pi^2}x^2}) = 2831.917$, and $|\Delta|/(2\sqrt{\frac{3}{\pi^2}x^2}) = 4.702307$. For $x = 1000$, $|\Delta|/(2\sqrt{\frac{3}{\pi^2}x^2}) = 3.961776$. This is some indication that $\sqrt{A(x)}$ is not growing due to the $\log_e x$ term in Mertens' result. Also, there is no apparent reason to expect that $\sqrt{8\sum_{v=1}^{A(x)/2} |\gamma_v|}$ or $\sqrt{A(x)\sum_{v=1}^{A(x)} \beta_v^2 - \frac{1}{\pi\sqrt{3}}}$ will become non-linear as x increases.) The Schwarz inequality gives $A(x)/\sqrt{x(x+1)(2x+1)/6}$ as a lower bound of $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2}$. For a linear least-squares fit of $\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2$ versus x for $x = 2, 3, 4, ..., 1000$, $p_1 = 1.47$ with a 95% confidence interval of (1.463, 1.477), $p_2 = -6.204$ with a 95% confidence interval of (−10.26, −2.146), SSE=1.06e+6, R-square-0.9941, and RMSE=32.6. For a linear least-squares fit of $\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 1.518$ with a 95% confidence interval of (1.515, 1.521), $p_2 = -25.46$ with a 95% confidence interval of (−34.21, −16.71), SSE=1.307e+8, R-square-0.995, and RMSE=159.7. For x values up to 5128, $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2}$ is between 2 and 3 times as large as $A(x)/\sqrt{x(x+1)(2x+1)/6}$. $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2}$ is greater than $|M(x)|$, indicating that the Stieltjes hypothesis is true. $\sum_{i=1}^{x} \log i/d(i)$ appears to be consistently greater than $\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2$ for $x > 20$. For $x \le 5128$, $\sqrt{\sum_{i=1}^{x} M(\lfloor x/i\rfloor)^2}$ is about half-way between $A(x)/\sqrt{x(x+1)(2x+1)/6}$ and $\psi(x)/(A(x)/\sqrt{x(x+1)(2x+1)/6})$.

$\sum_{i=1}^{x} M(\lfloor x/i\rfloor)$ is less than or equal to $\sum_{i=1}^{x} \phi(i)/i$. For a linear least-squares fit of $\sum_{i=1}^{x} \phi(i)/i$ versus x for $x = 2, 3, 4, ..., 1000$, $p_1 = 0.6079$ with a 95% confidence interval of (0.6079, 0.608), $p_2 = 0.3041$ with a 95% confidence of (0.2867, 0.3216), SSE=19.58, R-square=1, and RMSE=0.1401. For

$x > 13$ and $x <= 5128$, $\sqrt{\sum_{i=1}^{x} \phi(i)/i}$ has been confirmed to be greater than $|M(x)|$. ($\sum_{i}^{x} \phi(i)/i$ is analogous to $\sum_{i=1}^{x} \log i/d(i)$. i is another way of expressing $\sum_{d|i} \phi(d)$ and $d(i) \log i$ is another way of expressing $\sum_{d|i} \log d$.)

For a linear least-squares fit of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where $M(1)$ is set to 0, $p_1 = 0.1885$ with a 95% confidence interval of (0.1885, 0.1885) and $p_2 = 0.09297$ with a 95% confidence interval of (0.08563, 0.1003). SSE=91.91, R-square=1, and RMSE=0.1339. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 1.0 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 1.0 less than that of the least-squares fit. A $\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i$ value is less than or equal to the previous value (a value is equal to the previous value only if x is a power of 2). For a linear least-squares fit of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where $M(1)$ and $M(3)$ are set to 0, $p_1 = 0.1529$ with a 95% confidence interval of (0.1529, 0.1529) and $p_2 = 0.0746$ with a 95% confidence interval of (0.06539, 0.08382). SSE=145, R-square=1, and RMSE=0.1682. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 1.0 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 1.0 less than that of the least-squares fit. A $\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i$ value is greater than the previous value if 12 divides $x - 6$, otherwise the value is less than the previous value. For a linear least-squares fit of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where $M(1)$, $M(3)$, and $M(4)$ are set to 0, $p_1 = 0.1332$ with a 95% confidence interval of (0.1332, 0.1332) and $p_2 = 0.06404$ with a 95% confidence interval of (0.05169, 0.0764). SSE=260.6, R-square=1, and RMSE=0.2255. The values are bounded above by a line where the slope is the same as that of the least-squares fit and the y-intercept is 1.0 more than that of the least-squares fit. Similarly, the values are bounded below by a parallel line where the y-intercept is 1.0 less than that of the least-squares fit. Except when $x = 8$, a $\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i$ value is less than the previous value if 6 does not divide x. When 6 divides x, a $\frac{1}{2}\sum_{i}^{x} M(\lfloor x/i \rfloor)i$ value is less than or equal to the previous value only if 5 also divides x. For linear least-squares fits of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where the M values up to and including $M(5)$, $M(6)$, $M(7)$, ..., $M(15)$ are set to 0, the slopes are 0.1078, 0.09893, 0.086, 0.07587, 0.0677, 0.06441, 0.05907, 0.05455, 0.04862, 0.04511, and 0.0436 respectively and the y-intercepts are 0.05121, 0.04607, 0.03871, 0.033, 0.02939, 0.02691, 0.02301, 0.01939, 0.01661, 0.01567, and 0.01452 respectively (all the R-square values equal 1). Except for two x values (2263 and 4199) for the linear least-squares fit of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where the M values up to and including $M(12)$ are set to 0, the values are bounded above by a line having the same slope as the least-squares fit and a y-intercept 1.0 more than that of the least-squares fit. The values are bounded below by a line having the same slope as the least-squares fit and a y-intercept 1.0 less than that of the least-squares fit. For a quadratic least-squares fit of these slopes (0.1885, 0.1529, 0.1332, 0.1078, 0.09893, 0.086, 0.07587, 0.0677, 0.06441, 0.05907, 0.05455, 0.04862, 0.04511, and 0.0436) versus $\sqrt{x}$ for $x = 1, 2, 3, ..., 14$, $p_1 = 0.0374$ with a 95% confidence interval of (0.01188, 0.01559), $p_2 = -0.1178$ with a 95% confidence interval of (-0.127, -0.1087), and $p_3 = 0.2926$ with a 95% confidence interval of (0.2821, 0.3031). SSE=4.463e-5, R-square=0.9982, and RMSE=0.002014.

For a linear least-squares fit of $f(x) := \sum_{i=1}^{x}(m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor})$ versus x for $x = 2, 3, 4, ..., 101$, $p_1 = -0.167$ with a 95% confidence interval of (-0.1701, -0.164) and $p_2 = 0.1822$ with a 95% confidence interval of (0.0006782, 0.3637). SSE=19.6, R-square=0.9916, and RMSE=0.4472. The least-squares fit has a slope of about $-\frac{1}{6}$ since $f(x + 12) = f(x) - 2$ for $x = 2, 3, 4, ...$ For a linear least-squares fit of $\sqrt{-\sum_{i=1}^{x}(m_{\lfloor x/i \rfloor} - n_{\lfloor x/i \rfloor})i}$ versus x for $x = 2, 3, 4, ..., 5128$, $p_1 = 0.1549$ with a 95% confidence interval of (0.1549, 0.1549) and $p_2 = 0.07496$ with a 95% confidence interval of (0.05814, 0.09178). SSE=483.2, R-square=1, and RMSE=0.3071. This is about the same slope and y-intercept as that for the linear least-squares fit of $\sqrt{-\frac{1}{2}\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i}$ versus x for $x = 2, 3, 4, ..., 5128$ where $M(1)$ and $M(3)$ were set to 0.

4 A Measure Associated with the Farey Sequence Polygon

A simple polygon is generated when the denominators of the fractions in a Farey sequence are mapped to the y axis of a rectangular coordinate system, the numerators of the fractions are mapped to the x axis, and the points corresponding to the successive fractions are connected (no proof that the polygon is simple is given here). The point corresponding to 0/1 is included so that the point corresponding to 1/1 can be connected to it, thereby closing the polygon. In the following, the square roots of the lengths of the sides of this polygon (excluding the side corresponding to 1/1 and 0/1) are used as a measure in lieu of $|\delta_v|$. The two greatest lengths of sides of the polygon are $((x-1)^2 + (x-2)^2)^{1/2}$ and $((x-1)^2 + 1)^{1/2}$. The average square root of the lengths is then less than $\sqrt{x}$. When plotted against $\sqrt{x}$, the "curve" of average square roots of lengths is then bounded above by a line having a slope of 1. For a linear least-squares fit of the average length of a side of the polygon versus x for $x = 2, 3, 4, ...900$, $p_1 = 0.3826$ with a 95% confidence interval of (0.3826, 0.3826) and $p_2 = 0.1994$ with a 95% confidence interval of (0.1867, 0.2122). SSE=8.511, R-square=1, and RMSE=0.09741. For a linear least-squares fit of the average square root of the length of a side of the polygon versus $\sqrt{x}$ for $x = 2, 3, 4, ..., 900$, $p_1 = 0.5689$ with a 95% confidence interval of (0.5687, 0.569) and $p_2 = 0.04642$ with a 95% confidence interval of (0.04261, 0.05024). SSE=0.3349, R-square=1, and

RMSE=0.01932. When plotted against $\sqrt{x}$, the "curve" of average square roots of lengths has a greater slope than the "curve" of $\sqrt{A(x)\sum_{v=1}^{A(x)} \delta_v^2}$ values.

A sequence x_n of real numbers is uniformly distributed (mod 1) if and only if for every Riemann-integrable function f on [0, 1] one has $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} f([x_n]) = \int_0^1 f(x)dx$. (The brackets "[ ]" denote the fractional part of the operand.) Some re-ordering of the sequence of square roots of the lengths of the sides of the polygon generated by a Farey sequence (probably corresponding to the lexicographic ordering used for the Farey fractions) appears to be uniformly distributed (mod 1). (See the section "Farey Points" in Kuipers and Niederreiter's [6] book.) For a few functions such as sine, cosine, square, cube, etc., the sums have been computed and confirmed to approach the expected values.

$[r_v]$ (used to define β_v) is uniformly distributed (mod 1). Assuming that $A(x)$ approaches $\frac{3x^2}{\pi^2}$ as $x \to \infty$, it should be possible to prove that $\sqrt{\sum_{v=1}^{A(x)} \beta_v^2}$ approaches 1 as $x \to \infty$.

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5 Miscellaneous

2000 Mathematics Subject Classification: 11B57

Keywords: Riemann hypothesis, Farey sequence, Stieltjes hypothesis